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A tale of two cities:

Urban spatial structure and mode of transportation

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Abstract

This paper discusses the interdependence of the spatial structure of a city and the transport mode used there to obtain two types of equilibria. One is the "auto city equilibrium" at which workers, distributed over a city thinly, use an automobile for their commutes. The other is the "rapid transit city equilibrium" at which rapid transit services are provided for the commutes of workers, who are distributed densely. We have derived and characterized the conditions for each type of equilibrium. Furthermore, the possibility of multiple equilibria has been studied.

Keywords: auto city; density; multiple equilibria; rapid transit city; spatial structure of a city, transport firm

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1 Introduction

The spatial distribution of economic activities within a city significantly varies across cities. Notably, some cities extend over a quite broad plain where population and employment spread out more or less evenly, whereas others are compact with a large part of economic activities taking place in a fairly small area. This variation is attributable to a number of factors such as histories, social compositions, political environments and the policies for urban planning. Most people would agree, however, that one of the most important is a difference in the modes of transportation used in a city. It can be argued, for instance, that the spatial structure of Los Angeles has been shaped based on the use of automobile while that of Paris on the use of horse-drawn wagons, buses, trams or subways. The question of how the level of transport costs affects the spatial structure of a city is well answered by a standard economic theory on urban structure. The Alonso-Mills-Muth model of land use patterns, for instance, demonstrates that lower transport costs bring about a more outspread city with a lower population density in its inner area as long as its population is fixed. This proposition will explain the difference in spatial structures between the two cities currently observed, if Los Angels was built when transport costs were low while Paris was built when they were high.[1]

For all that, this explanation tells only one side of a story. The immediate question is why such a divergence in urban structure *persists*. Why did Paris, for example, not switch to an automobile city like today's Los Angels in the latter half of the 20th century? Obvious answer would be that *the mode of transportation used in a city depends on the spatial structure of that city*: In Paris, a highly dense concentration of population and employment along with insufficient total length and area of roads makes the commutes by automobile infeasible. In Los Angels, contrastingly, the provision of a rapid transit is not profitable enough because of relatively thin population and employment in a central business district and/or in suburban subcenters (it is only recently that a rapid transit system was constructed there, although it carries only a quite small portion of commuters compared to automobile). The key observation is that *not only is the spatial structure of a city determined by the mode of transportation used there, but also the mode of transportation used in a city is determined by the spatial structure of that city.* In other words, the causality between the spatial structure and the mode of transportation used goes *both way*.

Acknowledging this interdependence, we can understand the persistence of a certain spatial structure as a result of a lock-in effect arising from multiple equilibria. That is, in one equilibrium, referred to as an "auto city equilibrium", population density is low and workers use automobile to go to work. In the other equilibrium, a "rapid transit city equilibrium", population density is high enough for a transport firm to raise a sufficient amount of revenue to cover the construction

[1] It would be fair to point out that the population density becomes *higher*, to the contrary, in the city with lower transport costs, if the city is open and intercity migration is allowed. In this "small open city" case, therefore, the today's difference between the two cities would be explained from the hypothesis that Los Angeles was built when transport costs were *high* while Paris was built when they were *low*.

and operation of a rapid transit system. Here, a particular urban structure is supported as an equilibrium outcome when a particular mode of transportation is used; but it is no longer so when a different mode is used. At the same time, the use of a particular mode of transportation is supported as an equilibrium outcome when a city has a particular spatial structure; but it is no longer so when a city has a different spatial structure. It is this property of multiple equilibria that generates the lock-in effect. Furthermore, what is responsible for the property is that it takes quite long, sometimes infinite, time for an urban spatial structure to adjust to a change in its environments such as a change in the mode of transportation used there. There are mainly two reasons. For one thing, the coordination among the agents whose behaviors result in forming an urban spatial structure is usually highly difficult or even impossible because their number is enormous. What is more, the most important element of an urban structure is buildings, which are physically durable and whose economic values decline only slowly.

The aim of this paper is to examine such a possibility of multiple equilibria through a rigorous analysis. For that purpose, we construct a model of urban land use of the Alonso-Miills-Muth type incorporating a transport firm that may operate a rapid transit system. First of all, we examine the condition for each type of the equilibrium mentioned above. Then, we pay a special attention to the situation where both types of equilibria emerge for the same set of parameters. In this situation of multiple equilibria, even if two cities were endowed equally by nature, it may happen that one develops to an auto city and the other to a rapid transit city by, for instance, a historical accident. This is our tale of two cities.

One significance of this study concerns a recent tendency in urban policies toward a more compact city. In many cities, especially those in developed countries, the attempts to maintain and to accumulate residences and work places in a narrower area of a traditional city center have becoming more and more common. It is expected that they help us take advantage of the benefits of agglomeration economies further and reduce environmental burdens. What is important here is that such attempts often accompany new construction or re-development of a public transportation system represented by subways and trams. Thus, we can regard them as the attempts to switch the equilibrium where a city falls from the auto city equilibrium to the rapid transit city equilibrium. In this context, the newly developed mass transit system probably suffers losses, *given the spatial structure of the auto city, that is, sparse distributions of population and employments*. Therefore, the transit system cannot be run by a private company, and a government needs to continue to pay the losses until the city becomes concentrated enough to yield a sufficient amount of demand for mass transit services. That day may or may not going to come, and even if it comes, it will be in a fairly distant future, because the spatial structure changes only gradually. In this way, our perspective will provide a good conceptual framework to understand and evaluate the attempts to make a city more compact.

The rest of this paper consists of five sections. In Section 2, the model is presented. We first explain basic settings and then introduce transportation technologies. The next section defines two types of equilibria mentioned above, namely, the auto city equilibrium and the rapid transit city equilibrium. In Section 4, we derive the conditions for each of the two types of equilibria. The condition for both types co-existing is also derived. In Section 5, we explore the two types of equilibria by simulation analyses specifying functional forms and parameters. Section 6 concludes.

2 Model

2.1 Basic settings

We consider a linear city with its width being unity in a homogenous plane. Note that it is a set of the locations within 0.5 miles from the straight line penetrating the middle of the city. Therefore, we can interpret the city as the area served by a rapid transit line running along that midway line when the maximum walking distance for commuters is 0.5 mile (here, we are ignoring the fact that stations are not continuously but discretely placed). With this interpretation, assuming a linear city is rather more natural than assuming a disk-shaped city as in a standard model as long as only one transit line is concerned.

The city is monocentric and the "center" is located at its endpoint. Locations within the city is identified by the distance from the city center, $d \geq 0$.

There are workers endowed with 1 unit of time. Spending some portion of it, they work at the city center and earn a wage, whose rate is fixed at w. They live somewhere in the city and commute to the center, which takes both monetary and time costs. The monetary cost for a worker who lives at d is denoted by $m(d)$. For the time cost, we denote the required time for a d-mile commute by $t(d)$. We assume that both $m(\cdot)$ and $t(\cdot)$ are increasing functions.

Workers consume land for housing, a composite good and leisure, whose amounts are denoted by x, z and e, respectively. Here, we take the approach of Train and McFadden (1978) in which workers, facing a trade-off between consuming a greater amount of goods and enjoying longer time for leisure, freely choose the length of leisure time and thus work hours, denoted by n.[2] Their time constraint is given by $n + e + t(d) = 1$. Furthermore, assuming that the composite good is a numeraire and that land rent is taken by absentee landlords, we obtain their budget constraint as $r(d)x + z + m(d) = wn$, where $r(d)$ is the land rent at d. It is convenient to combine these two constraints into

$$r(d)x + z + we = y(d) \equiv w - c(d), \tag{1}$$

where $c(d) \equiv wt(d) + m(d)$ is a "generalized cost" of a commute inclusive of both monetary and time costs. $c(\cdot)$ is an increasing function. Furthermore, $y(d)$, a decreasing function of d, is the

[2]Another approach involves fixed work hours. The length of leisure time is automatically determined as a residual of the total available time after subtracting the work hours and commuting time. The reality is probably in between the worlds depicted by the two approaches. For the differences in their implications, see Jara-Diaz (2007), for instance.

income that would be earned by a worker who worked for all the available time, $w[1-t(d)]$, subtracted by the monetary cost of commute, $m(d)$. That is, it is the measure of a "potential" disposable income.

Workers have the same preference represented by a utility function, $U(x,z,e)$, which they maximize subject to (1). Because the division of spendings between the two goods and the leisure is not a main concern in this paper, we assume that the utility function can be written as follows:

$$U(x,z,e) = \beta \ln u(x,z) + (1-\beta) \ln e. \tag{2}$$

Then, the amount of leisure consumption is

$$e = \frac{(1-\beta)y(d)}{w}. \tag{3}$$

Note that β becomes equal to the ratio of an actually earned disposable income to the potential income, that is,

$$\beta = \frac{wn - m(d)}{y}. \tag{4}$$

Thus, we can interpret it as a relative measure of work hours.

Choosing x and z, workers maximize the sub-utility function, $u(x,z)$, subject to $r(d)x + z = \beta y(d)$. First order necessary condition is given by

$$u_1\Big(x,\ \beta y(d) - r(d)x\Big) - r(d)u_2\Big(x,\ \beta y(d) - r(d)x\Big) = 0, \tag{5}$$

where $u_i(\cdot)$ denotes a partial derivative of $u(\cdot)$ with respect to its i'th argument.

The city is small and open, that is, in the long run, migration occurs between this city and the outside regions so that the level of utility for each worker in the city coincides with that prevailing outside the city, denoted by $\bar{u}$. That is,

$$U\left(x,\ \beta y(d) - r(d)x,\ \frac{(1-\beta)y(d)}{w}\right) = \bar{u} \tag{6}$$

at a *locational equilibrium*.

Solving (5) and (6) simultaneously, we can obtain the demand for land by each worker and the land rent. Because these variables are functions of only $y(d)$, we denote them as $\widetilde{x}(d)$ and $\widetilde{r}(d)$ (their arguments will be dropped if doing so causes no confusion).

In addition, outside the city extends agricultural land, which is rented out at a fixed rent, r_a. Then, the city's boundary, b, is given by a solution to

$$\widetilde{r}(b) = r_a. \tag{7}$$

Finally, one qualification needs to be satisfied. The sum of the spendings on land and on a composite good must be nonnegative. Because (1) and (3) imply $r(d)x + z = \beta\,[w - c(d)]$, this qualification is expressed as $c(d) \leq w$. We concentrate on the case where r_a is so high and therefore, the city is so small that any $d \in [0,b]$ satisfies this qualification, that is,

$$c(b) \leq w. \tag{8}$$

Before turning to transport technologies, it is useful to review the impacts of changes in exogenous variables upon the land consumption because it will play a key role in the determination of the transport mode used in a city. For that purpose, let us define Υ as $\Upsilon \equiv u_{11} - 2u_{12}\widetilde{r} + u_{22}\widetilde{r}^2$ and Ψ as $\Psi \equiv u_{12} - u_{22}\widetilde{r}$, where $u_{ij}(\cdot)$ denotes the second-order partial derivative of $u(\cdot)$ with respect to the i'th and j'th arguments (its arguments are omitted for the clarity). Note that $\Upsilon < 0$ as long as the utility function is strictly quasi-concave and the first order condition, (5), is satisfied. Furthermore, we assume that the land is a normal good, that is, $\partial\widetilde{x}/\partial\left[\beta y(d)\right] \geq 0$. Because differentiating (5) yields $\partial\widetilde{x}/\partial\left[\beta y(d)\right] = -\Psi/\Upsilon$, the assumption is equivalent to

$$\Psi \geq 0. \tag{9}$$

Now, since $\widetilde{x}(d)$ and $\widetilde{r}(d)$ are determined by equations (5) and (6), we can derive the impacts on $\widetilde{x}(d)$ by totally differentiating them and eliminating a change in $\widetilde{r}(d)$. From (9) and the condition that the marginal rates of substitution become equal to relative prices, that is, $U_x(x,z,e)/U_z(x,z,e) = \widetilde{r}(d)$ and $U_x(x,z,e)/U_e(x,z,e) = \widetilde{r}(d)/w$, the next results follow:

$$\begin{cases} \widetilde{x}'(d) &= -\dfrac{c'(d)}{\Upsilon\widetilde{x}}\left[u_2 + (1-\beta)\Psi\widetilde{x}\right] > 0, \\ \dfrac{\partial\widetilde{x}(d)}{\partial c(d)} &= -\dfrac{1}{\Upsilon\widetilde{x}}\left[u_2 + (1-\beta)\Psi\widetilde{x}\right] > 0, \\ \dfrac{\partial\widetilde{x}(d)}{\partial\bar{u}} &= -\dfrac{\widetilde{r}}{\Upsilon\widetilde{x}}\left(u_2 + \Psi\widetilde{x}\right) > 0, \\ \dfrac{\partial\widetilde{x}(d)}{\partial\beta} &= -\dfrac{\Psi y}{\Upsilon} > 0, \\ \dfrac{\partial\widetilde{x}(d)}{\partial w} &= \dfrac{1}{\Upsilon w\widetilde{x}}\left[\Big\{\beta y + m(d)\Big\}u_2 + (1-\beta)\Upsilon\widetilde{x}m(d)\right] < 0. \end{cases} \tag{10}$$

We can explain these results intuitively. First, *the piece of land demanded by each worker expands as we move farther away from the city center*. As transport costs increase, a disposable income decreases. To accomplish the same utility level, therefore, workers need to increase the consumption of either land or composite good, or both. Second, *the land lot size at a given location shrinks as the transport costs from that location to the city center decline*. The intuition behind this result is the same as before: for the locational equilibrium, it is necessary to allocate a larger land lot to workers who pay higher transport costs. In addition, the finding implies that population density rises as transport costs decline. One might wonder if this result is inconsistent with the stylized fact that the decline in transport costs, mainly brought about by the use of automobile, is a main reason for the decentralization of a city. This is partly true but partly false. Our model demonstrates that the decline causes not only the increase in population density but also a spatial expansion of a city. Although the former effect may not be observed in the actual decentralization process, the latter effect is indeed one of the most salient features of the process. Third, as the target utility level rises, a worker needs to raise the amount of land consumption. Fourth, the rise in the relative measure of work hours, β, needs to be accompanied with a change that will raise a utility level, namely, the increase in the consumption of land. Finally, the impact of a change in wage rate is twofold. First, the rise in wage rate brings about an increase in wage income. To keep the utility level constant, therefore, each worker needs to reduce the amount of land consumption other things being equal. This is a standard channel of effect that one can see in the traditional land use theory. What is unique in this paper is, however, that the rise in wage rate implies the increase in a time cost of commute, because a higher value is now attached to a given length of time. As the second line in (10) indicates, this indirect effect is positive. The last line in (10) shows that the former direct effect more than offset the latter indirect effect.

2.2 Transport modes

In the economy, there is a transport firm, which constructs and operates a rapid transit system if it decides to do so. To make the analysis simple, we confine our discussion to the case where the firm can construct only one rapid transit line running from the city center to a terminal station located somewhere in a city. The location of the terminal station, l, is decided by the firm. The city with such a rapid transit line is referred to as an "*l-mile rapid transit city*".

The costs of the construction and operation of the rapid transit system depend on the length of the transit line and are expressed by $\Gamma(l)$. The function is increasing and $\Gamma(0) > 0$ because there are some fixed inputs. It is true that the number of passengers affects the costs in the reality, but we disregard it for two reasons. First, the change in variable costs arising from higher ridership is usually much smaller than the change in fixed costs arising from the extension of the line. To a railroad company, in other words, constructing and maintaining infrastructure is a much heavier burden than daily operations. Second, it makes the analysis considerably simple without changing main results.

In order to make the analysis tractable, we assume that the firm charges for its transport services a price proportional to the distance of a trip. (Although it is not difficult to assume other functional forms for the price, the analysis would become much more complicated without yielding additional insights.) The price of the round trip between the location at d and the city center is $m_R d$, where m_R is determined by the firm. Here recall that the aim of this paper is to discuss the relationship between the urban spatial structure and the mode of transport to be used. In this respect, what matters is whether a rapid transit can be successfully introduced, that is, whether the transport firm can raise a sufficient amount of revenue to cover the necessary costs. However, it can be discussed without going through at what level the price of transport services is determined. Thus, we have the determination of m_R unspecified. The price may be equal to the marginal cost, the average cost or something else.[3]

[3]Our assumption that the production cost does not depend on the number of passengers implies that the rapid transit industry exhibits decreasing costs. It is well-known that some regulations are necessary for decreasing cost industries in order to reduce the loss from natural monopoly. One of such regulations is to require an average cost pricing of transport firms. This pricing policy is not only actually observed in a number of cities in the world, but also well justified theoretically: To maximize social welfare under the constraint that a transport firm not suffer losses, the price must be

Workers possibly choose a mode of transport for commutes from the following two.

First, every worker can drive a car to go to her office. We refer to this mode as mode "auto," or mode A. The time necessary for a commute is proportional to its distance: $t(d) = t_A d$ for $t_A > 0$. The monetary cost is, in contrast, not proportional to the distance of commute but contains a fixed term: $m(d) = m_A d + \mu$ for $m_A > 0$. The fixed term, $\mu > 0$, contains various costs arising from owning an automobile, such as the costs of purchasing a car, most importantly, an insurance fee, a tax payment and a fee for a parking lot. The money paid by workers for their commutes is taken by someone outside the city, a petroleum industry in Texas, say, or simply burned up as iceberg transport costs, which is often assumed in the literature of new economic geography. The generalized costs is equal to $c_A(d) \equiv \eta_A d + \mu$ where $\eta_A \equiv m_A + wt_A$.

Second, workers may a rapid transit, a mode "rapid transit," or mode R. We assume that when a rapid transit is available, the worker living at d can take it from that point. To put it another way, "stations" are placed continuously along the transit line. The time taken for that mode is assumed to be proportional to the distance of a commute: $t(d) = t_R d$. Since the price of the rapid transit services from d is $m_R d$, furthermore, the monetary cost is also proportional to the distance: $m(d) = m_R d$. The generalized cost is equal to $c_R(d) \equiv \eta_R d$ where $\eta_R \equiv m_R + wt_R$.

3 Definitions of two types of equilibria

In order to define an equilibrium, we need to answer two questions.

The first question is which transport mode workers choose when both modes are available. To answer this question, let us consider a short-run period in which no migration takes place between the city in consideration and the outside economy, and its urban spatial structure is fixed. That is, the size of the land lot consumed by each worker, $\bar{x}(d)$, and the land rent at each location, $\bar{r}(d)$, have been already determined and can be regarded as constants in this short-run period. If a worker experiences a change in her potential disposable income during this period, she adjusts the consumptions of composite good and leisure. In other words, a worker who pays $c(d)$ for transport services consumes $\beta\left[w - c(d)\right] - \bar{r}(d)\bar{x}(d)$ units of composite good and $(1-\beta)\left[w - c(d)\right]/w$ units of leisure. Now, suppose that both mode A and mode R are available for a worker at d. Then, she prefers mode R to mode A if

$$\begin{aligned} U\left(\bar{x}(d),\ \beta\big[w - c_R(d)\big] - \bar{r}(d)\bar{x}(d),\ \frac{(1-\beta)}{w}\big[w - c_R(d)\big]\right) \\ > U\Big(\bar{x}(d),\ \beta\big[w - c_A(d)\big] - \bar{r}(d)\bar{x}(d),\ \frac{(1-\beta)}{w}\big[w - c_A(d)\big]\Big). \end{aligned} \tag{11}$$

Since the utility function is increasing in each argument, (11) is equivalent to $c_R(d) < c_A(d)$. Furthermore, the worker is indifferent between the two modes if $c_R(d) = c_A(d)$. Here, we assume ad hoc that workers take mode R when indifferent between the two modes.[4] Finally, she prefers

equal to the average cost because a consumer surplus decreases with the price.

[4]This assumption is made only for a technical reason. We can obtain similar results assuming otherwise.

mode A to mode R if $c_R(d) > c_A(d)$. By definition, therefore, when both modes are available, a worker at d takes mode R if and only if

$$m_R \leq \widehat{m}(d) \equiv \lambda + \frac{\mu}{d}, \tag{12}$$

where $\lambda \equiv m_A + w(t_A - t_R)$: $\widehat{m}(d)$ gives the highest price that induces the workers living *at d* to take mode R. Since $\widehat{m}(\cdot)$ is a decreasing function, however, any $d < d'$ satisfies (12) if d' satisfies it. Consequently, it is actually the highest price that induces all the workers living *within d miles from the city center* to take mode R. In this sense, we call it an *upper limit price* for d. Moreover, those workers are referred to as *d-mile inner city workers*.

We can view this result from the opposite perspective, using a new variable $\widehat{d}(m_R)$:

$$\widehat{d}(m_R) \equiv \begin{cases} \dfrac{\mu}{m_R - \lambda} & \text{if} \quad m_R > \lambda \\ \infty & \text{otherwise.} \end{cases}$$

There are two cases to consider. First, suppose that $m_R > \lambda$. Then, (12) is rewritten as $d \leq \widehat{d}(m_R)$. Therefore, the workers living at $d \leq \widehat{d}(m_R)$ take mode R if that mode is available at d (see Figure 1). In other words, the workers living at $d \leq \min\left[\widehat{d}(m_R), l\right]$ take mode R since the rapid transit line is constructed up to the terminal at l. In contrast, the workers living at $d \in \left(\widehat{d}(m_R), l\right]$, if any, and those living at $d > l$ take mode A because the former do not want to use it even though it is available and the latter have no access to it. With a new notation $\widehat{D}(l, m_R) \equiv \min\left[\widehat{d}(m_R), l\right]$, we can say that the $\widehat{D}(l, m_R)$-mile inner city workers take mode R. Second, suppose that $m_R \leq \lambda$. In that case, the workers at $d \leq l$ have an access to mode R and take that mode because (12) is satisfied. The workers at $d > l$ do not have an access to it and therefore take mode A. For this case, too, we can say by the definition of $\widehat{d}(m_R)$ that the $\widehat{D}(l, m_R)$-mile inner city workers take mode R and the rest take mode A.

Figure 1: Workers' choices of modes

One may wonder if the workers living farther than the terminal station carry out a "park-and-ride", or drive a car from home to a certain station of a rapid transit and then take it to the city center. It turns out, however, that this is not the case when the fixed monetary cost of an automobile use, μ, is sufficiently high. We will show this at the end of this section. In what follows, thus, we do not consider the possibility of the park-and-ride.

The above argument enables us to distinguish two types of city; an "*auto city*", where all the workers use automobile, and the l-mile rapid transit city, where the $\widehat{D}(l, m_R)$-mile inner city workers use a rapid transit and the rest use automobile. The size of land lot each worker consumes is determined based on the amounts of transport costs she pays. To express this relationship explicitly, we use superscript $s \in \{A, R\}$, which describes the spatial structure of a city: $s = A$ represents the auto city with $c(d) = c_A(d)$ for all $d \geq 0$, while $s = R$ represents the l-mile rapid transit city with $c(d) = c_R(d)$ for $d \leq \widehat{D}(l, m_R)$ and $c(d) = c_A(d)$ for $d > \widehat{D}(l, m_R)$. $\widetilde{x}(d)$ is now written as $\widetilde{x}^s(d)$. Here, it would be worth adding that the size of a land lot at a certain point does not depend on the values of variables at another location. In particular, it is independent of macro-scale variables such as a population distribution over a city and a geographical size of a city. This is a consequence of our assumption of a small open city.

The second question to ask is under what circumstances a transport firm goes into the business. Since the population density at d is equal to $1/\widetilde{x}(d)$ and the demand for its service comes from the $\widehat{D}(l, m_R)$-mile inner city workers, the revenue of a transport firm in a city with its spatial structure being s is equal to

$$\Lambda^s(l, m_R) = m_R \int_0^{\widehat{D}(l,m_R)} \frac{d}{\widetilde{x}^s(d)} \mathrm{d}d$$

for $s \in \{A, R\}$. The firm enters the industry if and only if its profit, $\Pi^s(l, m_R) \equiv \Lambda^s(l, m_R) - \Gamma(l)$, is nonnegative.[5]

Now we are ready to define two types of equilibria.

The first type is the equilibrium at which all the workers use automobile. This is the case if a transport firm would suffer a loss by entering the industry. Formally, we say that the auto city is supported as an equilibrium outcome if

$$\Lambda^A(l, m_R) < \Gamma(l) \tag{13}$$

for *any* $l \in [0, b^A]$ and *any* $m_R \geq 0$, where b^s denotes the location of the boundary of a city with spatial structure s ($s \in \{A, R\}$). It is important to note that in this definition, an entering transport firm is to compute its revenue on the supposition that the spatial structure of the auto city will remain unchanged, that is, $\widetilde{x}(d)$ will be kept at $\widetilde{x}^A(d)$, even after it begins to operate a rapid transit system.

At the second type of equilibrium, on the other hand, some workers use a rapid transit and a transport firm can earn nonnegative profit. We say that the l-mile rapid transit city is supported as an equilibrium outcome if there exists $m^* \geq 0$ such that

$$\Lambda^R(l,\ m^*) \geq \Gamma(l). \tag{14}$$

Two comments follow. First, this definition takes into account two channels through which the price affects the revenue. One is a direct channel, which involves a change in the payment by each worker, and the other is an indirect channel, which involves a change in the spatial structure of a city. Second, as has been argued above, we do not discuss the choice of a price by a transport firm. Indeed, (l, m^*) satisfying (14) does not necessarily maximize its profit.

In the rest of this section, we demonstrate that the park-and-ride does not occur when μ is sufficiently high. Consider a worker who drives a car from her home to $d^0 \leq l$ and then takes a rapid transit from there to the city center. The cost of such an itinerary is equal to $c_A(d - d^0) +$

[5] Here, we assume that the firm enters when the cost equals the revenue. This assumption is arbitrary and can be changed.

$c_R(d^0) = (m_R - \lambda)d^0 + \eta_A d + \mu$. If $m_R > \lambda$, it is minimized at $d^0 = 0$ and a worker does not conduct a park-and-ride. If $m_R \leq \lambda$, instead, it is minimized at $d^0 = l$ and she carries out a park-and-ride parking a car at a terminal station.[6]

Now, suppose that $m_R \leq \lambda$. We know that the l-mile inner city workers use a rapid transit. Furthermore, the above argument implies that *all the workers living beyond the terminal station* park a car there and then take a rapid transit. The total revenue is therefore equal to

$$\Lambda^s_{PKRD}(l, m_R) = m_R \int_0^l \frac{d}{\widetilde{x}^s(d)} \mathrm{d}d + l\, m_R \int_l^{b^s} \frac{1}{\widetilde{x}^s(d)} \mathrm{d}d$$

for $s \in \{A, R\}$. For a given spatial structure, this revenue is increasing in m_R and therefore, maximized at $m_R = \lambda$. When there is no restriction on m_R so that the firm can charge $m_R > \lambda$, on the other hand, it may charge $\widehat{m}(l) > \lambda$ to earn $\Lambda^s(l, \widehat{m}(l))$. If μ is sufficiently high, however,

$$\Lambda^s\big(l, \widehat{m}(l)\big) - \Lambda^s_{PKRD}(l, \lambda) = \frac{\mu}{l} \int_0^l \frac{d}{\widetilde{x}^s(d)} \mathrm{d}d - \lambda l \int_l^{b^s} \frac{1}{\widetilde{x}^s(d)} \mathrm{d}d$$

is positive. In that case, therefore, the firm prefers charging $\widehat{m}(l)$, which discourages a park-and-ride, to charging λ, which gives rise to a park-and-ride. In this paper, we concentrate on this case: μ is sufficiently high that the firm does not charge too low a price that induces a park-and-ride.

4 Analysis of the conditions for each type of equilibria

In this section, we explore the conditions for each type of equilibrium.

4.1 Condition for the auto city equilibrium

It is obvious that (13) holds for any $l \in [0, b^A]$ and any $m_R \geq 0$ if and only if

$$\max_{l \in [0, b^A],\ m_R \geq 0} \Pi^A(l, m_R) < 0, \tag{15}$$

where $\Pi^A(l, m_R) \equiv \Lambda^A(l, m_R) - \Gamma(l)$. Two observations are important. First, suppose that a price is so high that $\widehat{d}(m_R) < l$. Because the $\widehat{d}(m_R)$-mile inner workers use a rapid transit, the profit increases as the length of a rapid transit line is curtailed from l miles to $\widehat{d}(m_R)$ miles. Second, suppose that a price is so low that $\widehat{d}(m_R) > l$. Then, we can raise the price up to the upper limit price, $\widehat{m}(l)$, without changing the number of passengers, which yields a higher profit. It follows from these two observations that $m_R = \widehat{m}(l)$ at the maximum of $\Pi^A(l, m_R)$ (for more rigorous derivation, see the proof of the subsequent proposition). Let us denote $\Lambda^A\big(l, \widehat{m}(l)\big) = \widehat{m}(l) \int_0^l d / \widetilde{x}^A(d)\ \mathrm{d}d$ by $\Omega^A(l)$. We have established the following result:

[6] We arbitrarily assume that workers do a park-and-ride when they are indifferent between doing it and not doing it.

Proposition 1

The auto city is supported as an equilibrium outcome if and only if

$$\Omega^A(l) < \Gamma(l) \tag{16}$$

holds for any $l \in [0, b^A]$.

Proof. Let (l^0, m^0) be the solution to $\max_{l\in[0,b^A],\, m_R\geq 0} \Pi^A(l, m_R)$. First, suppose that $\widehat{d}(m^0) < l^0$. Then, $\Pi^A\left(\widehat{d}(m^0), m^0\right) - \Pi^A(l^0, m^0) = \Gamma(l^0) - \Gamma\left(\widehat{d}(m^0)\right) > 0$ since $\Gamma(\cdot)$ is an increasing function. This contradicts the supposition that l^0 maximizes $\Pi^A(l, m_R)$, and consequently $\widehat{d}(m^0) \geq l^0$. Second, suppose that $\widehat{d}(m^0) > l^0$. Then,

$$\Pi^A\left(l^0, \widehat{m}(l^0)\right) - \Pi^A(l^0, m^0) = \left[\widehat{m}(l^0) - m^0\right] \int_0^{l^0} \frac{d}{\widetilde{x}^A(d)} \mathrm{d}d > 0,$$

which contradicts the supposition that m^0 maximizes $\Pi^A(l, m_R)$, and consequently, $\widehat{d}(m^0) \leq l^0$. Hence, we have $\widehat{d}(m^0) = l^0$, or $m^0 = \widehat{m}(l^0)$, which implies that $\max_{l\in[0,b^A],\, m_R\geq 0} \Pi^A(l, m_R) = \max_{l\in[0,b^A]} \Omega^A(l) - \Gamma(l)$. The proposition immediately follows. ■

The impact of l on $\Omega^A(l)$ is ambiguous:

$$\Omega^{A\prime}(l) = \frac{\widehat{m}(l)l}{\widetilde{x}^A(l)} - \frac{\mu}{l^2} \int_0^l \frac{d}{\widetilde{x}^A(d)} \mathrm{d}d. \tag{17}$$

The first term of the right hand side represents a marginal effect. As the rapid transit line is extended, additional workers with their mass equal to $1/\widetilde{x}^A(l)$ come to use it, each paying $\widehat{m}(l)l$. The second term, furthermore, shows a effect through the change in price. In order to tempt workers to use a longer transit line, a transport firm needs to lower its price by the amount equal to $|\widehat{m}'(l)| = \mu/l^2$.

It is worthwhile examining the impacts of changes in parameters upon the condition.

First, the auto city becomes more likely to be supported by an equilibrium as the time cost of the use of a rapid transit, t_R, rises because

$$\frac{\partial\Omega^A(l)}{\partial t_R} = -w \int_0^l \frac{d}{\widetilde{x}^A(d)} \mathrm{d}d < 0.$$

The rise in t_R, making a rapid transit less attractive, lowers the upper limit price, $\widehat{m}(l)$, which reduces the revenue of a transport firm.

Second, the directions of the impacts of changes in the costs of automobile use are ambiguous. As an example, consider the change in m^A:

$$\frac{\partial\Omega^A(l)}{\partial m_A} = \int_0^l \frac{d}{\widetilde{x}^A(d)} \mathrm{d}d - \widehat{m}(l) \int_0^l \frac{d^2}{[\widetilde{x}^A(d)]^2} \frac{\partial\widetilde{x}^A(d)}{\partial c_A(d)} \mathrm{d}d. \tag{18}$$

For one thing, as a result of the rise in m_A, the upper limit price rises, which directly raises the revenue of a transport firm. This is represented by the first term of the right hand side. At the same time, the higher m_A brings about a more sparse population distribution, which reduces the profitability of a rapid transit (remember that $\partial\tilde{x}^A(d)/\partial c_A(d) > 0$ by (10)). This is captured by the second term. If the latter indirect effect dominates the former direct effect, the overall impact is negative. In that case, the auto city becomes more likely to be supported by an equilibrium as m_A rises. If the indirect effect is dominated by the direct effect, instead, the opposite result holds. The changes in t_A and μ have similar impacts.

Third, recall that the decline of a prevailing utility level ($\bar{u}$) and that of the relative measure of work hours (β) bring about a denser population distribution at a given location (see (10)). Therefore, these changes result in a higher revenue of a transport firm, which makes it less likely for the auto city to be supported by an equilibrium.[7]

Fourth and last, the effect of a change in the wage rate (w) is ambiguous. It affects the revenue of a transport firm through two channels. First, it alters a spatial structure: as the wage rate rises, the population density goes up (remember (10)). Second, it changes the upper limit price. Recall that the rise in the wage rate implies the increases in time costs. If the time cost of the use of a rapid transit is relatively lower compared to that of automobile, the former mode becomes further attractive as a result of the rise in the wage rate, which raises the upper limit price. If the opposite holds, the upper limit price falls as the wage rate rises. Indeed, we have

$$\frac{\partial\Omega^A(l)}{\partial w} = (t_A - t_R)\int_0^l \frac{d}{\tilde{x}^A(d)}\,\mathrm{d}d - \widehat{m}(l)\int_0^l \frac{d}{[\tilde{x}^A(d)]^2}\,\frac{\partial\tilde{x}^A(d)}{\partial w}\,\mathrm{d}d. \tag{19}$$

The two terms of the right hand side represent the ambiguous effect through the change in the upper limit price and the positive effect through the change in a spatial structure (recall that $\partial\tilde{x}^A(d)/\partial w < 0$), respectively. As long as the time cost of a rapid transit use is not too high compared to that of automobile, the first effect is dominated by the second, and total effect becomes positive.

4.2 Condition for the rapid transit city equilibrium

Let us turn our eyes to a rapid transit city equilibrium. In what follows, the arguments of $\widehat{d}(m_R)$ and $\widehat{D}(l, m_R)$ will be often omitted for the sake of clarity.

There exists $m^* \geq 0$ that satisfies (14) if and only if

$$\Omega^R(l) \equiv \max_{m_R \geq 0}\ \Lambda^R(l, m_R) \geq \Gamma(l), \tag{20}$$

which leads to the following result:

Proposition 2

The l-mile rapid transit city is supported as an equilibrium outcome if and only if (20) holds.

[7] For $\chi \in \{\bar{u}, \beta\}$, (10) implies that $\dfrac{\partial\Omega^A(l)}{\partial\chi} = -\widehat{m}(l)\displaystyle\int_0^l \frac{d}{[\tilde{x}^A(d)]^2}\,\frac{\partial\tilde{x}^A(d)}{\partial\chi}\,\mathrm{d}d < 0$.

It is straightforward to see $\Omega^{R\prime}(l) \geq 0$ for almost all l, because the envelope theorem implies that it is equal to $m_R l/\tilde{x}^R(l)$ if $l < \widehat{d}$ and 0 if $l > \widehat{d}$.[8]

Furthermore, two observations follow.

First, the solution to this maximization problem can be lower than the upper limit price, $\widehat{m}(l)$, that is, $\widehat{d}(m_R)$ can exceed l. This exhibits a sharp contrast to a finding for the auto city equilibrium, the finding that raising a price up to the upper limit price results in the increase in the revenue. The reason is that the change in a price invokes a change in the spatial structure of a city. To see this, suppose that the price falls from the upper limit price. Because the users of a rapid transit are still the l-mile inner city workers, the revenue declines *other things being equal*. As a result of the fall in the price, however, the population distribution becomes denser, which gives a positive impact on the revenue.

Second, the solution to the maximization problem can be greater than the upper limit price. This finding is also different from that obtained for the auto city equilibrium. Here, our concern is to maximize the revenue for a *given* length of a rapid transit line. Consequently, there is a possibility that a part of the line is left unused at the maximum.

These two observations are easily verified by computing the following derivative:

$$\frac{\mathrm{d}\Lambda^R(l, m_R)}{\mathrm{d}m_R} = \int_0^{\widehat{D}} \frac{d}{\tilde{x}^R(d)} \cdot \left[1 - \frac{\partial \ln \tilde{x}^R(d)}{\partial \ln m_R}\right] \mathrm{d}d \; + \; \frac{\widehat{D}^2}{\tilde{x}^R(d)} \cdot \frac{\partial \ln \widehat{D}}{\partial \ln m_R}. \tag{21}$$

The right hand side shows three components of the effects of the rise in a price. The first, represented by the first term in the square brackets ("1"), is a positive direct effect. The second component, captured by the second term in the brackets, is a negative effect by the decrease in the population density. The last component, appearing at the end, is a nonpositive effect that the range of the locations of passengers is curtailed. The first observation above concerns the case with $m_R < \widehat{m}(l)$, or equivalently, $l < \widehat{d}(m_R)$. In that case, $\widehat{D}(l, m_R) = l$ and therefore, the last term of (21) disappears. The overall derivative can be negative because we have the negative density effect. If it is negative, the maximum may be attained at $m_R < \widehat{m}(l)$. Instead, the second observation above concerns the case with $m_R > \widehat{m}(l)$, or equivalently, $l > \widehat{d}(m_R)$. In this case, $\widehat{D}(l, m_R) = \widehat{d}(m_R)$ and therefore, the negative last term remains. Nonetheless, the overall derivative can be positive. Then, $m_R > \widehat{m}(l)$, or equivalently, $l > \widehat{d}(m_R)$ can give the maximum.

Furthermore, it is straightforward to examine the impacts of changes in parameters upon $\Omega^R(l)$.

First, $\Omega^R(l)$ increases as the transport cost of an automobile use rises, that is, m_A, μ and/or t_A rises. This is because the change is transmitted only through $\widehat{d}(m_R)$, and

$$\frac{\partial \Omega^R(l)}{\partial \widehat{d}} = \begin{cases} \dfrac{m_R}{\tilde{x}^R(\widehat{d})} > 0 & \text{if} \quad \widehat{d}(m_R) < l \\ 0 & \text{if} \quad \widehat{d}(m_R) > l. \end{cases} \tag{22}$$

[8] $\Omega^{R\prime}(l)$ is not defined at $l = \widehat{d}$.

Second, $\Omega^R(l)$ decreases as the time cost of a rapid transit use, t_R, rises. For this change, we obtain

$$\frac{\partial \Omega^R(l)}{\partial t_R} = \begin{cases} -w\left[\dfrac{\mu m_R}{(m_R-\lambda)^2 \widetilde{x}^R(\widehat{d})} + \displaystyle\int_0^{\widehat{d}} \left\{\frac{d}{\widetilde{x}^R(\widehat{d})}\right\}^2 \frac{\partial \widetilde{x}^R(\widehat{d})}{\partial c_R(d)} \mathrm{d}d\right] < 0 & \text{if} \quad \widehat{d}(m_R) < l \\ -w m_R \displaystyle\int_0^{l} \frac{d}{\widetilde{x}^R(\widehat{d})} \frac{\partial \widetilde{x}^R(\widehat{d})}{\partial c_R(d)} \mathrm{d}d < 0 & \text{if} \quad \widehat{d}(m_R) > l. \end{cases} \tag{23}$$

The first term of the right hand side in the first line represents the effect of the fall in the number of users ($\widehat{d}$). The second term captures the effect of the change in a spatial structure to that with a lower population density (recall that we have seen in (10) that $\partial \widetilde{x}^R(\widehat{d})/\partial c_R(d) > 0$).

Third, changes that lead to a denser population distribution, namely, the decrease in a prevailing utility level and that in a relative work hours, give a favorable effect on the rapid transit city equilibrium through a rise in $\Omega^R(l)$.[9]

Fourth and finally, the effect of a change in the wage rate is ambiguous.

$$\frac{\partial \Omega^R(l)}{\partial w} = \begin{cases} m_R\left[\dfrac{\mu \widehat{d}(t_A - t_R)}{(m_R-\lambda)^2 \widetilde{x}^R(\widehat{d})} - \displaystyle\int_0^{\widehat{d}} \left\{\frac{d}{\widetilde{x}^R(\widehat{d})}\right\}^2 \frac{\partial \widetilde{x}^R(\widehat{d})}{\partial w} \mathrm{d}d\right] & \text{if} \quad \widehat{d}(m_R) < l \\ -m_R \displaystyle\int_0^{l} \frac{d}{\widetilde{x}^R(\widehat{d})} \frac{\partial \widetilde{x}_R(\widehat{d})}{\partial w} \mathrm{d}d > 0 & \text{if} \quad \widehat{d}(m_R) > l. \end{cases} \tag{24}$$

The two terms of the right hand side in the first line represent the ambiguous effect of the change in the number of users and the positive effect of the change in a spatial structure (recall that $\partial \widetilde{x}^R(d)/\partial w < 0$), respectively. As long as the time cost of a rapid transit use is not too high compared to that of automobile, the first effect is dominated by the second, and the total effect becomes positive.

4.3 Condition for the multiple equilibria

One of the interesting consequences of our model is a possibility that *both* the auto city equilibrium and the rapid transit city equilibrium are realized for the same set of parameters. In this case of multiple equilibria, an identical city becomes an auto city in some circumstances and a rapid transit city in others, depending on the factors outside economic considerations such as historical accidents and expectations.

A necessary and sufficient condition for the multiple equilibria immediately follows from Proposition 1 and Proposition 2:

Corollary 1

Both the auto city and the l^*-mile rapid transit city are supported as equilibrium outcomes for the same parameter set if both the following conditions are met: (16) is satisfied for any $l \in [0, b^A]$,

[9] Since $\partial\Omega^R(l)/\partial\chi = \partial\Lambda^R(l, m_R)/\partial\chi$ for $\chi = \{\bar{u}, \beta\}$, (10) implies that $\partial\Omega^R(l)/\partial\bar{u} = \Psi^{\bar{u}} < 0$ and $\partial\Omega^R(l)/\partial\beta = \Psi^{\beta} < 0$, where $\Psi^{\chi} \equiv -m_R \displaystyle\int_0^{\widehat{d}} \frac{d}{[\widetilde{x}^R(d)]^2} \cdot \frac{\partial \widetilde{x}^R(d)}{\partial \chi}\, \mathrm{d}d$.

and

$$\Omega^R(l^*) \geq \Gamma(l^*) \tag{25}$$

holds.

We have seen above that for some parameters, the directions of the effects of their changes on at least $\Omega^A(l)$ or $\Omega^R(l)$ are ambiguous. For other parameters, specifically, t_R, $\bar{u}$ and β, on the other hand, those on both $\Omega^A(l)$ and $\Omega^R(l)$ are unambiguous. However, the directions are the same. Therefore, it is not straightforward to decide analytically the directions of the impacts of changes in parameters on the multiple equilibria.

5 Numerical simulations

The key functions, $\Omega^A(l)$ and $\Omega^R(l)$, are too complicated to characterize the two types of equilibria analytically. In this section, we, specifying functional form, explore them through numerical simulations.

5.1 Specifications of functional forms and parameters

5.1.1 Utility function and cost function

Suppose that the preference over the consumptions of land and composite good is represented by a log linear utility function, $u(x, z) \equiv \alpha \ln x + (1-\alpha) \ln z$. The demand for land and the land rent at d that simultaneously solve (5) and(6) become

$$\widetilde{x}(d) = \frac{\theta}{\widetilde{r}(0)} \cdot \left[\frac{w - c(0)}{\{w - c(d)\}^{1-\theta}} \right]^{\frac{1}{\theta}} \quad \text{and} \quad \widetilde{r}(d) = \widetilde{r}(0) \left[\frac{w - c(d)}{w - c(0)} \right]^{\frac{1}{\theta}},$$

respectively, where $\theta \equiv \alpha\beta$ and $\widetilde{r}(0) \equiv \theta \big[(1-\alpha)\beta\big]^{\frac{1-\alpha}{\alpha}} (1-\beta)^{\frac{1-\beta}{\theta}} \bar{u}^{-\frac{1}{\theta}} w^{-\frac{1-\beta}{\theta}} [w - c(0)]^{\frac{1}{\theta}}$ are positive constants. Moreover, we can derive from (7) the location of a city boundary as a solution to

$$c(b) = w - \big[w - c(0)\big] \left[\frac{r_a}{\widetilde{r}(0)} \right]^{\theta},$$

which implies that the boundary qualification (8) is always satisfied for this preference.

For the sake of exposition, furthermore, we focus on the benchmark case of linear $C(l)$, i,e., $\Gamma(l) = \bar{\gamma} + \gamma l$ for some $\bar{\gamma} > 0$ and $\gamma > 0$, which are respectively referred to as a fixed cost and a marginal cost *not with respect to the amount of products but with respect to the length of a rapid transit line*.

In the rest of the paper, we evaluate the revenue of a transport firm in terms of the total land rent at the city center in the auto city, i.e., $\widetilde{r}^A(0)$:

$$\frac{\Lambda^s(l, m_R)}{\widetilde{r}^A(0)} = \frac{m_R}{\theta\,[w - c_A(0)]^{\frac{1}{\theta}}} \int_0^{\widehat{D}} d \cdot [w - c_s(d)]^{\frac{1-\theta}{\theta}}\,\mathrm{d}d$$
$$= \begin{cases} \dfrac{m_R}{\eta_A^2(1+\theta)(w-\mu)^{\frac{1}{\theta}}} \cdot \left[\theta(w-\mu)^{\frac{1+\theta}{\theta}} - (w - \eta_A \widehat{D} - \mu)^{\frac{1}{\theta}}\{\theta(w-\mu) + \eta_A \widehat{D}\}\right] & \text{for} \quad s = A, \\ \dfrac{m_R}{\eta_R^2(1+\theta)(w-\mu)^{\frac{1}{\theta}}} \cdot \left[\theta w^{\frac{1+\theta}{\theta}} - (w - \eta_R \widehat{D})^{\frac{1}{\theta}}(\theta w + \eta_R \widehat{D})\right] & \text{for} \quad s = R. \end{cases}$$

Here, the assumption that the transport costs are linear in d, that is, $c_A(d) = \eta_A d + \mu$ with $\eta_A \equiv m_A + w t_A$ and $c_R(d) = \eta_R d$ with $\eta_R \equiv m_R + w t_R$, enables us to calculate the integral. Furthermore, the boundary qualification (8), which we have shown to be satisfied, guarantees that $w - c_s(d)$ in the integral is greater than 0.

5.1.2 Parameters

For the time cost coefficients, Brueckner's model case furnishes a clue (Brueckner (2001)). He considers a worker who commutes 250 times a year by driving a car at 30 miles per hour.[10] However, one may argue that the driving speed of 30 miles per hour is rather too high in many cities in the world: In most of European and Japanese big cities, for example, the average driving speed is much lower. Thus, we use the speed of 25 miles per hour instead of 30 miles. Computing time costs for such a worker, we obtain $t_A = \widetilde{t}_A \equiv 0.002283$.[11] Furthermore, we use $\widetilde{t}_R \equiv 0.8\,\widetilde{t}_A$ as a benchmark value.

For the monetary costs of an automobile use, furthermore, we can use the estimates by the American Automobile Association (AAA) and the Automobile Association in the United Kingdom (AA).[12] The AAA reports that the variable costs are equal to \$0.1454 for a small sedan and \$0.1718 for a medium sedan, both per vehicle-mile.[13] The figures of the AA are much higher:

[10]The figure of 250 for the yearly work days is not far from that indicated by Japanese data (*Monthly Labor Survey*), which is 248.4 in 2000, 253.2 in 2005, and 228 in 2010.

[11]The worker spends $250 \cdot 2/25 = 20$ hours on a 1-mile commute (2 miles for a round trip) in one year, which occupies $20/(365 \cdot 24) = 0.002283$ of all the available time.

[12]Several economists have also estimated the costs. Small and Verhoef (2007) estimate that the private cost of operation and maintenance, the vehicle capital cost, roadway cost and parking cost are \$0.141, \$0.170, \$0.016 and \$0.007 per vehicle-mile, respectively, in the United States, which totals \$0.334. Moreover, Roy (2000) cites figures of €0.462 and €0.449 per vehicle-km (or, equivalently \$0.752 and \$0.731 per vehicle-mile) at peak hours and off-peak hours, respectively, for a small gasoline car in the urban areas of the United Kingdom excluding London area. Here, the exchange rate of \$1 = €0.987 in January of the year 2000, which is taken from Shams (2005), is used. Also he estimates the cost at €0.497 and €0.310 per vehicle-km (or, equivalently \$0.810 and \$0.505 per vehicle-mile) for a small gasoline car and diesel car, respectively, in the urban areas of France.

[13]The figures presented here are those in 2015 for the AAA and in 2014 for the AA. They are obtainable at their web sites. Furthermore, the variable costs in this paper correspond to the "operating costs" in AAA's report and the "running costs" in AA's, whereas the fixed costs correspond to the "ownership costs" in AAA's and the "standing charge" in AA's. Their definitions are, however, quite similar. The variable costs include the costs of fuel, tires and maintenance; and the fixed costs include insurance, taxes, depreciation and finance charge. In addition, AA's original data are in terms of pounds, which are converted to the American dollars based on the exchange rate in 2015 provided by the IMF.

\$0.3466 for a car whose price ranges from £13,000 to £18,000, and \$0.3743 for a car whose price ranges from £18,000 to £25,000, both per vehicle-mile. In this paper, we adopt a figure in between those in the two countries, \$0.3.[14] Then, assuming that a typical worker works 250 days a year, we can obtain the parameter for the yearly variable costs as $m_A = \widetilde{m}_A \equiv 150$ $(= 3 \cdot 2 \cdot 250)$. For the fixed cost, the AAA gives \$4,548 for a small sedan and \$6,139 for a large sedan, both per year. The figures in AA's are not too different: \$4,936 for a car from £13,000 to £18,000, and \$6,030 for a car from £18,000 to £25,000, both per year. Here, we take \$5,000, that is, $\mu = \widetilde{\mu} \equiv 5000$. In the following simulations, benchmark figures of $t_A = \widetilde{t}_A$, $m_A = \widetilde{m}_A$, $\mu = \widetilde{\mu}$ and $t_R = \widetilde{t}_R$ are used unless otherwise mentioned.

In the United States, household median income is equal to \$53,657 (the US Census) and the average annual hours actually worked per worker are 1,789 (the OECD), which occupies 20.42% of all the time, both in 2014. From these numbers, we estimate $w = 53657/0.2042 = 262,767$ (which is the amount of money that one could earn in one year by working 24 hours a day).

Finally, we estimate preference parameters. First, the share of spending on housing is usually 10% to 20%.[15] Thus, we assume that the share of spending on land, α, is equal to 15%. Second, for β, we evaluate (4) for an "average" worker. Our computation on a hypothetical average worker suggests that the reasonable size of β is 18% to 21%; and thus we assume that $\beta = 0.2$.[16] Lastly, $\alpha = 0.15$ and $\beta = 0.2$ yields $\theta = 0.03$.

Now, we are ready to conduct numerical simulations based on these benchmark values of parameters.

5.2 Simulation analysis of the two types of equilibria

To begin with, let us examine function $\Omega^A(l)$. Figure 2 depicts it as *an Ω^A revenue curve*. Each of the five curves corresponds to a different value of t_A. It is apparent that for smaller values of l, the curve is upward sloping. In the benchmark case with $t_A = \widetilde{t}_A$, for instance, it is upward sloping for $l \in [0, 39.43)$. For such values of l, the positive effect of the extension of a rapid transit line due to the increase in passengers (the first term of the right hand side in (17)) dominates the negative effect arising from the decline in a price (the second term).

Figure 2: Ω^A revenue curve

[14]Here, we take into account the fact that the price of gasoline is exceptionally low in the United States compared to those in other advanced countries.

[15]In Japan, for example, it is 16.2% for owner-occupied housing and 13.5% for rented housing in 2014 (*Family Income and Expenditure Survey*).

[16]Suppose that the average worker living at 10 miles away from the city center commutes by a car. We can obtain the estimates of $m(10)$ and $t(10)$ using $t = \widetilde{t}_A$, $m = \widetilde{m}_A$ and $\mu = \widetilde{\mu}$. Furthermore, wn is estimated at \$53,657, the household median income. Using these figures, we obtain $\beta = 0.1884$. In some countries, a rapid transit is widely used for commutes. If the marginal cost parameters for the rapid transit use are the same for the automobile use, that is, $t_R = \widetilde{t}_A$ and $m_R = \widetilde{m}_A$, β is equal to 0.2043. The estimate is not sensitive to the changes in cost parameters: If $t_R = 0.8\widetilde{t}_A$ and $m_R = 0.8\widetilde{m}_A$, the corresponding β becomes 0.2043. If $t_R = 0.5\widetilde{t}_A$ and $m_R = 0.5\widetilde{m}_A$, the corresponding β becomes 0.2043.

Next, let us turn our attention to function $\Omega^R(l)$. We have seen that the impact of a change in m_R on the revenue at the rapid transit city, $\Lambda^R(l, m_R)$, can be positive or negative, and therefore, there is a possibility that the maximum of $\Lambda^R(l, m_R)$ is attained at $m_R \neq \widehat{m}(l)$. However, simulation analyses show that such a case is not plausible to occur. For plausible values of parameters, in other words, $\mathrm{d}\Lambda^R(l, m_R)/\mathrm{d}m_R$ is positive for $m_R < \widehat{m}(l)$ and negative for $m_R > \widehat{m}(l)$, which implies that the maximum is attained at $m_R = \widehat{m}(l)$, the upper limit price for l.

To see this, we can take a look at Figure 3, which describes the relationship between m_R and $\Lambda^R(l, m_R)$. The upward sloping curves represent the revenues for $m_R < \widehat{m}(l)$. Because $\widehat{D}(l, m_R)$ is equal to l in this case, the revenue depends on l and thus, each curve corresponds to a particular value of l. As l rises, the revenue of a transport firm increases for given m_R and the curve shifts upward. Furthermore, the dashed curve represents the revenue for $m_R > \widehat{m}(l)$. For such m_R, $\widehat{D}(l, m_R)$ is equal to $\widehat{d}(m_R)$ and does not depend on l. This curve intersects each of the above upward sloping curve at $\widehat{m}(l)$. For instance, the intersection of that curve and the upward-sloping curve for $l = 15$, point A, is at $\widehat{m}(15)$. Putting these two kinds of curves altogether, the revenue is described by a mountain-shaped kinked curve with its summit at $\widehat{m}(l)$. Again for $l = 15$, it is described by curve OAB. As long as the dashed curve is downward sloping at $\widehat{m}(l)$, the maximum revenue is attained at that point. For the parameters we are considering, it is the case for $l \leq 61.6$, that is, the length of a rapid transit line is equal to or shorter then 61.6 miles, which is likely to hold in a real city. In the rest of the paper, therefore, we concentrate on this case, that is,

$$\Omega^R(l) = \Lambda^R\big(l,\ \widehat{m}(l)\big). \tag{26}$$

Figure 3: The relationship between the price and the revenue in a rapid transit city

Then, we have

$$\Omega^R(l) - \Omega^A(l) = \widehat{m}(l) \int_0^l d\left[\frac{1}{\widetilde{x}^R} - \frac{1}{\widetilde{x}^A}\right] \mathrm{d}d. \tag{27}$$

That is, the difference between the values of Ω functions for the two types of equilibria depends only on the difference in population densities. Furthermore, note that for $m_R = \widehat{m}(l)$, $c_R(d) - c_A(d)$ is equal to $\mu(d-l)/l$, which is negative for $d < l$. Because higher $c(d)$ implies higher $\widetilde{x}(d)$ (see (10)), it follows that $\Omega^R(l) - \Omega^A(l) > 0$. In other words, the lower transport cost at a rapid transit city brings about a denser spatial structure, which yields a higher revenue. This result implies that for any $l \geq 0$, there exists some $\Gamma(l)$ that satisfies both (16) and (25), that is, there is always a possibility of multiple equilibria.

Figure 4 depicts $\Omega^R(l)$ as an Ω^R *revenue curve* as well as the Ω^A revenue curve. Also described in the figure are three dashed straight lines, which represent $\Gamma(l)$'s. We call them *cost curves*. Here, their intercepts with the vertical axis are equal to the fixed cost, i.e., the cost when the length of a transit line is zero.

Consider the situation where the fixed cost gradually declines. First, when it is sufficiently high that the corresponding cost curve is given by $\Gamma_1(l)$ in Figure 4, the auto city is supported by

Figure 4: The revenue curves and the cost curve

an equilibrium because the cost curve lies above the Ω^A revenue curve for the entire range of l (see Proposition 1). At the same time, no rapid transit city can be supported by an equilibrium because the cost curve is situated also above the Ω^R revenue curve for the entire range of l (see Proposition 2). Second, when the fixed cost declines and the cost curve shifts to $\Gamma_2(l)$, the auto city is still an equilibrium outcome. However, a rapid transit city can be also an equilibrium outcome because the cost curve now intersects the Ω^A revenue curve. Indeed, any rapid transit city where a transit line is constructed up to a terminal located in between l_3 and l_5 is supported by an equilibrium. In this case with a medium size fixed cost, therefore, there are multiple equilibria (see Corollary 1). Third and finally, when the fixed cost further declines to the extent that the cost curve is described by $\Gamma_3(l)$, the auto city is no longer supported as an equilibrium outcome. It is because the cost curve now intersects the Ω^A revenue curve and a rapid transit line with length l_2 to l_4 can be constructed in an auto city. At the same time, any rapid transit city with a transit line extending to l_1 to l_6 is supported as an equilibrium outcome.

6 Concluding remarks

In this paper, we have discussed the relationship between the spatial structure of a city and the transport mode used there. Paying special attention to the fact that they are interdependent with each other, we have defined two types of equilibria. One is the "auto city equilibrium" at which workers, distributed over a city thinly, use an automobile for their commutes. The other is the "rapid transit city equilibrium" at which rapid transit services are provided for the commutes of workers, who are distributed densely. We have derived and characterized the conditions for each type of equilibrium. Furthermore, the possibility of multiple equilibria, that is, the possibility that both types of equilibria emerge for the same set of parameters, has been studied. When such multiple equilibria emerge, the same city may become an auto city or a rapid transit city depending on, for example, historical paths and expectations.

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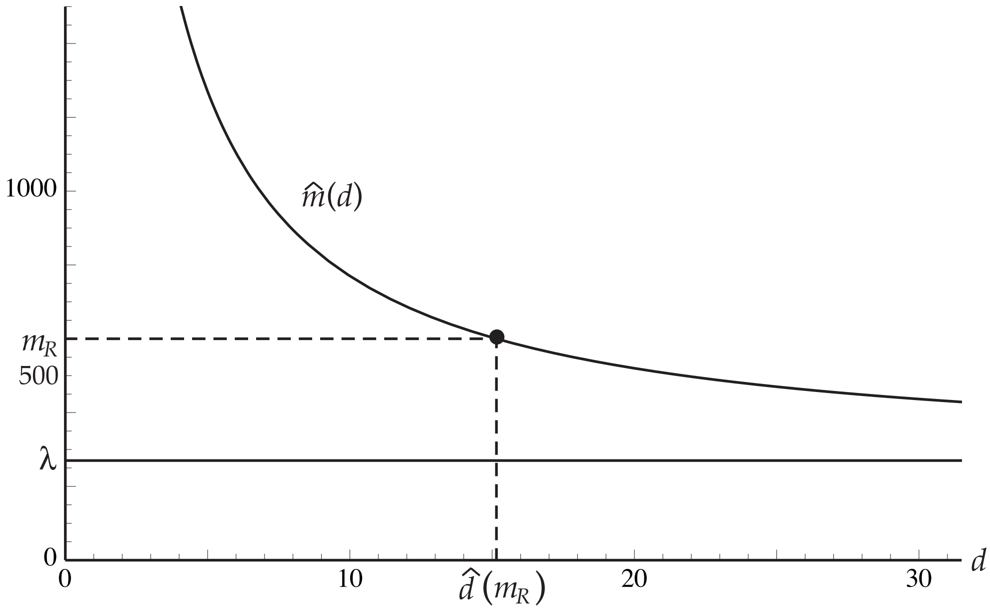

Figure 1: Workers' choices of modes

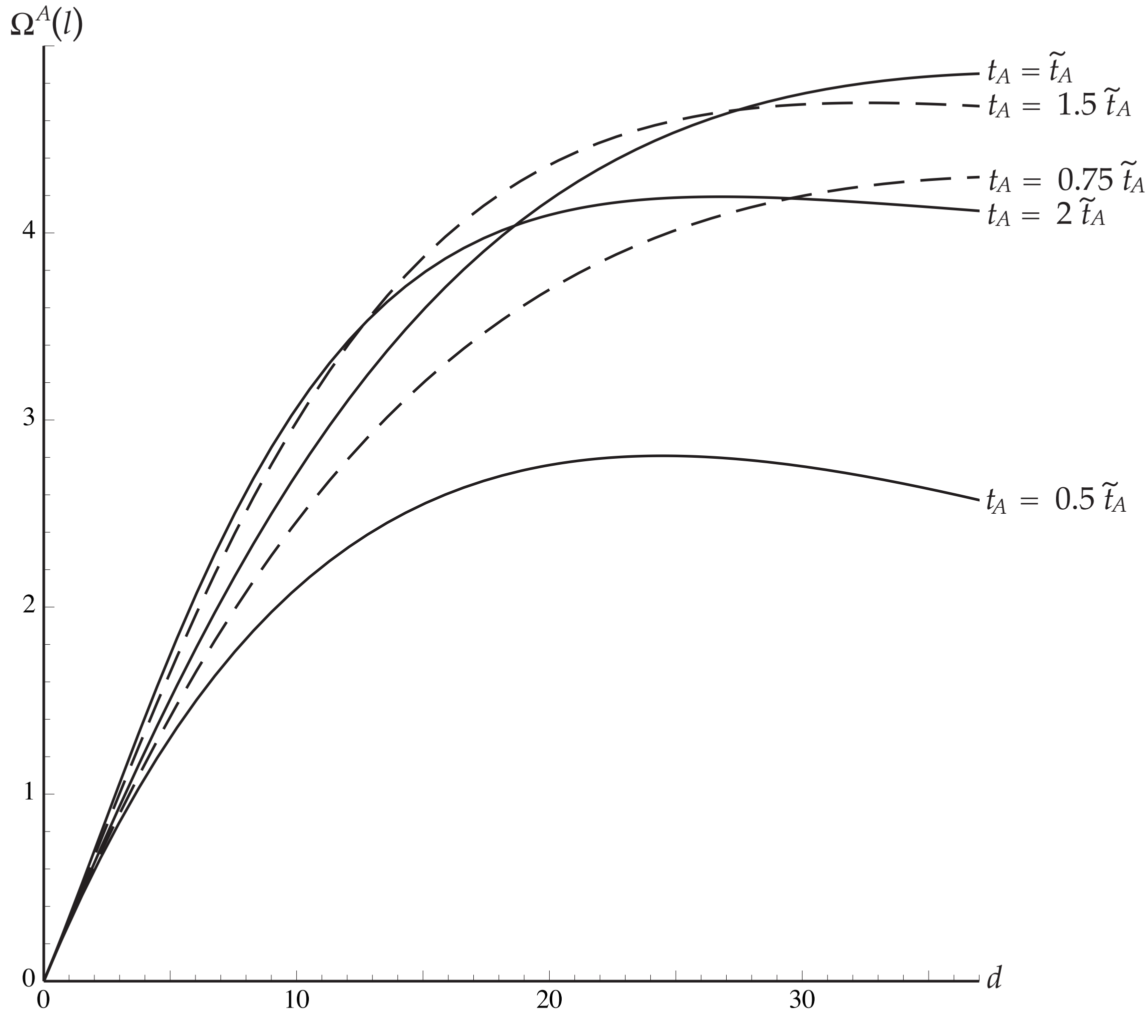

Figure 2: Ω^A revenue curve

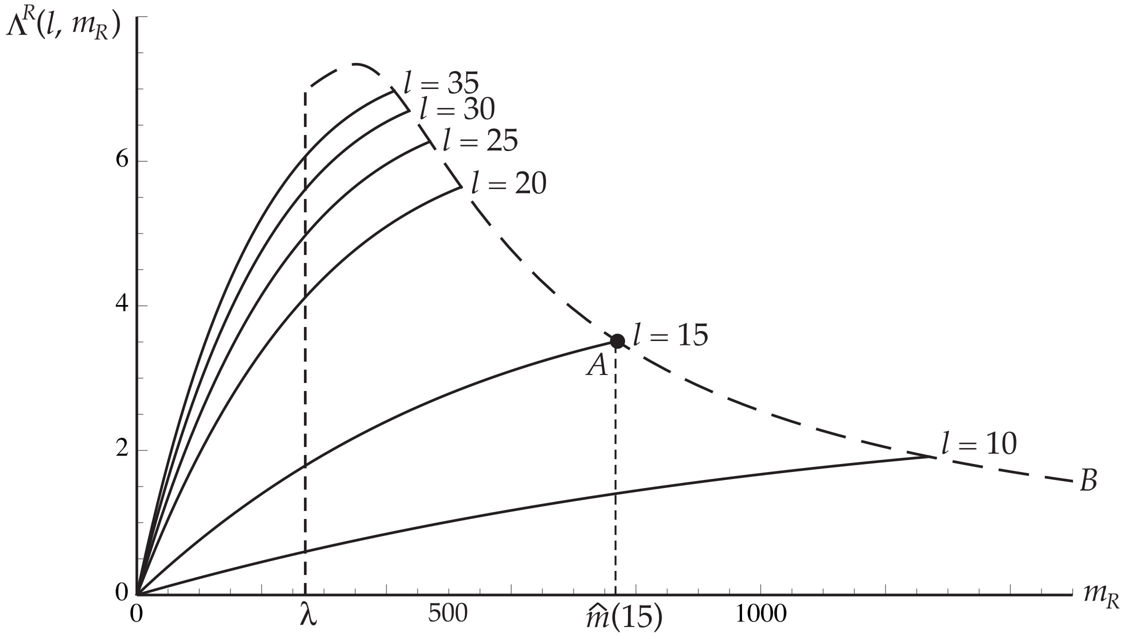

Figure 3: The relationship between the price and the revenue in the rapid transit city

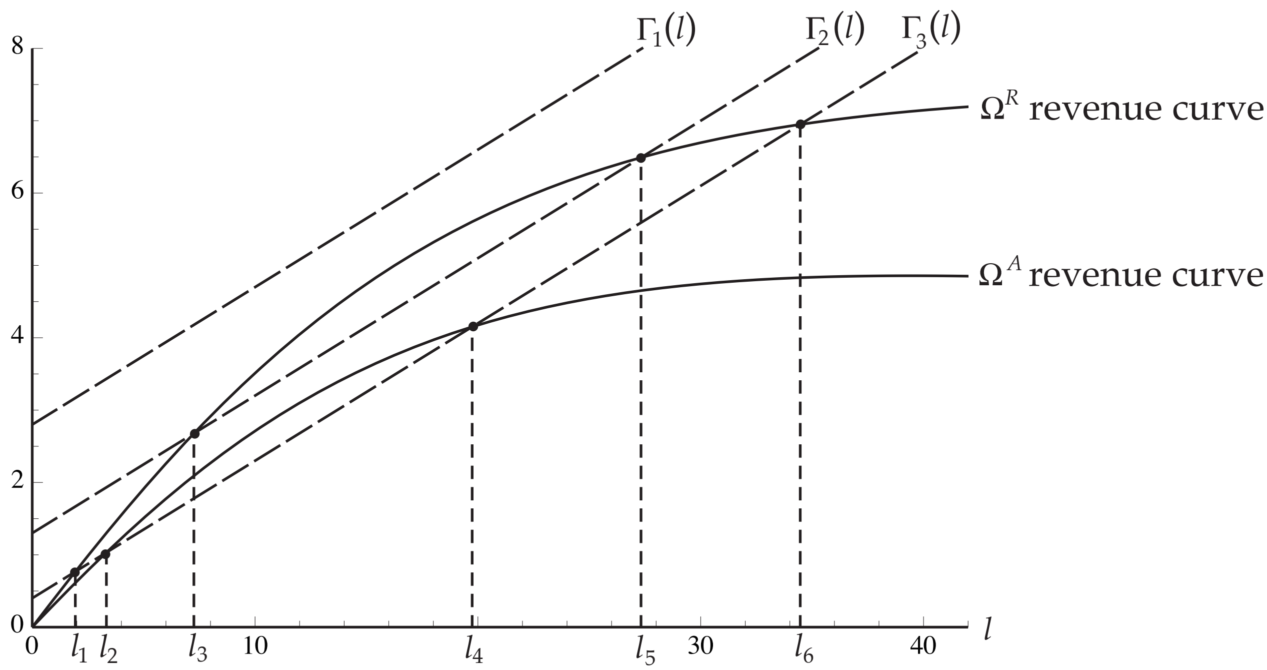

Figure 4: The revenue curves and the cost curve