# Non-Smooth Integrability Theory

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**Abstract**

This paper studies a reverse calculation method for consumer's preference from demand function that is not necessarily differentiable. Compared with the classical theory, the feature of this study is to avoid the use of the Slutsky matrix. Instead, we assume that the solution of the partial differential equation corresponding with the Shephard's lemma has a concave solution. If the demand function is continuously differentiable, then this assumption is equivalent to the negative semi-definiteness and symmetry of the Slutsky matrix. Further, we demonstrate that our result is applicable for a demand function with quasi-linear preference by showing an example.

**Keywords**: demand function, integrability, income-Lipschitzian, expenditure function, Shephard's lemma.

**JEL Classification Numbers**: D11.

# 1 Introduction

In this paper, we construct a reverse calculation method for consumer's preference from demand function that is not necessarily differentiable. This study is concerned with the integrability theory, and non-parametric estimation theory on consumer's preference.

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In traditional consumer theory, a consumer has two theoretical components: the demand function and the preference relation. The demand function is treated as an observable thing, and the preference relation as an unobservable one. If there is a method for calculating the preference relation from the demand function, then the preference relation becomes to be observable. This is the original motivation of the integrability theory.

However, as time goes on, economists found that the demand function is also unobservable——the observable things are only finite purchase data, and to obtain the demand function, we must **estimate** it. In this view, the meaning of the integrability theory should be changed. The preference relation is unobservable: but the demand function is also unobservable. Is this theory nonsense? Our answer is **NO**. The difficulties to estimate these are different. The demand function corresponds with the purchase data, and it is observable. On the other hand, the preference relation corresponds with no observable data. The preference is hidden in their mind. Therefore, if there is a method for calculating the preference relation from the demand function, then **such a method decreases the difficulty to estimate the preference relation**. This is our purpose to study such a reverse calculation method.

To archieve this purpose, however, we must answer the following five question.

1) When does the demand function have the corresponding preference relation? (rationalizability, which is a classical theme)

2) How do we calculate the preference relation actually? (constructivity)

3) Is the corresponding preference relation unique? (recoverability)

4) Is the corresponding preference relation continuous?

5) When is the estimate error small?

The most important questions are 2) and 5). To use this method for econometrics, we must prepare an easy and simple method for calculating the preference relation. Hence, 2) is important. Moreover, we hope that if the estimate error of the demand function is sufficiently small, then the estimate error of the corresponding preference relation is also small. Thus, 5) is also important. However, to show 5), clearly 3) is needed: if 3) is violated, then we can miss the true preference relation even if we can detect the true demand function. Moreover, to argue 5), we must determine the topology of the space of both demand functions and preference relations. We should use the closed convergence topology for the topology of the latter space, and thus 4) is needed.

Hosoya (2016a) argues the above five questions, and answers all questions under the continuous differentiability of demand functions. However, there are three reasons why this result is not sufficiently good. First, Hosoya (2016a) uses the local $C^1$ topology as the topology of the space of demand functions. However, under this topology, the true demand function is assumed to be differentiable, which is a too strong assumption. Second, almost every quasi-linear preference relation has a non-smooth demand function. Therefore, Hosoya (2016a) could not treat such preference relations. Third, this theory could not treat Katzner's (1968) counterexample.

The main result is theorem 1. Theorem 1 shows that for a demand function $f$ with some mild requirements, if for every $(p, m)$, there is a concave solution of the following partial differential equation

$$DE(q) = f(q, E(q)),\ E(p) = m,$$

then the demand function $f$ can be rationalized by a utility function $u_{f,\bar{p}}$. This theorem answers the questions 1) and 2) simultaneously.

Hosoya (2016b) shows that if $f$ is continuously differentiable, then the existence of such $E$ is equivalent to the negative semi-definiteness and symmetry of the Slutsky matrix. However, if $f$ is not differentiable, then the Slutsky matrix cannot be defined. If $f$ is locally Lipschitz, then by Rademacher's theorem, the Slutsky matrix can be defined at **almost every point**. However, this condition is far from the continuous differentiability, and we must add an assumption for ensuring conditions in theorem 1. Fortunately, we can obtain a reasonable sufficient condition for this assumption. (theorems 2-3) However, Katzner's example is not locally Lipschitz, though it is income-Lipschitzian, and thus the above theorems cannot be used. Instead, our theorems can be used in almost all quasi-linear cases, and thus, by using our theorems, we can recover a quasi-linear preference relation from the corresponding demand function (example 1). Note that the above partial differential equation corresponds with the Shephard's lemma, and therefore, if $f$ is rationalizable, then such a $E$ exists and it is the expenditure function.

The questions 3)-5) is not treated in this paper. Hosoya (2016c) treats 3) and 4) partially.

The next section treats our main results. The proof is in section 3.

# 2 Results

## 2.1 Definitions of Notations

We consider that the notation $\Omega$ denotes the consumption space, and assume that $\Omega$ is a subset of $\mathbb{R}^n_+$, where $n \ge 2$ be given. We write $x \gg y$ if $x_i > y_i$ for any $i$.

Choose any binary relation $\succsim$ on $\Omega$, that is, $\succsim \subset \Omega^2$. We write $x \succsim y$ if $(x,y) \in \succsim$ and $x \not\succsim y$ if $(x,y) \notin \succsim$. We say that $\succsim$ is

- **complete** if for any $x, y \in \Omega$, either $x \succsim y$ or $y \succsim x$,
- **transitive** if for any $x, y, z \in \Omega$, $x \succsim y$ and $y \succsim z$ imply $x \succsim z$,
- **continuous** if $\succsim$ is closed in $\Omega^2$,
- **upper semi-continuous** if for any $x \in \Omega$, the set $\{y \in \Omega | y \succsim x\}$ is closed in $\Omega$,
- **monotone** if for any $x, y \in \Omega$, $x \succsim y$ and $y \not\succsim x$ when $x \gg y$,
- **strictly convex** if for any $x, y \in \Omega$ with $x \succsim y$ and $x \neq y$, and $t \in ]0, 1[$, $(1-t)x + ty \succsim y$ and $y \not\succsim (1-t)x + ty$.

We call a binary relation $\succsim$ on $\Omega$ a **preference relation** if it is complete and transitive. If $\succsim$ is a preference relation, then we write $x \succ y$ if $x \succsim y$ and $y \not\succsim x$, and $x \sim y$ if $x \succsim y$ and $y \succsim x$.

Suppose that $u : \Omega \to \mathbb{R}$ satisfies the following condition:

$$u(x) \ge u(y) \Leftrightarrow x \succsim y.$$

Then, we say that $u$ represents $\succsim$, or $u$ is a **utility function** of $\succsim$. Note that if some function $u$ represents $\succsim$, then $\succsim$ is a preference relation, and $\succsim$ is continuous (resp. upper semi-continuous) if $u$ is continuous. (resp. upper semi-continuous.)[1]

Next, we call a function $f : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \Omega$ a *demand function* if it satisfies budget inequality: that is,

$$p \cdot f(p, m) \le m,$$

for any $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$. If

$$p \cdot f(p, m) = m$$

[1]Conversely, if a preference relation $\succsim$ is continuous, (resp. upper semi-continuous,) then there is a continuous (resp. upper semi-continuous) function $u$ that represents $\succsim$. This result is obtained by the second countability of $\Omega$. See Debreu (1954).

for any $(p,m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$, then this demand function is said to satisfy the Walras' law.

Suppose that $f$ is a demand function. Then, the following relations can be defined.

$$\begin{aligned} x \succ_r y &\Leftrightarrow x \neq y, \exists (p,m), x = f(p,m) \text{ and } p \cdot y \le m, \\ x \succ_{ir} y &\Leftrightarrow \exists x_0, ..., x_k \in \Omega, x_0 = x, x_k = y, \\ &\qquad \text{and } x_{i+1} \succ_r x_i \text{ for any } i = 0, ..., k-1. \end{aligned}$$

Then, $f$ satisfies the *weak axiom* if $\succ_r$ is asymmetric (that is, $x \succ_r y$ implies $y \not\succ_r x$), and $f$ satisfies the *strong axiom* if $\succ_{ir}$ is asymmetric. Clearly, the strong axiom implies the weak axiom.

Now, let $\succsim$ be a binary relation on $\Omega$ and define

$$f^{\succsim}(p,m) = \{x \in \Omega | \forall y, p \cdot y \le m \Rightarrow x \succsim y\}.$$

If $\succsim$ is strongly monotone, then $f^{\succsim}$ satisfies Walras' law. We call $f^{\succsim}$ a *demand relation induced by* $\succsim$ and say that $\succsim$ corresponds with $f$ (or, $f$ corresponds with $\succsim$) if $f = f^{\succsim}$. If $u$ represents $\succsim$, then $f^{\succsim}$ is sometimes written as $f^u$, and we say that $u$ corresponds with $f$ (or, $f$ corresponds with $u$) if $f^u = f$. It is well known that for any demand function $f$, $f = f^{\succsim}$ for some preference relation $\succsim$ if and only if $f$ satisfies the strong axiom.[2]

Finally, suppose that $f$ is a demand function. $f$ is said to be **income-Lipschitzian**[3] if for any compact set $C \subset \mathbb{R}^n_{++} \times \mathbb{R}_{++}$, there exists $L > 0$ such that for any $p \in \mathbb{R}^n_{++}$ and $m_1, m_2 > 0$ with $(p, m_i) \in C$,

$$\|f(p,m_1) - f(p,m_2)\| \le L|m_1 - m_2|.$$

Similarly, $f$ is said to be **locally Lipschitz** if for any compact set $C \subset \mathbb{R}^n_{++} \times \mathbb{R}_{++}$, there exists $L > 0$ such that for any $(p_1, m_1), (p_2, m_2) \in C$,

$$\|f(p_1,m_1) - f(p_2,m_2)\| \le L\|(p_1,m_1) - (p_2,m_2)\|.$$

## 2.2 Main Result

**Theorem 1**. Suppose that $f : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \Omega$ is a continuous, income-Lipschitzian demand function that satisfies the Walras' law. Further, the following two condition holds.

[2]See Richter (1966) or section 3.J of Mas-Colell, Whinston, and Green (1995).

[3]This name is in Mas-Colell (1977).

(I) for any $(p,m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$, there exists a concave solution $E : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ of the following partial differential equation:

$$DE(q) = f(q, E(q)), \tag{1}$$

with the initial value condition $E(p) = m$, and

(II) If $x \neq y, x = f(p,m), y = f(q,w)$, $E$ is a solution of (1) with $E(p) = m$, and $w \geq E(q)$, then $p \cdot y > m$,

Let $\bar{p} \gg 0$ and define $u_{f,\bar{p}}(x) = 0$ if $x$ is not in the range of $f$, and $u_{f,\bar{p}}(x) = E(\bar{p})$ if $x = f(p,m)$ and $E$ is the solution of the above equation. Then, this definition of $u_{f,\bar{p}}(x)$ is independent of the choice of $(p,m)$, and $f = f^{u_{f,\bar{p}}}$.

**Remarks**. Under the income-Lipschitzian property, the solution $E$ of (1) is unique. To verify this, let $E_1, E_2$ be solutions of (1) with $E_1(p) = E_2(p) = m$. Choose any $q \in \mathbb{R}^n_{++}$, and define $c_i(t) = E_i((1-t)p + tq)$. Then, $c_i$ are solutions of the following ordinary differential equation:

$$\dot{c}(t) = f((1-t)p + tq, c(t)) \cdot (q - p),\ c(0) = m.$$

By Picard-Lindelöf theorem, the solution of this equation is unique. Hence, the income-Lipschitzian property implies that $c_1(t) \equiv c_2(t)$, and thus $E_1(q) = c_1(1) = c_2(1) = E_2(q)$, as desired.

Note that, if $f = f^{\succsim}$ for some preference relation, then we have (I) holds by Shephard's lemma. Moreover, if the range of $f = f^{\succsim}$ is either open or convex, then we can show that (II) holds. Therefore, (I) and (II) is a necessary and sufficient condition of the strong axiom under some additional condition.

In general, to obtain the solution $E$ is not so easy. However, we can use the following "guess and verify" method. First, if theorem 1 can be applied, then $E$ must be the expenditure function of $u_{f,\bar{p}}$ (by Shephard's lemma), and thus it must be **homogeneous of degree one**. Second, if there exists the solution $E$ of (1), then $c(t) = E((1-t)p + tq)$ is the solution of

$$\dot{c}(t) = f((1-t)p + tq, c(t)),\ c(0) = m.$$

In particular, $c(1) = E(q)$. Therefore, to obtain a candidate of $E$, the following method is useful. First, we solve the above equation, and define $E(q) = c(1)$. Second, we examine whether this function $E$ is actually the concave solution of (1). If either $E$ cannot be defined for some $q$ or $E$ is not the concave solution of (1), then there is no concave solution of (1). In this case, the expenditure function is absent, and thus $f$ must violate the strong axiom. If $E$ can be defined and is the concave solution of (1), then theorem 1 can be applied. In next subsection, we demonstrate an example, and calculate $E$ explicitly.

Actually, to verify (II) of theorem 1 is also not so easy. Therefore, we want to obtain a sufficient condition of (II) that is easier to check than (II) itself. The following theorem answers this question. Note that, if $f$ is locally Lipschitz, then by Rademacher's theorem, it is differentiable at almost every point.

**Theorem 2**. Suppose that $f : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \Omega$ is a locally Lipschitz demand function that satisfies the Walras' law. Moreover, suppose that (I) of theorem 1 holds. Define

$$df(p, m; q, w) = \limsup_{t \downarrow 0} \frac{f(p + tq, m + tw) - f(p, m)}{t},$$

for every $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$ and $(q, w) \in \mathbb{R}^n \times \mathbb{R}$, and suppose that

(*) for every $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$ and $(q, w) \in \mathbb{R}^n \times \mathbb{R}$, there exists a convergent sequence $((p_k, m_k))$ to $(p, m)$ such that $f$ is differentiable at $(p_k, m_k)$, and

$$df(p, m; q, w) = \lim_{k \to \infty} df(p_k, m_k; q, w).$$

Then, (II) of theorem 1 holds.

**Theorem 3**. Suppose that $f : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \Omega$ is a continuous demand function that satisfies the Walras' law. Moreover, suppose that there exists a partition $A_1, ..., A_N$ of $\mathbb{R}^n \times \mathbb{R}_{++}$ and continuously differentiable functions $f^i : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \mathbb{R}^n$ such that $f(p, m) = f^i(p, m)$ for $(p, m) \in A_i$. Further, suppose that (I) of theorem 1 holds. Then, (*) holds.

## 2.3 Examples

Consider the problem

$$\begin{aligned} \max \quad & u(x) = \sqrt{x_1} + x_2, \\ \text{subject to.} \quad & x \geq 0, \\ & p \cdot x \leq m. \end{aligned}$$

Then, the solution of this problem is

$$f(p,m)=\begin{cases}(\frac{m}{p_1},0) & \text{if } p_2^2\ge 4p_1m,\\ (\frac{p_2^2}{4p_1^2},\frac{4p_1m-p_2^2}{4p_1p_2}) & \text{otherwise.}\end{cases}$$

This function is not differentiable if $p_2^2=4p_1m$. However, it satisfies the requirement of theorem 3. We shall guess the solution of the equation (1) with above $f$, and verify that it is actually the concave solution. First, choose any $q=(q_1,q_2)$. If $E$ is a solution of (1), then $c(t)=E((1-t)p+tq)$ satisfies the following ordinary differential equation:

$$\dot{c}(t)=f((1-t)p+tq,c(t))\cdot(q-p),\ c(0)=m. \tag{2}$$

Therefore, we can guess that if $c(t)$ is a solution of (2) defined on $[0,1]$, then $c(1)$ coincides with $E(q)$. Second, define

$$f^1(p,m)=(\frac{m}{p_1},0),\ f^2(p,m)=(\frac{p_2^2}{4p_1^2},\frac{4p_1m-p_2^2}{4p_1p_2}),$$

and consider

$$\dot{c}_i(t)=f_i((1-t)p+tq,c_i(t))\cdot(q-p). \tag{3}$$

To solve (3), we have

$$c_1(t)=c_1(s)\frac{p_1+t(q_1-p_1)}{p_1+s(q_1-p_1)},$$

and if $q_2=p_2$, then

$$c_2(t)=c_2(s)-\frac{1}{4}[\frac{(p_2+t(q_2-p_2))^2}{p_1+t(q_1-p_1)}-\frac{(p_2+t(q_2-p_2))^2}{p_1+s(q_1-p_1)}].$$

Third, suppose that $p_2=q_2$, $4p_1m\le p_2^2$, and $4q_1c_1(1)\le q_2^2$, where $c_1(0)=m$. Note that $4(p_1+t(q_1-p_1))c_1(t)$ is monotone. Thus, in this case we have $c(t)=c_1(t)$ on $[0,1]$, and

$$4q_1c_1(1)=\frac{4q_1^2m}{p_1}\le q_2^2.$$

Therefore, we obtain a candidate of $E(q)$: that is,

$$E(q)=\frac{q_1m}{p_1}, \tag{4}$$

if $\frac{q_1}{q_2} \leq \sqrt{\frac{p_1}{4m}}$. We can guess that $E$ is homogeneous of degree one, we can remove the assumption $p_2 = q_2$. By easy calculation, we can confirm that $E$ is actually the concave solution of equation (1) on the set $\{q | \frac{q_1}{q_2} \leq \sqrt{\frac{p_1}{4m}}\}$.

Fourth, suppose that $p_2 = q_2$, $4p_1 m \leq p_2^2$, and $4q_1 c_1(1) > q_2^2$, where $c_1(0) = m$. Note that $q_1 \neq p_1$. We can guess that $c(t) = c_1(t)$ on $[0, t^*]$, and $c(t) = c_2(t)$ on $[t^*, 1]$, where $c(t^*) = c_1(t^*) = c_2(t^*)$ and $\dot{c}_1(t^*) = \dot{c}_2(t^*)$. Then,

$$\frac{c_1(t^*)(q_1 - p_1)}{p_1 + t^*(q_1 - p_1)} = \dot{c}_1(t^*) = \dot{c}_2(t^*) = \frac{p_2^2(q_1 - p_1)}{4(p_1 + t^*(q_1 - p_1))^2},$$

and thus

$$c(t^*) = c_1(t^*) = c_2(t^*) = \frac{p_2^2}{4(p_1 + t^*(q_1 - p_1))}.$$

Then,

$$c_1(t^*) = \frac{m(p_1 + t^*(q_1 - p_1))}{p_1} = \frac{p_2^2}{4(p_1 + t^*(q_1 - p_1))},$$

and hence, we obtain

$$t^* = \frac{1}{q_1 - p_1}[\sqrt{\frac{p_1 p_2^2}{4m}} - p_1].$$

Check that $t^* \in [0, 1]$ if and only if $\frac{q_1}{q_2} \geq \sqrt{\frac{p_1}{4m}}$, which is equivalent to $4q_1 c_1(1) \geq q_2^2$. We have assumed $4q_1 c_1(1) > q_2^2$, this assumption holds, and $q_1 < p_1$. Then,

$$c(t) = \begin{cases} m\frac{p_1 + t(q_1 - p_1)}{p_1} & \text{if } t \leq t^*, \\ (p_2 + t(q_2 - p_2))\sqrt{\frac{m}{p_1}} - \frac{(p_2 + t(q_2 - p_2))^2}{4(p_1 + t(q_1 - p_1))} & \text{if } t \geq t^*. \end{cases}$$

Particularly,

$$c(1) = q_2\sqrt{\frac{m}{p_1}} - \frac{q_2^2}{4q_1}.$$

Therefore, we obtain a candidate of $E$: that is,

$$E(q) = q_2\sqrt{\frac{m}{p_1}} - \frac{q_2^2}{4q_1},$$

where this form is homogeneous of degree one. Thus, we can guess that

$$E(q) = \begin{cases} \frac{q_1 m}{p_1}, & \text{if } \frac{q_1}{q_2} \leq \sqrt{\frac{p_1}{4m}}, \\ q_2\sqrt{\frac{m}{p_1}} - \frac{q_2^2}{4q_1}, & \text{otherwise.} \end{cases} \tag{5}$$

We can check that this $E$ is actually the concave solution of (1) with $E(p) = m$.

By similar arguments, we obtain a candidate of the solution $E$ of (1) even if $4p_1m > p_2^2$: that is,

$$E(q) = \begin{cases} q_1\left(\frac{2p_2}{p_1} + \frac{4p_1m - p_2^2}{4p_1p_2}\right)^2, & \text{if } \frac{q_2}{q_1} \ge 2\left(\frac{2p_2}{p_1} + \frac{4p_1m - p_2^2}{4p_1p_2}\right), \\ q_2\left(\frac{2p_2}{p_1} + \frac{4p_1m - p_2^2}{4p_1p_2}\right) - \frac{q_2^2}{4q_1}, & \text{otherwise.} \end{cases} \tag{6}$$

It can easily verified that this $E$ is actually the concave solution of (1). Therefore, theorem 1 can be applied.

Set $\bar{p} = (1, 1)$, and choose any $x \in \mathbb{R}^n_+$ with $x_1 > 0, x_2 = 0$. If $2\sqrt{x_1} \le 1$, then $x = f(\bar{p}, x_1)$, and thus $u_{f,\bar{p}}(x) = x_1$. If $2\sqrt{x_1} > 1$, then $x = f(\frac{1}{2\sqrt{x_1}}, 1, \sqrt{x_1}/2)$, and thus $u_{f,\bar{p}}(x) = \sqrt{x_1} - \frac{1}{4}$.

Next, choose any $x \in \mathbb{R}^n_+$ with $x_1, x_2 > 0$. Set $p_2 = 1, p_1 = \frac{1}{2\sqrt{x_1}}$, and $m = x_2 + \frac{\sqrt{x_1}}{2}$. Then, $x = f(p, m)$. If $2(\sqrt{x_1} + x_2) \le 1$, then

$$u_{f,\bar{p}}(x) = (\sqrt{x_1} + x_2)^2,$$

and if $2(\sqrt{x_1} + x_2) > 1$, then

$$u_{f,\bar{p}}(x) = \sqrt{x_1} + x_2 - \frac{1}{4}.$$

To summarize, we have

$$u_{f,\bar{p}}(x) = \begin{cases} (u(x))^2, & \text{if } 2u(x) \le 1, \\ u(x) - \frac{1}{4}, & \text{if } 2u(x) > 1. \end{cases} \tag{7}$$

Clearly, $u_{f,\bar{p}}$ represents the same preference as $u$.

# 3 Proofs

## 3.1 Proof of Theorem 1

Suppose that (I) and (II) holds. Let $x \ne y, x = f(p, m), y = f(q, w), p \cdot y \le m$, and $E$ (resp. $F$) be the solution of (1) with $E(p) = m$ (resp. F(q)=w). By contraposition of (II), we have $E(q) > w = F(q)$, and thus, we have $E(r) > F(r)$ for every $r \in \mathbb{R}^n_{++}$. Particularly, $m = E(p) > F(p)$, and hence, we have $q \cdot x > w$ by (II), which implies that the weak axiom holds.

Suppose that $x = f(p, m) = f(q, w)$. Let $p(t) = (1 - t)p + tq$ and $m(t) = (1 - t)m + tw$. If $f(p(t), m(t)) = y \ne x$ for $t \in [0, 1]$, then $p(t) \cdot y =$

$m(t) = p(t) \cdot x$, and thus by the weak axiom, $p \cdot y > m$ and $q \cdot y > w$, which implies that $p(t) \cdot y > m(t)$, a contradiction. Therefore, $f(p(t), m(t)) \equiv x$ on $[0, 1]$. Then,

$$\dot{m}(t) = \dot{p}(t) \cdot x = (q - p) \cdot f(p(t), m(t)), \ m(0) = m.$$

Meanwhile, if $E$ is a solution of (1) with $E(p) = m$, and $c(t) = E((1-t)p+tq)$, then

$$\dot{c}(t) = (q - p) \cdot f(p(t), c(t)), \ c(0) = m.$$

Therefore, both $c(t)$ and $m(t)$ are the solution of the same ordinary differential equation. By income-Lipschitzian assumption and Picard-Linderöf theorem, such a solution is unique, and thus $c(t) \equiv m(t)$. Particularly,

$$E(q) = c(1) = m(1) = w,$$

which implies that the definition of $u_{f,\bar{p}}(x)$ does not depend on the choice of $(p, m)$.

Next, let $x = f(p, m), x \neq y$ and $p \cdot y \leq m$. If $y$ is not in the range of $f$, then $u_{f,\bar{p}}(y) = 0 < u_{f,\bar{p}}(x)$. Otherwise, let $E$ (resp. $F$) be the solution of (1) with $E(p) = m$ (resp. $F(q) = w$). By contraposition of (II), we have $E(q) > w = F(q)$. This implies that $E(\bar{p}) > F(\bar{p})$, and thus $u_{f,\bar{p}}(x) > u_{f,\bar{p}}(y)$. Therefore, we have $f^{u_{f,\bar{p}}}(p, m) = x = f(p, m)$. This completes the proof. ■

## 3.2 Proof of Theorem 2

First, we introduce a lemma.

**Lemma 1**. Choose any $p, q \in \mathbb{R}^n_{++}$ and $m > 0$, and let $E$ be the solution of (1) and $p(t) = (1 - t)p + tq, d(t) = p \cdot f(p(t), E(p(t)))$. Then, $d(t)$ is nondecreasing on $[0, 1]$.

**Proof of lemma 1**. Suppose not. Then, there exists $t_1, t_2 \in [0, 1]$ such that $t_1 < t_2$ and $d(t_1) > d(t_2)$. Let

$$c(t) = (t - t_1)(d(t_2) - d(t_1)) - (t_2 - t_1)(d(t) - d(t_1)).$$

Then, $c(t_1) = c(t_2) = 0$. Because $c(t)$ is continuous on $[t_1, t_2]$, there exists $t^* \in ]t_1, t_2[$ such that $c(t^*)$ attains either the maximum or the minimum on $[t_1, t_2]$. If $c(t^*)$ attains the minimum, then

$$\liminf_{t \downarrow t^*} \frac{c(t) - c(t^*)}{t - t^*} \geq 0.$$

This implies that

$$\limsup_{t\downarrow t^*} \frac{d(t)-d(t^*)}{t-t^*} \le \frac{d(t_2)-d(t_1)}{t_2-t_1} < 0.$$

Now, suppose that $t > t^*$. Then,

$$\left|\frac{d(t)-d(t^*)}{t-t^*} - p\cdot \frac{f(p(t),E(p(t^*))+(t-t^*)\ \frac{d}{ds}E(p(s))\big|_{s=t^*}) - f(p(t^*),E(p(t^*)))}{t-t^*}\right|$$

$$\le \|p\| \left\|\frac{f(p(t),E(p(t))) - f(p(t),E(p(t^*))+(t-t^*)\ \frac{d}{ds}E(p(s))\big|_{s=t^*})}{t-t^*}\right\|$$

$$\le L\|p\| \left|\frac{E(p(t))-E(p(t^*))-(t-t^*)\ \frac{d}{ds}E(p(s))\big|_{s=t^*}}{t-t^*}\right|$$

$$\to 0 \text{ as } t\downarrow t^*,$$

where $L > 0$ is some positive constant whose existence is ensured by the local Lipschitz property. We abbriviate $\frac{d}{ds}E(p(s))$ by $w^*$. By (*), there exists a sequence $(p_k, m_k)$ such that $(p_k, m_k) \to (p(t^*), E(p(t^*)))$, and

$$p\cdot df(p_k,m_k;(q-p),w^*) \to p\cdot df(p(t^*),E(p(t^*));(q-p,w^*)) = \limsup_{t\downarrow t^*} \frac{d(t)-d(t^*)}{t-t^*}.$$

However,

$$\begin{aligned} df(p_k,m_k;(q-p),w^*) &= [D_pf(p_k,m_k) + D_mf(p_k,m_k)f^T(p(t^*),E(p(t^*)))](q-p) \\ &= S_f(p_k,m_k)(q-p) \\ &\quad + D_mf(p_k,m_k)(f^T(p(t^*),E(p(t^*))) - f^T(p_k,m_k))(q-p), \end{aligned}$$

where $S_f(p_k,m_k) = D_pf(p_k,m_k) + D_mf(p_k,m_k)f^T(p_k,m_k)$ is a Slutsky matrix of $f$ at $(p_k,m_k)$. Clearly, if $E_k$ is a solution of (1) with initial value condition $E_k(p_k) = m_k$, then $D^2E_k(p_k) = S_f(p_k,m_k)$. Because $E_k$ is concave, it is negative semi-definite. Moreover, to differentiate the both side of the Walras' law, we have

$$p_k^TD_pf(p_k,m_k) = -f^T(p_k,m_k),\ p_k^TD_mf(p_k,m_k) = 1,$$

and thus,

$$p_k^TS_f(p_k,m_k) = 0.$$

Therefore,

$$\begin{aligned} p^Tdf(p_k,m_k;(q-p),w^*) &= -t^*(q-p)^TS_f(p_k,m_k)(q-p) + (p(t^*)-p_k)^TS_f(p_k,m_k)(q-p) \\ &\quad + p^TD_mf(p_k,m_k)(f^T(p(t^*),E(p(t^*))) - f^T(p_k,m_k))(q-p) \\ &\ge (p(t^*)-p_k)^TS_f(p_k,m_k)(q-p) \\ &\quad + p^TD_mf(p_k,m_k)(f^T(p(t^*),E(p(t^*))) - f^T(p_k,m_k))(q-p), \end{aligned}$$

where the left-hand side goes to zero as $k \to \infty$ because of the local Lipschitz condition of $f$. This implies that

$$\limsup_{t \downarrow t^*} \frac{d(t) - d(t^*)}{t - t^*} \geq 0,$$

a contradiction. This completes the proof of lemma 1. ■

Now, let $x \neq y, x = f(p, m), y = f(q, w)$ and $E$ (resp. $F$) be the solution of (1) with $E(p) = m$ (resp. $F(q) = w$).

If $F(q) = w > E(q)$, then $F(p) > m$, and thus by lemma 1,

$$p \cdot y = p \cdot f(q, F(q)) = d(1) \geq d(0) = p \cdot f(p, F(p)) = F(p) > m,$$

where $p(t) = (1-t)p + tq$ and $d(t) = p \cdot f(p(t), F(p(t)))$.

Next, suppose that $F(q) = w = E(q)$. Then, we have $E \equiv F$. Let $p(t) = (1-t)p + tq, d(t) = p \cdot f(p(t), E(p(t)))$. It suffices to show $m = d(0) < d(1)$. Suppose not. By lemma 1, we have $d(t) \equiv d(0)$ on $[0, 1]$. Let $X(r) = f(r, E(r))$ and $Y(t) = X(p(t))$. Because $Y(0) = x \neq y = Y(1)$ and $Y$ is absolutely continuous, there exists $t^* \in ]0, 1[$ such that $\dot{Y}(t^*) \neq 0$.

Let $w^*$ and $(p_k, m_k)$ be the same as in the proof of lemma 1, and $S_k$ denote $S_f(p_k, m_k)$. Note that by Young's theorem, we have $S_k$ is symmetric and negative semi-definite, and thus there exists a symmetric and positive semi-definite matrix $A_k$ such that $S_k = -A_k^2$.[4] Then,

$$-t^*(q-p)S_k(q-p) = t^*\|A_k(q-p)\|^2.$$

Because $\dot{d}(t^*) = 0$, we have $A_k(q-p) \to 0$ as $k \to \infty$.

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[4]If

$$S_k = P^T \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} P,$$

where $P$ is an orthogonal matrix and $\lambda_1, ..., \lambda_n$ are eigenvalues of $S_k$, then

$$A_k = P^T \begin{pmatrix} \sqrt{-\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{-\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{-\lambda_n} \end{pmatrix} P.$$

Meanwhile, by the same arguments as in lemma 1, we have

$$\left|\frac{Y(t)-Y(t^*)}{t-t^*} - \frac{f(p(t),E(p(t^*))+(t-t^*)w^*)-f(p(t^*),E(p(t^*)))}{t-t^*}\right|$$
$$\leq L\left|\frac{E(p(t))-E(p(t^*))+(t-t^*)\left.\frac{d}{ds}E(p(s))\right|_{s=t^*}}{t-t^*}\right|$$
$$\to 0 \text{ as } t\downarrow t^*,$$

where $L>0$ is some constant. Therefore, we have

$$\dot{Y}(t^*) = \lim_{k\to\infty} df(p_k,m_k;(q-p),w^*).$$

This implies that

$$\dot{Y}(t^*) = \lim_{k\to\infty} S_k(q-p) = \lim_{k\to\infty} A_k(A_k(q-p)) = 0,$$

which is absurd.[5] This completes the proof. ■

## 3.3 Proof of Theorem 3

Clearly, this function $f$ is locally Lipschitz, and by Rademacher's theorem, it is differentiable at almost every point. If $f$ is differentiable at $(p,m)$, then for any $(q,w)$, there exists $i$ and a sequence $(t_k)$ of positive real numbers such that $t_k\downarrow 0$ and $(p+t_kq,m+t_kw)\in A_i$. By continuity, we have $f(p,m)=f^i(p,m)$ and thus,

$$\begin{aligned}
Df(p,m)(q,w) &= \lim_{k\to\infty}\frac{f(p+t_kq,m+t_kw)-f(p,m)}{t_k}\\
&= \lim_{k\to\infty}\frac{f^i(p+t_kq,m+t_kw)-f^i(p,m)}{t_k}\\
&= Df^i(p,m)(q,w).
\end{aligned}$$

Therefore, if we define

$$B_i = \{(p,m)\in\mathbb{R}^n_{++}\times\mathbb{R}_{++}|f(p,m)=f^i(p,m),\ Df(p,m)(q,w)=Df^i(p,m)(q,w)\},$$

then $\cup_i B_i$ is dense in $\mathbb{R}^n_{++}\times\mathbb{R}_{++}$.

Choose any $(p,m)\in\mathbb{R}^n_{++}\times\mathbb{R}_{++}$ and $(q,w)\in\mathbb{R}^n\times\mathbb{R}$, and a sequence $(t_k)$ of positive real numbers such that $t_k\downarrow 0$ and

$$df(p,m;q,w) = \lim_{k\to\infty}\frac{f(p+t_kq,m+t_kw)-f(p,m)}{t_k}.$$

[5] Note that the operator norm of $A_k$ is less than the square root of the operator norm of $S_k$, which is bounded by the local Lipschitz condition.

Taking a subsequence, we can assume that there exists $i$ such that for every $k$, $(p+t_kq, m+t_kw)$ is in the closure of $B_i$. Then, $(p,m)$ is also in the closure of $B_i$, and by continuity of $f$, we have $f(p,m) = f^i(p,m)$ and $f(p+t_kq, m+t_kw) = f^i(p+t_kq, m+t_kw)$. Clearly,

$$df(p,m;q,w) = Df^i(p,m)(q,w),$$

and thus, if we choose any sequence $((p_k, m_k))$ in $B_i$ such that $(p_k, m_k) \to (p,m)$, then

$$df(p_k, m_k; q, w) = Df^i(p_k, m_k)(q,w) \to Df^i(p,m)(q,w) = df(p,m;q,w),$$

which completes the proof. ■

# References


[1] Debreu, G., 1954. Representation of a Preference Ordering by a Numerical Function. In: Thrall, R. M., Coombs, C. H., Davis, R. L. (Eds.), Decision Processes. Wiley, pp.159-165.

[2] Hosoya, Y. (2016a) "The Relationship between Revealed Preference and the Slutsky Matrix", mimeo.

[3] Hosoya, Y. (2016b) "On First-Order Partial Differential Equations: An Existence Theorem and Its Applications", Advances in Mathematical Economics 20, 77-87.

[4] Hosoya, Y. (2016c) "The Recoverability Revisited", mimeo.

[5] Katzner, D. (1968) "A Note on the Differentiability of Consumer Demand Function", Econometrica 36, 415-418.

[6] Mas-Colell, A., 1977. The Recoverability of Consumers' Preferences from Market Demand Behavior. Econometrica 45, 1409-1430.

[7] Mas-Colell, A., Whinston, M. D., Green, J. R., 1995. Microeconomic Theory. Oxford University Press, Oxford.

[8] Richter, M. K., 1966. Revealed Preference Theory. Econometrica 34, 635-645.