

# Innovation, Delegation, and Asset Price Swings\*

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## Abstract

This paper studies a dynamic asset-market equilibrium model in which (1) an “innovative” asset with as-yet-unknown average payoff is traded, and (2) investors delegate investment to experts. Experts secretly renege on investors’ orders and take on leveraged positions in the asset to manipulate investors’ beliefs, thereby attracting more orders and fees. Despite agents’ full rationality, the combination of experts’ moral hazard and investors’ learning generates bubble-like price dynamics: gradual upswing, overshoot, and reversal. Consistent with empirical observations, hedge funds “ride” price swings, adjusting holdings counter-cyclically to other financial intermediaries. Innovative assets with highly uncertain payoff characteristics have high trading volumes.

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# 1 Introduction

Bubble-like price movements have recurred in financial markets throughout history. Many of them followed a common pattern: upswings, triggered by technological innovation, are eventually followed by dramatic downswings, leading to persistent economic downturns.<sup>1</sup> The 2007–2009 financial crisis is no exception: the rise in the U.S. housing prices, fueled by financial innovation (i.e., securitization), and the subsequent reversal are the key factors behind the global turmoil. Given their serious impacts on the real economy, it is important to understand the mechanism of asset price swings. Especially, studying the entire cycle—emergence of upswing, its overshoot, and eventual reversal—coherently in a unified framework appears to be a critical task.

To tackle this problem, we develop a fully rational, dynamic asset-market equilibrium model with delegated investment. We consider a market for a new and “innovative” asset, whose average payoff is as-yet-unknown and subject to learning. Investors delegate their investment to financial experts. We highlight the roles of (1) moral hazard in delegated investment, and (2) investors’ learning about the asset’s average payoff. Despite full rationality of long-lived agents, the combination of these two elements generates endogenous bubble-like price dynamics: gradual upswing, overshoot, and eventual downswing.

Specifically, we consider a discrete time model with finite horizon. There are one risky asset and one riskless asset. Initially, the agents have large uncertainty about the risky asset’s average payoff. Over time, they learn about it based on the asset’s payoff history. This asset is interpreted as a financial asset backed by an unprecedented and/or hard-to-understand technology—such as Internet stocks, biotech stocks, or sophisticated structured products—whose underlying profitability is initially unknown to most investors due to the lack of track record and background knowledge. There is a continuum of investment funds, each with a financial expert and an investor. The investor can invest

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<sup>1</sup>See Brunnermeier and Oehmke (2013) for a historical overview of bubbles and crises.

directly in the riskless asset. However, investing in the risky asset requires that she submits to the expert a purchase order that specifies the number of shares of the asset to be purchased on her behalf. Each period, the expert earns a delegation fee proportional to the order.

There are two items the investor cannot directly observe. First, the expert's actual purchase of the asset is unobservable. The expert can secretly renege on the investor's purchase order (at a cost) and boost the asset purchase by using leverage. An example of such a fund can be a hedge fund adopting a flexible trading strategy that is not communicated to investors. Second, although fund returns are (obviously) observable, the periodic payoffs of the risky asset in the fund portfolio are unobservable.<sup>2</sup> These two layers of unobservability create a signal-jamming problem akin to Sato (2014). The investor tries to learn about the risky asset's average payoff from the observed fund returns: the higher the fund returns, the better the investor's assessment of the risky asset's average payoff, hence the larger her purchase orders (and thus fees). So the expert is inclined to boost the expected fund return by secretly leveraging up and increasing the purchase of the risky asset, inflicting excessive risk on the investor. The investor is not fooled in equilibrium because she is rational; nevertheless, the expert still reneges and levers up since otherwise his fund's future prospect would be underestimated by the investor who believes that the expert does renege secretly. The critical difference from Sato (2014)—which is silent about the dynamic patterns of asset prices and trades—is that the expert's signal-jamming behavior and the investors' optimal investments both change over time according to the progress of learning about the innovative asset. This feature allows us to study nonmonotonically time-varying equilibrium prices and trades.

In equilibrium, the risky asset's price path exhibits a bubble-like pattern on average:

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<sup>2</sup>An alternative—and perhaps more realistic—assumption yielding exactly the same results is that each investor can directly observe the asset's payoff by incurring a small effort cost  $\epsilon > 0$ . Even if  $\epsilon$  is very close to 0, it would be optimal for the investors to *not* observe the payoff directly because, in equilibrium, they learn it perfectly and costlessly from the fund return anyway.

it rises gradually, surpasses the benchmark level that would be obtained in the case in which the asset's average payoff is fully known, and eventually falls and converges to the benchmark level over time. Intuitively, these swings are caused by the combination of the following two effects that have opposing pressures on the asset's aggregate demand and thus on its market-clearing price.

1. *Learning effect.* Initially, the investors' estimate of the asset's average payoff has low precision. So, being risk averse, they hesitate to purchase the asset. The associated demand for the asset is weak; thus, ceteris paribus, the initial price is low. But, as the investors' learning progresses over time, the precision of their estimate increases. This leads them to increase purchase orders over time, having an upward pressure on the asset's demand and thus its price.
2. *Leverage effect.* Initially, the investors' estimate of the asset's average payoff has low precision and hence is susceptible to the experts' manipulation. This leads the experts to renege on purchase orders and choose high leverage. The associated aggregate demand is high; so, ceteris paribus, the initial price is high. But, as the investors' learning progresses, their estimate becomes precise and less subject to manipulation. Accordingly, the experts deleverage over time, having a downward pressure on the asset's demand and hence its price.

In early periods, where the investors still have large uncertainty about the asset, the learning effect dominates the leverage effect, initiating upswing in the price. On average, the price overshoots the level of the benchmark case, in which neither of the above effects is at work, because the experts' use of leverage pushes up the asset's aggregate demand and its market-clearing price. As investors' learning progresses, the learning effect weakens and is dominated by the leverage effect, leading to downswing of the price. At some point, the investors' estimate becomes so precise that it is no longer worthwhile for the experts

to attempt to manipulate it. The leverage effect disappears, and the price converges to the benchmark level over time.

The up-and-down swings are pronounced if the investors' initial estimate has low precision. This is because for these swings to occur we need both the learning and leverage effects to be strong, which is the case when the investors' estimate has very low precision. Thus, the model predicts that swings and overshooting of prices are more pronounced for new and innovative assets with highly uncertain payoff characteristics than for old-economy assets already familiar in the market. This prediction is consistent with the historical observation that bubble-like price movements tend to arise in times of technological change (e.g., railroads or the Internet) or financial innovation (e.g., securitization), as noted by Brunnermeier and Oehmke (2013).

To study the evolution of funds' holdings and trading volume over time, we extend the model to accommodate two types of funds: hedge funds (HFs), which can secretly renege on the investors' orders, and other funds (OFs), which cannot do so due to statutory disclosure requirements. Despite the same preferences and the same asset valuation, these funds trade the asset over time: the OFs serve as trading counterparties to the HFs who adjust holdings according to the evolution of the learning and leverage effects. We show that the HFs tend to increase their holdings in the price upturn and then decrease them in the downturn, consistent with the empirical finding of Brunnermeier and Nagel (2004) that hedge funds were "riding" the 1998–2000 dot-com bubble. In contrast, the OFs' holdings are negatively related to those of the HFs over time because they are counterparties to each other. This is consistent with Ang, Gorovyy, and van Inwegen (2011) who document that hedge funds' leverage was counter-cyclical to that of other financial intermediaries during the 2007–2009 crisis.

Our paper is related to the theoretical literature on price anomalies such as bubbles or momentum and reversal. Vayanos and Woolley (2013) also study momentum and re-

versal in a model with delegated investment. As noted in their paper, however, delegated investment is not essential for momentum and reversal to arise in Vayanos and Woolley (2013): the driving force is delay in the reaction of fund flows to returns. We do not assume delays in agents' reactions; in our model, delegation and the associated moral hazard problem are critical. The possibility that experts attempt to manipulate investors' beliefs by deviating from their equilibrium strategies—which Vayanos and Woolley (2013) exclude by assumption—is the key driver of price swings in our model. Like our paper, Pastor and Veronesi (2009) develop a fully rational model and obtain bubble-like patterns in the prices of “new-economy” assets. The key ingredients of their model are a time lag between the introduction and adoption of a new technology and investors' learning during that lag. While investors' learning plays a central role also in our model, we do not distinguish the introduction and adoption of a new technology; our results are driven by agency relationship in delegated investment in which investors' learning is potentially influenced by experts. DeMarzo, Kaniel, and Kremer (2007) also study bubbles caused by technological innovation. While their model is static and focuses on bubbles in real investment, our model studies dynamic bubble-like patterns—both upswings and reversals—in security prices. Hong, Scheinkman, and Xiong (2003) is related in that they also study bubbles stemming from the investor-expert relationship. They focus on the emergence of a bubble and do not analyze its burst; moreover, their results rely on the assumption that some investors are naive. In contrast, our purpose is to explain both upswing and downswing of asset prices in a fully rational framework. Doblas-Madrid (2012) is related to our paper as he also studies the endogenous price upswing and reversal. He extends the framework of Abreu and Brunnermeier (2003) by endogeneizing the growing bubble price, with a key assumption that investors' endowments grow over time. In contrast, in our model the upswing and overshooting are generated by investors' learning and experts' moral hazard.

Some theoretical works attribute price overshooting to moral hazard problems such

as risk shifting in debt contracts (Allen and Gorton 1993; Allen and Gale 2000; Barlevy 2013). While we also highlight moral hazard, our key mechanism is not risk shifting but signal jamming. Methodologically, our model is closely related to Sato (2014), which also features signal jamming associated with portfolio delegation. However, the economic questions and results are different. Sato (2014) studies the implications of opacity in financial markets for the stationary level of asset prices. By contrast, we study up-and-down dynamics of innovative assets' prices, as well as the evolution of funds' holdings and trading volume behind such price swings. All of these are absent in Sato (2014).<sup>3</sup>

In recent years, a theoretical literature studying the equilibrium implications of delegated portfolio management has been growing (Shleifer and Vishny 1997; Berk and Green 2004; Vayanos 2004; Cuoco and Kaniel 2011; Guerrieri and Kondor 2012; He and Krishnamurthy 2012, 2013; Malliaris and Yan 2012; Kaniel and Kondor 2013). To our knowledge, bubble-like price dynamics—upswing, overshoot, and reversal—are not yet discussed in this literature. Our paper contributes to this literature by showing that, despite agents' full rationality, portfolio delegation and the associated moral hazard generate such patterns of equilibrium asset prices.<sup>4</sup>

The paper proceeds as follows. Section 2 presents the model. Section 3 characterizes the equilibrium. Section 4 studies the price dynamics. Section 5 studies the evolution of holdings and trading volume. Section 6 concludes. All proofs are in the Appendix.

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<sup>3</sup>Moreover, our modeling approach is more standard and general than that of Sato (2014). While Sato (2014) assumes overlapping generations of short-lived risk-neutral investors whose decisions stem from a decreasing-returns-to-scale assumption à la Berk and Green (2004), we consider a standard dynamic programming of long-lived risk-averse investors.

<sup>4</sup>Our paper is also related to the literature on career concerns and asset prices (Dasgupta and Prat 2008; Guerrieri and Kondor 2012; Dasgupta, Prat, and Verardo 2011). In these papers, fund managers seek to influence investor beliefs about their ability. In our paper, experts try to influence investor expectations about the innovative asset's future prospects.

## 2 Model

Time  $t$  is discrete and finite:  $t = 0, \dots, T + 1$ . Period  $T$  is the last period in which the market is open and agents make decisions. In period  $T + 1$ , the agents just consume their entire wealth. There is a single risky asset and a riskless asset. The riskless asset has an infinitely elastic supply at an exogenous rate of return  $r > 0$  and is freely accessible to all agents. There are two classes of agents: investors and experts. The investors can purchase the risky asset only through investment funds run by the experts. Each investor provides capital to a fund, specifying the number of shares of the risky asset to be purchased on her behalf. The expert can secretly lever up the investor capital and buy a larger number of shares of the asset than asked by the investor. The experts' leverage, the investors' demand for the risky asset, and the asset's price are determined in equilibrium.

### 2.1 Risky asset

In period  $t = 1, 2, \dots, T + 1$ , the risky asset yields a per-share payoff  $\delta_t = \bar{\delta} + u_t$ . The transitory component  $u_t$  is i.i.d. across time, normally distributed with mean 0 and variance  $1/\eta_u$ , and unobservable to anyone. The average payoff  $\bar{\delta}$  is a constant drawn by nature in period 0 from a normal distribution with mean  $\hat{\delta}_0$  and variance  $1/\eta_0$ . The agents do not observe  $\bar{\delta}$  directly, and learn about it over time based on the payoff history  $\mathcal{H}_t \equiv \{\delta_\tau\}_{\tau=1}^t$ . We denote the period- $t$  estimate of  $\bar{\delta}$  given  $\mathcal{H}_t$  by  $\hat{\delta}_t \equiv E[\bar{\delta}|\mathcal{H}_t]$ . Parameter  $\eta_0$  measures (the inverse of) the risky asset's "innovativeness." The asset with small  $\eta_0$  is interpreted as an innovative asset whose cash flow is backed by an unprecedented and/or hard-to-understand technology—such as Internet stocks, biotech stocks, or sophisticated structured products—because most market participants initially have large uncertainty about such an asset's average payoff due to the lack of track record and background knowledge. The investors cannot directly observe the realized payoff  $\delta_t$  for all  $t$ , whereas



the experts can do so. Although it is unobservable, as shown later, in equilibrium the investors learn  $\delta_t$  perfectly from the observed fund return.<sup>5</sup>

In period  $t = 0, \dots, T$ , the asset is traded in the market at a publicly observable market-clearing price,  $P_t$ . The asset's supply  $S > 0$  is constant over time. Let

$$R_{t+1} \equiv \delta_{t+1} + P_{t+1} - (1 + r)P_t \quad (2.1)$$

denote the excess return on the risky asset per share. We denote the expected excess return conditional on  $\mathcal{H}_t$  by  $\hat{R}_{t+1} \equiv E[R_{t+1}|\mathcal{H}_t]$ .

## 2.2 Delegated investment

There is a continuum with mass one of investment funds, each indexed by  $i \in [0, 1]$ . Fund  $i$  consists of a risk-neutral expert and a risk-averse investor, who both live from  $t = 0$  to  $t = T + 1$ . For simplicity, we assume that investor  $i$  can neither invest in nor observe activities in the other funds. This assumption eliminates the possibility that an expert attracts an infinitely large amount of capital to have price impact in the asset market.

In each period  $t = 0, \dots, T$ , investor  $i$  submits to expert  $i$  a *purchase order*, which specifies the number of shares of the risky asset,  $y_{i,t} \in [0, \infty)$ , that the expert is supposed to purchase on behalf of the investor. On top of the capital needed for this purchase ( $P_t y_{i,t}$  dollars), the investor pays  $\phi y_{i,t}$  dollars to the expert as the *delegation fee*, where the fee rate  $\phi > 0$  is an exogenous parameter. The assumption of a proportionate fee is consistent with Deli (2002), who finds that the compensation contracts in most investment funds include a fixed percentage of fund assets.

After receiving capital and fee from the investor, the expert buys the asset. Despite

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<sup>5</sup>As noted in footnote 2, we could alternatively assume that each investor can observe  $\delta_t$  with a very small effort cost  $\epsilon > 0$ . However small  $\epsilon$  is, the investors would choose not to observe  $\delta_t$  because they learn it perfectly and costlessly from the fund return.

being asked to purchase only  $y_{i,t}$  shares, the expert can renege and buy  $(1 + \xi_{i,t})y_{i,t}$  shares of the asset, where  $\xi_{i,t} \in [0, \infty)$  is the expert's choice variable. We assume  $\xi_{i,t}$  is unobservable to the investor and the expert cannot commit to his choice of  $\xi_{i,t}$ . Since the expert has zero personal wealth, choosing  $\xi_{i,t} > 0$  requires borrowing funds from outside lenders. Thus,  $\xi_{i,t}$  is also a measure of the fund's *leverage*. To choose  $\xi_{i,t}$ , the expert incurs a nonpecuniary cost of renegeing,  $\kappa\xi_{i,t}$  with  $\kappa > 0$ . This represents the cost of effort to “cook the books” and/or manipulate the disclosure documents to make them look like the expert adhered to the investor's purchase order.<sup>6</sup>

In period  $t + 1$ , the fund's period- $t$  investment yields the total proceeds of

$$Q_{i,t+1} \equiv R_{t+1}(1 + \xi_{i,t})y_{i,t} + (1 + r)P_t y_{i,t} \quad (2.2)$$

dollars.<sup>7</sup> The investor can directly observe  $Q_{i,t+1}$ .

## 2.3 Maximization problems

Each expert maximizes his expected lifetime utility, where his within-period utility is the difference between the fee and the cost of renegeing. That is, expert  $i$ 's problem in period  $t = 0, \dots, T$ , denoted by  $\mathcal{P}_{i,t}^E$ , is to choose  $\xi_{i,t} \in [0, \infty)$  to maximize

$$\mathbb{E} \left[ \sum_{\tau=0}^{T-t} \beta^\tau (\phi y_{i,t+\tau} - \kappa \xi_{i,t+\tau}) \middle| \mathcal{F}_{i,t}^E \right], \quad (2.3)$$

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<sup>6</sup>It is not important that we allow only nonnegative  $\xi_{i,t}$ . The reason for assuming  $\xi_{i,t} \geq 0$  is that it is compatible with the proportional renegeing cost,  $\kappa\xi_{i,t}$ . If we allow  $\xi_{i,t} < 0$ , we would need a cost function of the form  $\kappa|\xi_{i,t}|$ . This would reduce tractability of analysis while the results remain unchanged because, in equilibrium, the experts may choose  $\xi_{i,t} > 0$  but never  $\xi_{i,t} < 0$ . An alternative cost function accommodating  $\xi_{i,t} < 0$  is a quadratic form,  $\kappa\xi_{i,t}^2/2$ . With this specification, the experts still do not choose  $\xi_{i,t} < 0$  in equilibrium and the model yields very similar results.

<sup>7</sup> $Q_{i,t+1}$  is computed as follows. In period  $t$ , the investor invests  $P_t y_{i,t}$  dollars in the fund. The expert borrows  $P_t \xi_{i,t} y_{i,t}$  dollars from outside lenders and invests  $P_t(1 + \xi_{i,t})y_{i,t}$  dollars in the risky asset (i.e., buys  $(1 + \xi_{i,t})y_{i,t}$  shares). In period  $t + 1$ , the fund receives  $\delta_{t+1}(1 + \xi_{i,t})y_{i,t}$  dollars of payoff from the asset, and obtains  $P_{t+1}(1 + \xi_{i,t})y_{i,t}$  dollars from selling the asset in the market. The expert pays back  $(1 + r)P_t \xi_{i,t} y_{i,t}$  dollars to the lenders. Thus, the total proceeds from the fund's investment is  $Q_{i,t+1} = \delta_{t+1}(1 + \xi_{i,t})y_{i,t} + P_{t+1}(1 + \xi_{i,t})y_{i,t} - (1 + r)P_t \xi_{i,t} y_{i,t} = R_{t+1}(1 + \xi_{i,t})y_{i,t} + (1 + r)P_t y_{i,t}$ .

where  $\beta \in (0, 1)$  is a discount factor common for all agents, and  $\mathcal{F}_{i,t}^E = \{Q_{i,\tau}, y_{i,\tau}, \xi_{i,\tau}, P_\tau, \delta_\tau : \tau \leq t\}$  is his information set in period  $t$ .

Investor  $i$ 's problem in period  $t = 0, \dots, T$ , denoted by  $\mathcal{P}_{i,t}^I$ , is to choose purchase order  $y_{i,t}$  and consumption  $c_{i,t}$  to maximize her expected lifetime utility

$$-E \left[ \sum_{\tau=0}^{T-t} \beta^\tau \exp(-\nu c_{i,t+\tau}) \middle| \mathcal{F}_{i,t}^I \right], \quad (2.4)$$

where  $\nu > 0$  is the coefficient of absolute risk aversion and  $\mathcal{F}_{i,t}^I = \{Q_{i,\tau}, y_{i,\tau}, P_\tau : \tau \leq t\}$  is her information set in period  $t$ , subject to her dynamic budget constraint

$$W_{i,t+1} = Q_{i,t+1} - \phi y_{i,t} + (1+r)(W_{i,t} - c_{i,t} - P_t y_{i,t}), \quad (2.5)$$

where  $W_{i,t}$  is her wealth in period  $t$ . Constraint (2.5) states that the investor's next-period wealth is the sum of the proceeds from the delegated investment net of fee and her own investment in the riskless asset. In the final period  $t = T + 1$ , in which there is no market for the asset, she simply consumes her entire wealth:  $c_{i,T+1} = W_{i,T+1}$ .

## 2.4 Definition of equilibrium

The equilibrium consists of the price  $P_t$ , the investor's purchase order  $y_{i,t}$ , and the expert's leverage  $\xi_{i,t}$  for  $i \in [0, 1]$  such that, for all  $t = 0, \dots, T$ ,

1. given  $P_t$  and the others' actions, investor  $i$  solves  $\mathcal{P}_{i,t}^I$ ,
2. given  $P_t$  and the others' actions, expert  $i$  solves  $\mathcal{P}_{i,t}^E$ , and
3. the risky asset's market clears:

$$\int_0^1 (1 + \xi_{i,t}) y_{i,t} di = S. \quad (2.6)$$

### 3 Analysis

This section characterizes the equilibrium of the model. We look for a linear equilibrium such that  $P_t$  is linear in  $\hat{\delta}_t$ . We follow the following steps to solve the model.

1. Characterize the evolution of the agents' estimates of  $\bar{\delta}$  (Section 3.1).
2. Conjecture a linear form of the equilibrium price. Also conjecture that the sequence of each expert's optimal leverage  $\{\xi_\tau^*\}_{\tau=0}^T \geq 0$  is deterministic (Section 3.2).
3. Specify the investors' out-of-equilibrium beliefs (Section 3.3).
4. Solve each investor's problem (Section 3.4) for her optimal purchase order  $y_{i,t}$ .
5. Solve each expert's problem (Section 3.5). Verify that  $\{\xi_\tau^*\}_{\tau=0}^T$  is indeed deterministic as conjectured in step 2, and obtain  $\xi_t^*$  as a function of  $\hat{R}_{t+1}$  (Eq.(3.16)).
6. Impose market clearing (Section 3.6) and obtain  $\hat{R}_{t+1}$  as a function of  $\xi_t^*$  (Eq.(3.17)).
7. Solve Eqs. (3.16) and (3.17) for two unknowns  $\xi_t^*$  and  $\hat{R}_{t+1}$  (Section 3.6; Figure 1). Verify that the resulting equilibrium price is indeed linear as conjectured in step 2.

#### 3.1 Evolution of estimates

Since  $\bar{\delta}$  is unobservable to anyone, all agents learn about it over time by Kalman filtering. The experts, who observe  $\mathcal{H}_t$  directly, update their period- $t$  estimate  $\hat{\delta}_t$  by

$$\hat{\delta}_t = \lambda_t \hat{\delta}_{t-1} + (1 - \lambda_t) \delta_t, \quad (3.1)$$

with the updating factor  $\lambda_t \in (0, 1)$  that increases over time deterministically according to  $\lambda_{t+1} = 1/(2 - \lambda_t)$  from its initial value  $\lambda_1 = \eta_0/(\eta_0 + \eta_u)$  (see Appendix A).

The investors, who cannot observe  $\mathcal{H}_t$  directly, estimate  $\bar{\delta}$  based on the  $\mathcal{H}_t$  that they *infer* from the available information. Let  $\mathcal{H}_{i,t}^I \equiv \{\delta_{i,\tau}^I\}_{\tau=1}^t$  denote the payoff history inferred by investor  $i$ , where  $\delta_{i,t}^I$  is the value of  $\delta_t$  inferred by her. Her period- $t$  estimate of  $\bar{\delta}$  is denoted by  $\hat{\delta}_{i,t}^I \equiv E[\bar{\delta}|\mathcal{F}_{i,t}^I]$ , and her conditional expectation of the excess return is denoted by  $\hat{R}_{i,t+1}^I \equiv E[R_{t+1}|\mathcal{F}_{i,t}^I]$ . If she infers that  $\delta_t$  is  $\delta_{i,t}^I$ , she updates  $\hat{\delta}_{i,t}^I$  as

$$\hat{\delta}_{i,t}^I = \lambda_t \hat{\delta}_{i,t-1}^I + (1 - \lambda_t) \delta_{i,t}^I, \quad (3.2)$$

where  $\hat{\delta}_{i,0}^I = \hat{\delta}_0$  is exogenously given, and the updating factor  $\lambda_t$  is the same as that of (3.1) (see Appendix A).

Importantly, the investor's learning (3.2) potentially departs from (3.1) on off-the-equilibrium paths because the experts can manipulate the investors' inference by secretly deviating from their equilibrium strategy (i.e., choosing  $\xi_{i,t}$  that is not anticipated by the investors). However, on the equilibrium path the investors correctly anticipate  $\xi_{i,t}$ ; consequently  $\mathcal{H}_{i,t}^I = \mathcal{H}_t$  holds, and thus (3.2) coincides with (3.1).

## 3.2 Conjectures

First, we propose and later verify the following conjecture about the equilibrium price.

**Conjecture 1** (Price). *The equilibrium price for  $t = 0, \dots, T$  is of the form*

$$P_t = a_t \hat{\delta}_t^I - b_t \quad \text{with} \quad \hat{\delta}_t^I \equiv \int_0^1 \hat{\delta}_{i,t}^I di, \quad (3.3)$$

where

$$a_t \equiv \sum_{\tau=1}^{T+1-t} \left( \frac{1}{1+r} \right)^\tau \quad (3.4)$$

is a riskless discount factor and  $\{b_\tau\}_{\tau=0}^T > 0$  is a deterministic sequence.

The first term of (3.3),  $a_t \hat{\delta}_t^I$ , is the present value of the *average* of all the investors'

expectations about the asset's future payoffs discounted at the riskless rate. The second term,  $b_t$ , involves risk premium. This term is time-varying because the premium demanded by the investors changes together with their learning about  $\bar{\delta}$ . In the last period  $t = T + 1$ , the agents cannot sell the asset because there is no market; so we set  $P_{T+1} \equiv 0$  (i.e.,  $a_{T+1} \equiv 0$  and  $b_{T+1} \equiv 0$ ).

Second, we conjecture and later verify the expert's equilibrium strategy as follows.

**Conjecture 2** (Expert's optimal strategy). *There exists a deterministic sequence  $\{\xi_\tau^*\}_{\tau=0}^T \geq 0$  such that, for all  $i \in [0, 1]$ , expert  $i$  optimally chooses  $\xi_{i,t} = \xi_t^*$  in period  $t = 0, \dots, T$  on the equilibrium path and also on off-the-equilibrium paths where  $\hat{\delta}_{i,t}^I \neq \hat{\delta}_t$ .*

Conjecture 2 states that the expert's optimal leverage in each period  $t$  is deterministic, irrespective of his potential deviations in the past,  $\{\xi_{i,\tau}\}_{\tau=0}^{t-1}$ .

### 3.3 Out-of-equilibrium beliefs

As shown later, all investors infer  $\mathcal{H}_t$  correctly on the equilibrium path and therefore have the same estimate of  $\bar{\delta}$ . So, on the equilibrium path, each investor observes  $P_t$  and confirms that the other investors' average estimate  $\hat{\delta}_t^I = \frac{P_t + b_t}{a_t}$  (computed from (3.3)) is the same as her estimate based on her inferred payoff history  $\mathcal{H}_{i,t}^I$ . That is,  $E[\bar{\delta} | \mathcal{H}_{i,t}^I] = \frac{P_t + b_t}{a_t}$  for all  $i$  and  $t$  on the equilibrium path. However, on off-the-equilibrium paths where some agents deviate from their equilibrium strategies, an investor may observe  $P_t$  and realize that her estimate  $E[\bar{\delta} | \mathcal{H}_{i,t}^I]$  does not equal  $\frac{P_t + b_t}{a_t}$ . For such a case, we specify the following out-of-equilibrium belief of investors.

$$\text{If } E[\bar{\delta} | \mathcal{H}_{i,t}^I] \neq \frac{P_t + b_t}{a_t} \text{ then } \hat{\delta}_{i,t}^I = E[\bar{\delta} | \mathcal{H}_{i,t}^I]. \quad (3.5)$$

That is, an investor whose estimate (based on her inferred history  $\mathcal{H}_{i,t}^I$ ) disagrees with the observed price  $P_t$  would stick with her own estimate.

**Remark (Comparison with REE models).** One may argue that (3.5) is implausible because, based on the rational-expectations equilibrium (REE) logic, each investor should revise her belief in favor of the price. Such an argument does not apply to our model, as the model setting and the nature of analysis are fundamentally different from those of the standard REE models. In asymmetric information models à la Grossman and Stiglitz (1980), uninformed investors should indeed revise their estimates in favor of the price because it reflects informed investors' superior signals. Also, in differential information models à la Grossman (1976), investors should revise their estimates in favor of the price because it aggregates all investors' signals and smooths out their idiosyncratic noises. In contrast, in our model, the price need not convey information superior to each investor's because no investor has information superior to other investors' and no one has private information that would be collectively useful. That is, each investor has no reason to believe that the others' average estimate is more informative than her own. On the equilibrium path, all investors infer the same payoff history  $\mathcal{H}_t$  and thus do not *learn* new information from  $P_t$ : each of them just *confirms* that her own estimate  $E[\bar{\delta}|\mathcal{H}_{i,t}^I]$  is equal to  $\frac{P_t+b_t}{a_t}$  (which equals  $\hat{\delta}_t^I$  on the equilibrium path). If  $E[\bar{\delta}|\mathcal{H}_{i,t}^I] \neq \frac{P_t+b_t}{a_t}$  on an off-the-equilibrium path, she may potentially attribute the discrepancy to the following events: (1) her estimate  $E[\bar{\delta}|\mathcal{H}_{i,t}^I]$  is biased because expert  $i$  has deviated from the equilibrium strategy, (2) the others' average estimate  $\hat{\delta}_t^I$  is biased because some other experts have deviated, (3)  $P_t$  does not reflect  $\hat{\delta}_t^I$  in the form of (3.3) because some other investors have deviated, or a combination of these three. The investor cannot conduct a statistical inference as to which of these three events is more likely than the others, because all of them are supposed to occur with probability zero. The out-of-equilibrium belief (3.5) states that each investor attributes the discrepancy to (2) or (3) instead of (1).<sup>8</sup>

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<sup>8</sup>Moreover, note that (3.5) ensures that  $P_t$  reflects the unbiased estimate  $\hat{\delta}_t$  on the equilibrium path. Indeed, somewhat paradoxically, it is precisely because each investor would prioritize her own estimate over  $P_t$  when there were a discrepancy between them that  $P_t$  reflects the unbiased  $\hat{\delta}_t$  on the equilibrium path. To see this point, suppose to the contrary that each investor would prioritize  $P_t$  over her own esti-

### 3.4 Investor's optimization

We conjecture and later verify that investor  $i$ 's value function in period  $t = 0, \dots, T + 1$  is, for all  $i$ ,

$$V_t(W_{i,t}) = -\exp(-A_t W_{i,t} - B_t), \quad (3.6)$$

where  $\{A_\tau\}_{\tau=0}^{T+1}$  and  $\{B_\tau\}_{\tau=0}^{T+1}$  are deterministic sequences obtained later. Function  $V_t(\cdot)$  is time-varying because the investor's maximized expected utility changes over time as their learning about  $\bar{\delta}$  progresses. The Bellman equation is

$$V_t(W_{i,t}) = \max_{c_{i,t}, y_{i,t}} \left\{ -\exp(-\nu c_{i,t}) + \beta \mathbb{E} [V_{t+1}(W_{i,t+1}) | \mathcal{F}_{i,t}^I] \right\}. \quad (3.7)$$

The following two lemmas characterize the investor's optimization.

**Lemma 1** (Investor's value function). *The investor's value function (3.6) satisfies the Bellman equation (3.7) if*

$$A_t = \frac{\nu}{1 + a_t} \quad \text{for } t = 0, \dots, T \quad (3.8)$$

and  $A_{T+1} = \nu$ , and

$$B_t = \sum_{s=t}^T \left( \prod_{k=t}^s \frac{a_k}{1 + a_k} \right) \left\{ \begin{array}{l} -\ln \beta + \frac{1}{2\chi_s} \left( \frac{\nu S}{(1+r)a_s} \right)^2 \\ + \frac{1}{a_s} \ln \left( \frac{1}{a_s} \right) - \frac{1+a_s}{a_s} \ln \left( \frac{1+a_s}{a_s} \right) \end{array} \right\} \quad \text{for } t = 0, \dots, T \quad (3.9)$$

and  $B_{T+1} = 0$ , where

$$\chi_t \equiv \frac{1}{\text{Var}_t[R_{t+1}]} = \frac{\eta_u \lambda_{t+1}}{(1 + a_{t+1}(1 - \lambda_{t+1}))^2} \quad (3.10)$$

---

mate (that is, consider the out-of-equilibrium belief of the form: if  $\mathbb{E}[\bar{\delta} | \mathcal{H}_{i,t}^I] \neq \frac{P_t + b_t}{a_t}$  then  $\hat{\delta}_{i,t}^I = \frac{P_t + b_t}{a_t}$ ). Under such a belief, there would be infinitely many equilibrium prices of the form (3.3). Specifically,  $P_t = a_t z - b_t$  for an arbitrary number  $z$  would support an equilibrium because each investor observes  $P_t$ , revises her estimate to  $\hat{\delta}_{i,t}^I = z$  and forms demand on that basis, which then consistently translates into the market-clearing price  $P_t = a_t z - b_t$  even on the equilibrium path. The out-of-equilibrium belief (3.5) eliminates this multiplicity and guarantees that  $P_t$  reflects only  $\hat{\delta}_t$  on the equilibrium path.



is the period- $t$  precision of the risky asset's excess return.

**Lemma 2** (Investor's purchase order). *Given  $P_t$ , investor  $i$  asks the expert to buy*

$$y_{i,t} = \frac{\chi_t(\hat{R}_{i,t+1}^I(1 + \xi_t^*) - \phi)}{A_{t+1}(1 + \xi_t^*)^2} \quad (3.11)$$

*shares of the risky asset.*

The investor's optimal order (3.11) can be viewed as a mean-variance solution, standard in the CARA-normal setting. It is increasing in the after-fee expected fund return,  $\hat{R}_{i,t+1}^I(1 + \xi_t^*) - \phi$ , and is decreasing in the volatility of fund return,  $(1 + \xi_t^*)^2/\chi_t$ , and the time-adjusted risk aversion,  $A_{t+1}$ . There are three points worth noting. First,  $y_{i,t}$  is increasing in the asset return precision  $\chi_t$ , which measures how much the investor's learning has progressed. Over time,  $\chi_t$  increases as the uncertainty about  $\bar{\delta}$  is unraveled, encouraging the risk-averse investor to increase  $y_{i,t}$ . As shown later, this upward pressure on  $y_{i,t}$  generates upswing in the equilibrium price  $P_t$ . Second,  $y_{i,t}$  depends on the term  $(1 + \xi_t^*)$  because the investor anticipates that the expert will renege on her purchase order and buy  $(1 + \xi_t^*)y_{i,t}$  shares. Importantly, it is the investor's *belief* ( $\xi_t^*$ ) about the expert's choice and not the choice itself ( $\xi_{i,t}$ ) that affects  $y_{i,t}$ , because  $\xi_{i,t}$  is unobservable. This is the source of the expert's moral hazard that is central to the following analyses. Third,  $y_{i,t}$  depends on the expected excess return from investor  $i$ 's point of view,  $\hat{R}_{i,t+1}^I$ , which is not necessarily equal to  $\hat{R}_{t+1}$  on some off-the-equilibrium paths. On the equilibrium path, of course,  $\hat{R}_{i,t+1}^I = \hat{R}_{t+1}$  for all  $i$  and  $t$  because all the investors infer  $\mathcal{H}_t$  correctly.

### 3.5 Expert's optimization

The expert's problem is solved in a similar fashion to Sato (2014). To verify that it is indeed optimal for expert  $i$  to choose  $\{\xi_\tau^*\}_{\tau=0}^T$  deterministically, we consider what would

happen if he deviated from his equilibrium play and instead chose an arbitrary sequence of leverage  $\{\xi_{i,\tau}\}_{\tau=0}^T \neq \{\xi_\tau^*\}_{\tau=0}^T$  even as investor  $i$  still believes that  $\{\xi_\tau^*\}_{\tau=0}^T$  is played.

What would be investor  $i$ 's inference  $\delta_{i,t+1}^I$  of the payoff  $\delta_{t+1}$  that is unobservable to her? The value of  $\delta_{i,t+1}^I$  solves

$$(\delta_{i,t+1}^I + P_{t+1} - (1+r)P_t)(1 + \xi_t^*) = (\delta_{t+1} + P_{t+1} - (1+r)P_t)(1 + \xi_{i,t}). \quad (3.12)$$

The right-hand side (RHS) is  $R_{t+1}(1 + \xi_{i,t})$ , whose value is known to investor  $i$  who observes  $Q_{i,t+1}$ . It depends on the expert's actual choice,  $\xi_{i,t}$ , and the true payoff,  $\delta_{t+1}$ . The left-hand side (LHS) is the decomposition of the RHS as inferred by investor  $i$ . It depends on her incorrect belief about the expert's action,  $\xi_t^*$ , and her erroneous inferred payoff,  $\delta_{i,t+1}^I$ . Rearranging (3.12), we have

$$\delta_{i,t+1}^I = \delta_{t+1} + \left( \frac{\xi_{i,t} - \xi_t^*}{1 + \xi_t^*} \right) R_{t+1}. \quad (3.13)$$

This implies that if the expert plays  $\xi_{i,t} > \xi_t^*$  and if  $R_{t+1} > 0$ , then the investor will overshoot her inference, i.e.,  $\delta_{i,t+1}^I > \delta_{t+1}$ .

The investor's erroneous perception about  $\delta_{t+1}$  biases her learning subsequently. Specifically, if  $\delta_{i,t+1}^I > \delta_{t+1}$  then her estimates of  $\bar{\delta}$  and asset return will be both biased upward in future periods, i.e.,  $\hat{\delta}_{i,t+\tau}^I > \hat{\delta}_{t+\tau}$  and  $\hat{R}_{i,t+\tau+1}^I > \hat{R}_{t+\tau+1}$  for  $\tau = 1, 2, \dots, T - t$  (see Appendix C). When choosing  $\xi_{i,t}$  in period  $t$ , the expert takes into account the fact that he can potentially inflate the investor's perceived expected returns,  $\hat{R}_{i,t+\tau+1}^I$  ( $\tau \geq 1$ ), and therefore her purchase orders,  $y_{i,t+\tau}$  ( $\tau \geq 1$ ), by choosing  $\xi_{i,t} > \xi_t^*$ . Lemma 3 characterizes his optimal choice in period  $t$ , both on and off the equilibrium path.

**Lemma 3** (Expert's leverage). *Taking  $\hat{R}_{t+1}$  and  $\xi_t^*$  as given, the expert's optimal choice*

of leverage  $\xi_{i,t}$  is as follows.

$$\text{If } \frac{\phi\Omega_t\hat{R}_{t+1}}{1+\xi_t^*} \begin{cases} > \kappa & \text{then } \xi_{i,t} \rightarrow \infty, \\ = \kappa & \text{then } \xi_{i,t} \in [0, \infty) \text{ (indifferent),} \\ < \kappa & \text{then } \xi_{i,t} = 0, \end{cases} \quad (3.14)$$

where

$$\Omega_t \equiv (1-\lambda_{t+1}) \sum_{\tau=1}^{T-t} \beta^\tau \frac{\chi_{t+\tau} (1 + a_{t+\tau+1}(1 - \lambda_{t+\tau+1}))}{A_{t+\tau+1}(1 + \xi_{t+\tau}^*)} \left( \prod_{k=t+2}^{t+\tau} \lambda_k \right) \text{ for } t = 0, \dots, T-1 \quad (3.15)$$

and  $\Omega_T = 0$ .

Lemma 3 states that the expert chooses  $\xi_{i,t}$  by weighing the marginal gain from influencing the investor beliefs in future periods,  $\phi\Omega_t\hat{R}_{t+1}/(1+\xi_t^*)$ , against the marginal cost,  $\kappa$ . The deterministic variable  $\Omega_t$  measures the sensitivity of the expert's expected future gain to an increase in  $\xi_{i,t}$ . In  $t = 0, \dots, T-1$ ,  $\Omega_t$  is positive because the expert can potentially gain from influencing the investor's future beliefs. However,  $\Omega_T$  is zero because there is no "future" in period  $T$ : since the investor makes no decisions in period  $T+1$ , the expert has no benefit from influencing the investor's belief in period  $T$ .

Note that Lemma 3 only characterizes the choice of  $\xi_{i,t}$  optimal from expert  $i$ 's personal perspective, taking the equilibrium level  $\xi_t^*$  as fixed. To determine  $\xi_t^*$ , we need to ensure that investor  $i$ 's belief about  $\xi_{i,t}$  is consistent with the expert's optimal choice. As will be shown in Section 3.6,  $\hat{R}_{t+1}$  is a deterministic variable; thus, the investor's belief is consistent (that is, Conjecture 2 is correct) if and only if  $\xi_{i,t} = \xi_t^*$  in (3.14), or

$$\xi_t^* = \max \left\{ 0, \frac{\phi\Omega_t\hat{R}_{t+1}}{\kappa} - 1 \right\}. \quad (3.16)$$

Note that  $\xi_t^* = 0$  if  $\Omega_t$  is small enough. That is, the experts do not renege on the purchase

orders if the benefit of manipulating investor beliefs is small. Indeed, in period  $T$  they will surely choose  $\xi_T^* = 0$  because  $\Omega_T = 0$ . If  $\Omega_t$  is large enough,  $\xi_t^*$  is positive and increases with  $\hat{R}_{t+1}$ . This makes sense: a large  $\hat{R}_{t+1}$  means a large marginal benefit for the experts from influencing the future investor beliefs by renegeing (i.e., the LHS of (3.14) is large), inducing them to increase leverage  $\xi_t^*$ . We will pin down the equilibrium values of  $\xi_t^*$  and  $\hat{R}_{t+1}$  explicitly in Section 3.6 by imposing market clearing.

### 3.6 Equilibrium

The market-clearing condition (2.6) determines the asset's expected return and the agents' actions. Plugging the investor's optimal policy (3.11) into (2.6), and noting that  $\hat{R}_{i,t+1}^I = \hat{R}_{t+1}$  holds for all  $i$  in equilibrium, we obtain  $\hat{R}_{t+1}$  as a function of  $\xi_t^*$ :

$$\hat{R}_{t+1} = \frac{A_{t+1}S}{\chi_t} + \frac{\phi}{1 + \xi_t^*}. \quad (3.17)$$

The first term on the RHS is the risk premium demanded by investors, which increases with the degree of their risk aversion  $\nu$  and decreases with the precision  $\chi_t$  of asset return. The second term  $\phi/(1 + \xi_t^*)$  is the “fee premium,” i.e., the return that compensates investors for the delegation fees they pay. Importantly, (3.17) implies that  $\hat{R}_{t+1}$  decreases with the expert's leverage  $\xi_t^*$  because the fee premium is decreasing in  $\xi_t^*$ . Why does the fee premium decrease with  $\xi_t^*$ ? The reason is that the fee effectively serves as a fixed cost of investment in the risky asset from the investors' perspective. An increase in  $\xi_t^*$  lowers the average cost per share of the asset purchased, leading to a lower fee premium demanded by investors. Note that the risk premium does not depend on  $\xi_t^*$  despite the fact that a rise in  $\xi_t^*$  increases the risk borne by investors, all else equal. This is because each investor responds to the higher  $\xi_t^*$  by decreasing her purchase order  $y_{i,t}$  so that the total risk she bears remains the same.

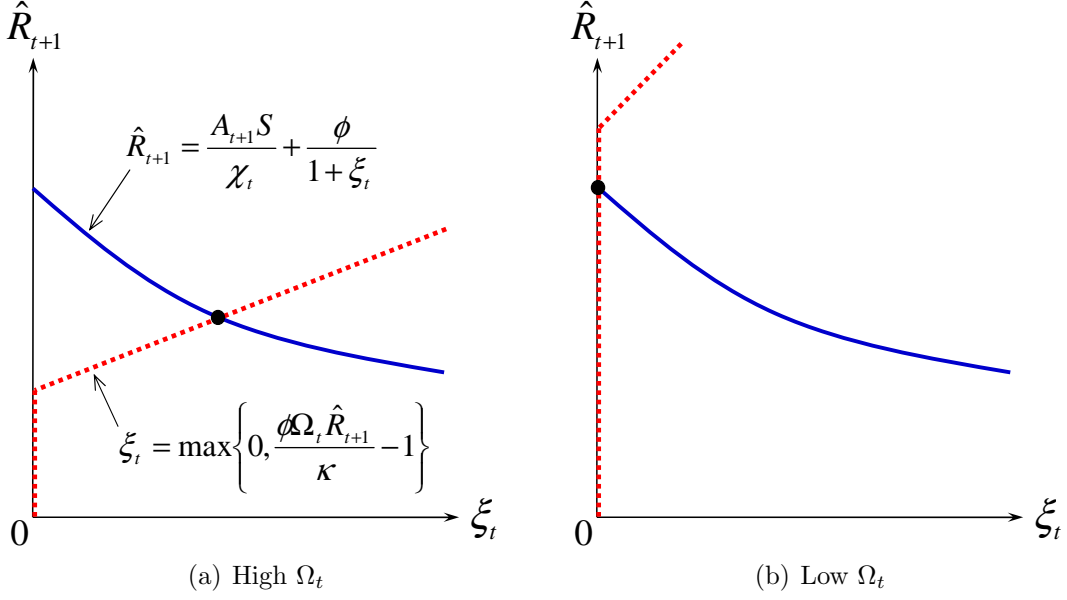


Figure 1: Determination of the risky asset's expected excess return  $\hat{R}_{t+1}$  and the fund leverage  $\xi_t^*$

Given  $\Omega_t$ , the equilibrium levels of  $\xi_t^*$  and  $\hat{R}_{t+1}$  are obtained by solving the system of equations (3.16) and (3.17). The solutions are

$$\xi_t^* = \max \left\{ 0, \frac{\phi \Omega_t}{2\kappa} \left( \frac{A_{t+1}S}{\chi_t} + \sqrt{\frac{A_{t+1}^2 S^2}{\chi_t^2} + \frac{4\kappa}{\Omega_t}} \right) - 1 \right\} \quad \text{and} \quad (3.18)$$

$$\hat{R}_{t+1} = \min \left\{ \frac{A_{t+1}S}{\chi_t} + \phi, \frac{1}{2} \left( \frac{A_{t+1}S}{\chi_t} + \sqrt{\frac{A_{t+1}^2 S^2}{\chi_t^2} + \frac{4\kappa}{\Omega_t}} \right) \right\}. \quad (3.19)$$

Figure 1 illustrates the determination of (3.18) and (3.19). Panel (a) shows the case with  $\Omega_t > \kappa / (\phi(A_{t+1}S/\chi_t + \phi))$ . Since the experts have strong desire to influence investors' future beliefs (i.e.,  $\Omega_t$  is large), they choose high leverage given  $\hat{R}_{t+1}$  (see (3.16)), leading to  $\xi_t^* > 0$  in equilibrium. The equilibrium level of  $\hat{R}_{t+1}$  decreases with  $\Omega_t$  for the following reason. For a given  $\hat{R}_{t+1}$ , a rise in  $\Omega_t$  induces the experts to increase  $\xi_t^*$ , which leads to a higher aggregate demand  $(1 + \xi_t^*)y_t$  for the risky asset. Thus,  $\hat{R}_{t+1}$  decreases (i.e.,  $P_t$  increases) to clear the market. Panel (b) is the case with  $\Omega_t \leq \kappa / (\phi(A_{t+1}S/\chi_t + \phi))$ .

Here,  $\Omega_t$  is so small that the equilibrium leverage is  $\xi_t^* = 0$ . The resulting small aggregate demand for the asset accompanies a low market clearing price and a high  $\hat{R}_{t+1}$ .

Once  $\hat{R}_{t+1}$  is identified, the equilibrium price is readily determined. Conjecture 1 implies that  $P_t$  can be written as  $P_t = a_t \hat{\delta}_t^I - (b_{t+1} + \hat{R}_{t+1})/(1+r)$ .<sup>9</sup> This implies that Conjecture 1 is correct if and only if  $b_t = (b_{t+1} + \hat{R}_{t+1})/(1+r)$ , or

$$b_t = \sum_{\tau=1}^{T+1-t} \left( \frac{1}{1+r} \right)^\tau \hat{R}_{t+\tau}. \quad (3.20)$$

That is,  $b_t$  is the present value of the future expected excess returns.

**Proposition 1.** *There is a linear equilibrium in which*

1. *the risky asset's excess return  $R_{t+1}$  is, conditional on  $t$ , normally distributed with mean  $\hat{R}_{t+1}$  and precision  $\chi_t$ , which are given by (3.19) and (3.10), respectively;*
2. *the risky asset's price is  $P_t = a_t \hat{\delta}_t^I - b_t$ , where  $a_t$  and  $b_t$  are given by (3.4) and (3.20), respectively;*
3. *each expert's leverage is  $\xi_{i,t} = \xi_t^*$ , given by (3.18);*
4. *each investor asks the expert to buy  $y_{i,t} = S/(1 + \xi_t^*)$  shares of the asset;*
5. *investor's consumption in  $t = 0, \dots, T$  is an affine function of wealth,*

$$c_{i,t} = \frac{W_{i,t}}{1 + a_t} + \frac{1}{\nu} \left( \frac{a_t}{1 + a_t} \right) \left( -\ln \beta + \frac{1}{2\chi_t} \left( \frac{\nu S}{(1+r)a_t} \right)^2 + B_{t+1} + \ln a_t \right),$$

where  $B_t$  is given by (3.9); in the final period  $t = T + 1$ , she consumes her entire wealth, i.e.,  $c_{i,T+1} = W_{i,T+1}$ .

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<sup>9</sup>This is shown as follows. From (C.7) in Appendix C, all investors' average expected excess return is  $\hat{R}_{t+1}^I \equiv \int_0^1 \hat{R}_{i,t+1}^I di = (1 + a_{t+1}(1 - \lambda_{t+1})) \int_0^1 \hat{\delta}_{i,t}^I di + a_{t+1} \lambda_{t+1} \hat{\delta}_t^I - b_{t+1} - (1+r)P_t = (1 + a_{t+1})\hat{\delta}_t^I - b_{t+1} - (1+r)P_t$ . Rearranging this and noting that  $a_t = (1 + a_{t+1})/(1+r)$ , we have  $P_t = a_t \hat{\delta}_t^I - (b_{t+1} + \hat{R}_{t+1}^I)/(1+r)$ . Since  $\hat{R}_{t+1}^I = \hat{R}_{t+1}$  in equilibrium for all  $t$ , the result follows.

Proposition 1 provides a characterization of the equilibrium in closed form. Although it is costly to renege on the investor's order, each expert chooses  $\xi_t^* > 0$  if it gives him sufficiently large marginal benefit through manipulating the investor's future beliefs (i.e., if  $\Omega_t$  is large enough). The investors understand the experts' desire to fool them and hence their beliefs are not manipulated on the equilibrium path; nonetheless, the experts still renege on the investors' orders and lever up. This is because, given the investors' beliefs that the experts will lever up, the experts indeed lever up optimally since otherwise the funds' future prospects would be underestimated by the investors. By itself, this over-leverage result may not be surprising: it is in line with standard signal-jamming models (Holmström 1999; Stein 1989). The primary purpose of this paper, which distinguishes ours from existing works, is to examine the implication of the experts' signal-jamming behavior for the dynamics of security prices. We explore this issue in Section 4.

## 4 Dynamics: Asset Price Swings

This section studies the dynamics of  $P_t$  determined in Proposition 1. First, we show that  $P_t$  on average exhibits a bubble-like pattern: gradual upswing, overshoot, and eventual reversal. Second, we show that such a pattern is pronounced when the asset is innovative in that its average payoff is highly uncertain when introduced to the market (i.e., low  $\eta_0$ ).

### 4.1 Price path

To obtain the sequence of  $P_t$ , first we need to obtain the sequences of  $\Omega_t$ ,  $\xi_t^*$ , and  $\hat{R}_{t+1}$ . As shown in Appendix E, the deterministic sequence  $\{\Omega_\tau\}_{\tau=0}^{T-1}$  is obtained by solving the following difference equation of  $\Omega_t$ , backward from the terminal values  $\Omega_T = 0$  and  $\xi_T^* = 0$ :

$$\Omega_t = \beta \left( \Omega_{t+1} + \left( \frac{1 - \lambda_{t+2}}{\lambda_{t+2}} \right) \frac{\chi_{t+1} (1 + a_{t+2} (1 - \lambda_{t+2}))}{A_{t+2} (1 + \xi_{t+1}^*)} \right). \quad (4.1)$$

We already know the deterministic values of  $\{\lambda_\tau\}_{\tau=1}^{T+1}$ ,  $\{a_\tau\}_{\tau=0}^{T+1}$ ,  $\{A_\tau\}_{\tau=0}^{T+1}$ , and  $\{\chi_\tau\}_{\tau=0}^T$  from (A.5), (3.4), (3.8), and (3.10), respectively. Thus, together with (4.1), we can identify the deterministic values of  $\{\xi_\tau^*\}_{\tau=0}^T$  and  $\{\hat{R}_\tau\}_{\tau=1}^{T+1}$  backward by (3.18) and (3.19), respectively, which then determine the deterministic values of  $\{b_\tau\}_{\tau=0}^{T+1}$  by (3.20). Generating a sequence of normal random payoffs  $\{\delta_\tau\}_{\tau=1}^{T+1}$ , we compute the associated estimates  $\{\hat{\delta}_\tau\}_{\tau=1}^{T+1}$  by (3.1). Then we can simulate a path of the price,  $\{P_\tau\}_{\tau=0}^{T+1}$ , by  $P_t = a_t \hat{\delta}_t + b_t$ .

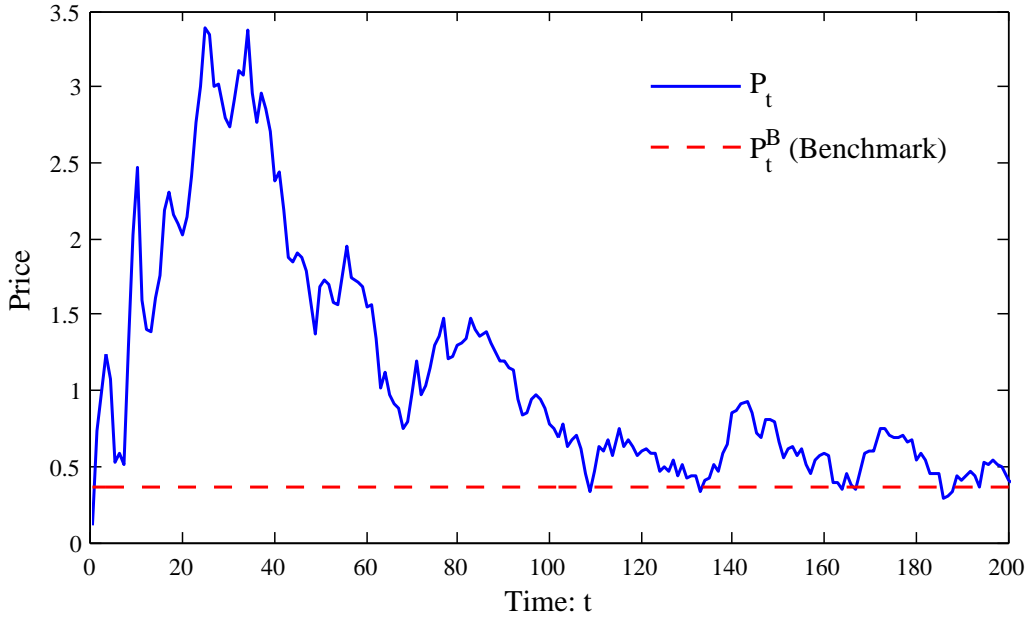
Panel (a) of Figure 2 plots a sample path of  $P_t$  (blue solid line). Here we assume that the agents have the correct prior expectation in  $t = 0$ , i.e.,  $\hat{\delta}_0 = \bar{\delta}$ . So the obtained price pattern is not driven by the agents' incorrect prior about  $\bar{\delta}$ . We also plot the benchmark price path, denoted by  $P_t^B$  (red dashed line), which corresponds to a “noninnovative” asset whose average payoff  $\bar{\delta}$  is fully known in period 0 (i.e.,  $\hat{\delta}_0 = \bar{\delta}$  and  $\eta_0 = \infty$ ). Although we set  $T = 1,000$ , we present only the first 200 periods in the figure because our focus is on the price dynamics of an innovative asset, whose  $\bar{\delta}$  is highly uncertain to the investors. For very large  $t$ , the asset is no longer “innovative” since the agents have already learned  $\bar{\delta}$  with high precision. Also, the price paths in later periods are not interesting economically:  $P_t$  simply converges to  $P_t^B$ .<sup>10</sup>

The innovative asset's price  $P_t$  is highly volatile in early periods. This makes sense. Since the precision of the agents' estimate  $\hat{\delta}_t$  is low in the early phase of learning, they update  $\hat{\delta}_t$  drastically when having a new realization of stochastic payoff  $\delta_t$  (that is,  $\lambda_t$  is small in early periods), making the price volatile. But the price becomes less volatile over time because, as learning progresses, the agents become more “confident” about their estimate: they do not change  $\hat{\delta}_t$  so much with a new realization of  $\delta_t$  (that is,  $\lambda_t$  increases

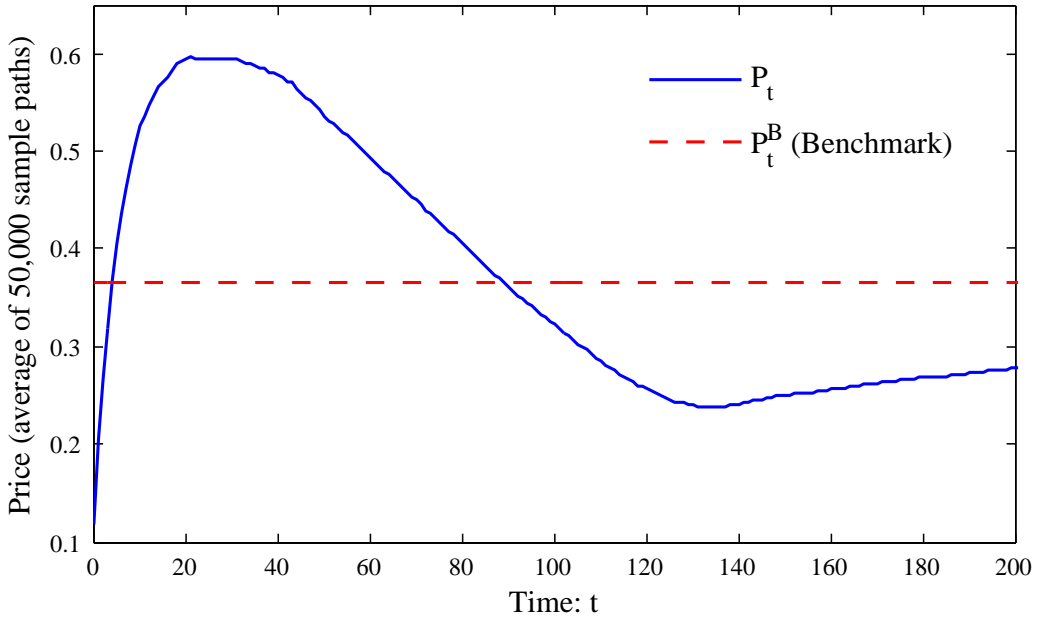
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<sup>10</sup>After the periods shown in the figure, the path of  $P_t$  approaches that of  $P_t^B$  and stays almost flat for most of the remaining periods. When the final period approaches, both of these price paths start to fall (around  $t = 900$ ) and reach zero in the final period ( $t = 1,001$ ). This price fall occurs because there is no market in the very last period  $t = T + 1$ —which is an inevitable assumption in this finite-period setting—and thus the investors cannot sell the asset in that period (i.e.,  $P_{T+1} = 0$ ). We do not view this price fall in the last periods as an economically relevant result because it is merely an artifact of the finite-horizon assumption. Thus, we do not report it in the figure and focus on early periods.





(a) Simulated price path.



(b) Average of 50,000 simulated price paths.

Figure 2: Price dynamics. We plot the time paths of  $P_t$  obtained in Proposition 1 and the benchmark price  $P_t^B$  that would be obtained in the case in which  $\bar{\delta}$  is fully known in  $t = 0$ . Panel (a) plots a simulated path. Panel (b) plots the average of 50,000 simulated paths. The parameter values are  $r = 0.04$ ,  $\phi = 0.11$ ,  $\kappa = 1.2$ ,  $\nu = 0.2$ ,  $\beta = 1/(1 + r)$ ,  $S = 10$ ,  $\eta_u = 5$ ,  $\bar{\delta} = 0.14$ ,  $\hat{\delta}_0 = 0.14$ , and  $T = 1,000$ . We set  $\eta_0 = 50$  for  $P_t$  and  $\eta_0 = \infty$  for  $P_t^B$ .

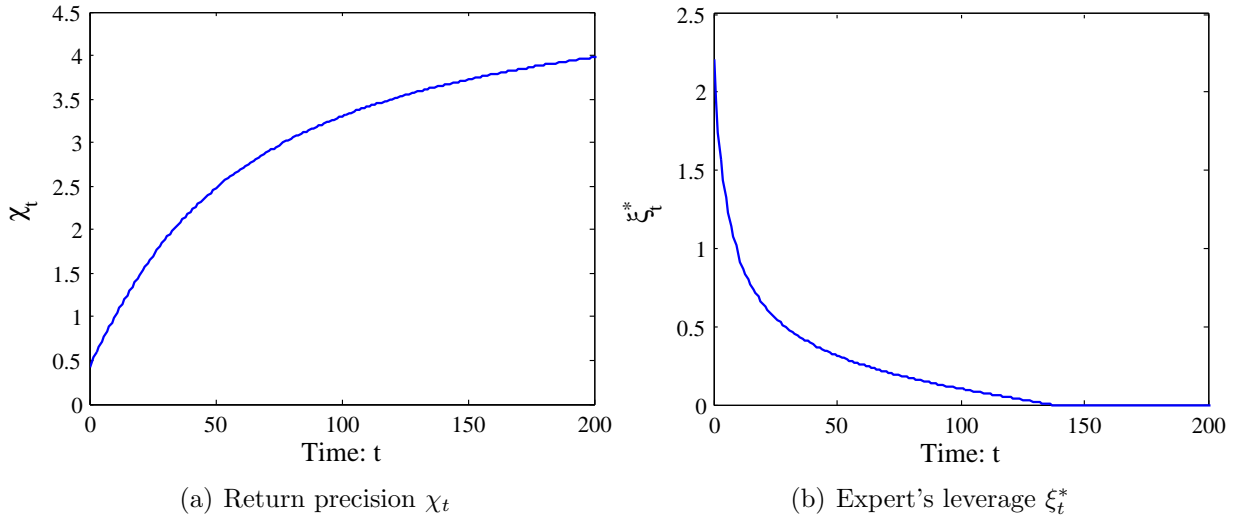


Figure 3: Time paths of the risky asset's return precision  $\chi_t \equiv 1/\text{Var}_t[\mathbf{R}_{t+1}]$  and the expert's equilibrium leverage  $\xi_t^*$ . The parameter values are the same as those of Figure 2.

over time). Indeed, the conditional price volatility  $\text{Var}_t[P_{t+1}] = a_{t+1}^2(1 - \lambda_{t+1})^2/(\eta_u \lambda_{t+1})$  decreases with  $t$ . By contrast, if the asset is noninnovative the agents do not update their estimate (i.e.,  $\hat{\delta}_t = \bar{\delta}$  for all  $t$ ), and hence  $P_t^B$  is deterministic and almost flat in early periods.

To see the overall trend of the price dynamics more clearly, we plot the average of 50,000 simulated price paths in panel (b) of Figure 2. The innovative asset's price  $P_t$  exhibits bubble-like dynamics on average: it rises gradually (“upswing”), surpasses the noninnovative-asset benchmark  $P_t^B$  (“overshoot”), and then falls gradually (“downswing”). Around  $t = 130$ , it starts increasing again and converges to  $P_t^B$  over time.

Intuitively, the initial up-and-down swings in  $P_t$  are caused by the combination of the following two effects that have opposing pressures on the asset's aggregate demand and therefore its market-clearing price.

1. *Learning effect.* Initially, the investors' estimate of the asset return has low precision; i.e.,  $\chi_t$  is small for small  $t$ . So, being risk averse, they hesitate to purchase the asset. Thus, ceteris paribus, the associated aggregate demand is weak and the price is low.

But, as time goes on, the investors learn about  $\bar{\delta}$  and thus  $\chi_t$  increases (Figure 3(a)). This encourages them to increase  $y_{i,t}$  over time, having an upward pressure on the aggregate demand and thus the price.

2. *Leverage effect.* Initially, the investors' estimate  $\hat{\delta}_{i,t}^I$  has low precision and is susceptible to manipulation; i.e.,  $\lambda_t$  is small for small  $t$ . So  $\Omega_t$  is large, leading the experts to choose high leverage  $\xi_t^*$ . Thus, ceteris paribus, the associated aggregate demand is high and the price is high. But, as time goes on, the investors learn about  $\bar{\delta}$  and thus  $\hat{\delta}_{i,t}^I$  becomes precise, lowering the experts' desire to manipulate it (i.e.,  $\Omega_t$  decreases). Accordingly, the experts deleverage over time (Figure 3(b)), having a downward pressure on the aggregate demand and thus the price.

In sum, due to the learning (leverage) effect, the price tends to be low (high) initially and then increases (decreases) over time; the combination of these two effects generates the inverse-U pattern of Figure 2(b). In the noninnovative-asset benchmark, neither of these effects exists because the agents do not conduct learning (i.e.,  $\chi_t = \eta_u \forall t$ ) and the experts do not use leverage (i.e.,  $\xi_t^* = 0 \forall t$ ). For the parameter values used in Figures 2 and 3, the learning effect dominates the leverage effect in early periods, initiating the upswing in  $P_t$ . It even surpasses  $P_t^B$ . This overshoot is caused by the experts' use of leverage: higher  $\xi_t^*$  is associated with larger aggregate demand for the risky asset, pushing up the market-clearing price  $P_t$ . As learning unravels  $\bar{\delta}$  over time, the learning effect fades out because the incremental increase in  $\chi_t$  diminishes over time towards 0; that is,  $\chi_t$  approaches  $\eta_u$  asymptotically (Figure 3(a)). At some point, the leverage effect dominates the weakened learning effect, leading to downswing in the average  $P_t$  (around  $t = 30$  in Figure 2(b)). Eventually, the investors' estimate becomes so accurate that it is no longer beneficial for the experts to use leverage to influence investor beliefs. That is,  $\xi_t^*$  reaches 0 and the leverage effect disappears (around  $t = 130$  in Figures 2(b) and 3(b)). Afterwards, the average  $P_t$  increases over time due to the learning effect (which is weakened but still at

work), and converges to  $P_t^B$  as  $\chi_t$  converges to  $\eta_u$ .

**Remark (Is this a bubble?).** Figure 2 resembles bubble-like price movements observed in reality. Is it a “bubble”? The answer depends on how we define a bubble. Some existing works define it as a situation in which an asset is overpriced *even though investors are contemporaneously aware that the price is too high* (Allen, Morris, and Postlewaite 1993; Abreu and Brunnermeier 2003). Taking this somewhat narrow definition, the price pattern in Figure 2 is not a bubble. The reason is as follows. The fact that  $P_t$  overshoots  $P_t^B$  *on average* means that all agents know at  $t = 0$  that  $P_t > P_t^B$  is *likely* to occur in the near future. Also, the agents are likely to realize *ex post*, after they have learned  $\bar{\delta}$  with a high precision, that  $P_t > P_t^B$  in early periods. However, when they are *actually* making investment decisions in early periods, they are not sure whether  $P_t > P_t^B$  or  $P_t < P_t^B$  because they do not know the benchmark level  $P_t^B$  that depends on  $\bar{\delta}$  they are still learning about. Indeed, while Figure 2(a) reports a typical sample path in which overshoot occurs, it is possible to obtain (rare) simulated paths such that  $P_t < P_t^B$  for all  $t$ .

## 4.2 Effect of asset’s innovativeness

Under what circumstances are price swings pronounced? This section examines how the price dynamics change in response to a change in  $\eta_0$  (inverse of the risky asset’s innovativeness). Figure 4 plots the average of 50,000 simulated paths of  $P_t$  for different levels of  $\eta_0$ . The average price exhibits up-and-down swings only if  $\eta_0$  is small enough ( $\eta_0 \leq 50$  in the figure), i.e., only if the agents have sufficiently large uncertainty about  $\bar{\delta}$  initially. For large  $\eta_0$  ( $\eta_0 \geq 300$  in the figure), the learning effect is so weak that it is already dominated by the leverage effect in  $t = 0$ , and thus the initial upswing does not occur. Moreover, if  $\eta_0$  is very large ( $\eta_0 \geq 600$ ), the experts’ leverage  $\xi_t^*$  is so low that even overshoot does not occur on average. As  $\eta_0$  increases further, the price swings are toned

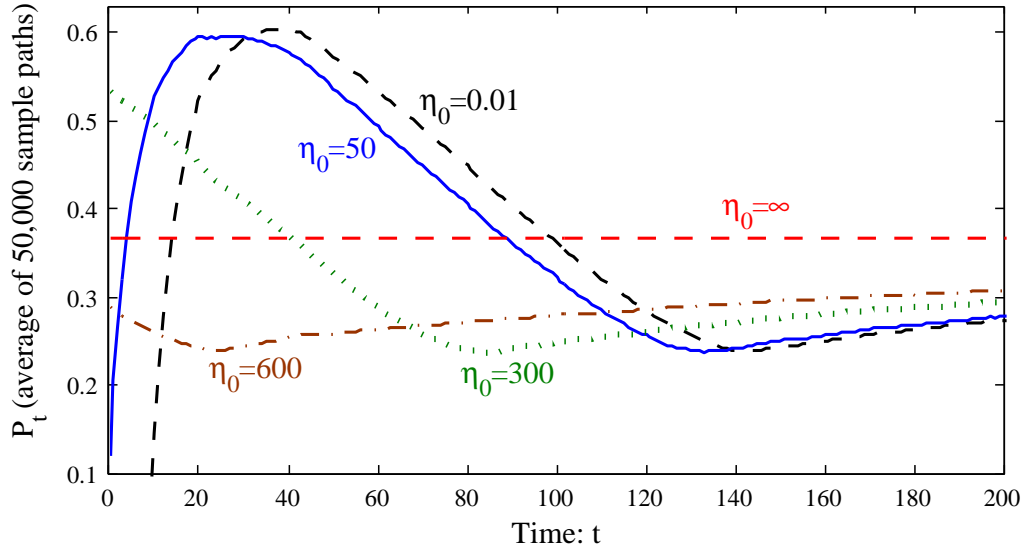


Figure 4: The average of 50,000 sample price paths for different levels of  $\eta_0$  (inverse of the risky asset’s innovativeness). The parameter values other than  $\eta_0$  are the same as those of Figure 2.

down and the path converges to that of the benchmark  $P_t^B$  as  $\eta_0 \rightarrow \infty$ .

Thus, the model predicts that swings and overshooting of prices are more pronounced for new and innovative assets with highly uncertain payoff characteristics than for old-economy assets already familiar in the market. This prediction is consistent with the historical observation that bubble-like price movements tend to arise in times of technological change (e.g., railroads or the Internet) or financial innovation (e.g., securitization), as noted by Brunnermeier and Oehmke (2013).

Note that price swings are pronounced with small  $\eta_0$  because *both* the learning and leverage effects are large when  $\eta_0$  is small. The intuition is as follows. Suppose that a financial asset backed by an unprecedented and/or hard-to-understand technology—such as Internet stocks, biotech stocks, or structured products—is newly introduced to the market. The investors have large uncertainty about such an asset’s average payoff due to the lack of track record and background knowledge (i.e.,  $\eta_0$  is small). On the one hand, the investors, being risk averse, hesitate to purchase such an asset initially; but they increase

demand gradually as learning progresses, generating a gradual upswing in the price (the learning effect). On the other hand, the experts, being motivated by career concerns, try to exploit the as-yet-unknown nature of the asset. Initially, they take on high leverage and invest in the asset aggressively in an effort to trick investors into believing that the asset is more profitable than it really is, causing the price overshoot; however, as the asset's true nature becomes known to investors, experts lose their desire to influence the investor beliefs and thus deleverage, causing a downswing in the price (the leverage effect).

## 5 Holdings and Trading Volume

The model of Section 2 provides an explanation of bubble-like price swings by shedding light on the role of leveraged, opaque financial companies such as hedge funds. However, the model is silent about how such funds' stock holdings evolve over time behind the price swings. Indeed, since all the funds are identical and the asset's supply is fixed at  $S$  shares, every fund's asset holding in equilibrium is constant at  $S$  for all  $t$  and thus the trading volume is zero for  $t \geq 1$ . This is clearly counterfactual. The empirical literature documents that hedge funds actively adjust their holdings in times of large price swings, and their trading activities in such times are different from other players in the market. Brunnermeier and Nagel (2004) find that hedge funds increased their holdings of the technology stocks during the upturn of the 1998–2000 dot-com bubble, but they decreased them before the bubble collapsed. Ang, Gorovyy, and van Inwegen (2011) report that hedge funds' leverage was counter-cyclical to that of other market participants during the 2007–2009 crisis. This section attempts to explain these empirical observations by rationalizing time-varying holdings of the asset.

## 5.1 Setup

We make a single modification to the model of Section 2. There are two types of funds: *hedge funds* (HFs), indexed by  $i \in [0, \gamma]$ , and *other funds* (OFs), indexed by  $i \in (\gamma, 1]$ . The proportion  $\gamma$  of HFs is exogenous. The HFs are the same as the funds of Section 2, whereas the OFs' experts cannot renege on the investors' purchase orders. That is, the only difference between these types is that HF experts can choose  $\xi_{i,t} \geq 0$  while OF experts cannot.<sup>11</sup> The OFs can be viewed as representing various financial institutions subject to statutory disclosure requirements, such as mutual funds, banks, or investment banks. Alternatively, since the OF experts just take the investors' orders passively with no agency frictions stemming from  $\xi_{i,t}$ , the OFs can also be interpreted as individual investors, each of whom incurs a cost (such as a brokerage fee) of  $\phi$  per share she purchases on her own.<sup>12</sup> For notational clarity, we append a tilde to the variables related to the OFs. We look for an equilibrium in which, for all  $t$ , every agent is optimizing and the risky asset's market clears, i.e.,  $\int_0^\gamma (1 + \xi_{i,t}) y_{i,t} di + \int_\gamma^1 \tilde{y}_{i,t} di = S$ , where  $\tilde{y}_{i,t}$  is OF investor  $i$ 's purchase order.

## 5.2 Equilibrium

The equilibrium of this economy is derived following steps similar to those in Section 3 (see Appendix F for details).

**Proposition 2.** *There is a linear equilibrium in which*

1. *the risky asset's excess return  $R_{t+1}$  is, conditional on  $t$ , normal with mean*

$$\hat{R}_{t+1} = \min \left\{ \frac{A_{t+1}S}{\chi_t} + \phi, \frac{1}{2} \left( \frac{A_{t+1}S}{\chi_t} + (1 - \gamma)\phi + \sqrt{\left( \frac{A_{t+1}S}{\chi_t} + (1 - \gamma)\phi \right)^2 + \frac{4\gamma\kappa}{\Omega_t}} \right) \right\}$$

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<sup>11</sup>An alternative assumption yielding the same results is that each OF's expert can choose  $\xi_{i,t} \geq 0$  but his choice is observable to the investor. In such a case, the OFs' experts choose  $\xi_{i,t} = 0$  for all  $t$  because  $\xi_{i,t} > 0$  would not influence the investors' behavior and yet is costly to choose.

<sup>12</sup>It is not important that the cost  $\phi$  is the same for both types of funds. Allowing for different values of  $\phi$  does not change the main results.

and precision  $\chi_t$ , where  $A_t$ ,  $\chi_t$ , and  $\Omega_t$  are given by (3.8), (3.10), and (3.15), respectively;

2. the risky asset's price is  $P_t = a_t \hat{\delta}_t - b_t$ , where  $a_t$  and  $b_t$  are given by (3.4) and (3.20), respectively;

3. each HF expert's leverage is, for all  $i \in [0, \gamma]$ ,  $\xi_{i,t} = \xi_t^*$  with

$$\xi_t^* = \max \left\{ 0, \frac{\phi \Omega_t}{2\kappa} \left( \frac{A_{t+1} S}{\chi_t} + (1 - \gamma) \phi + \sqrt{\left( \frac{A_{t+1} S}{\chi_t} + (1 - \gamma) \phi \right)^2 + \frac{4\gamma\kappa}{\Omega_t}} \right) - 1 \right\};$$

4. each HF investor asks the expert to buy  $y_{i,t} = \frac{S}{1 + \xi_t^*} + \frac{\phi(1 - \gamma)\chi_t \xi_t^*}{A_{t+1}(1 + \xi_t^*)^2}$  shares of the asset, and each OF investor asks the expert to buy  $\tilde{y}_{i,t} = S - \frac{\phi\chi_t \xi_t^*}{A_{t+1}(1 + \xi_t^*)}$  shares of the asset;

5. HF investor's value function is  $V_t(W_{i,t}) = -\exp(-A_t W_{i,t} - B_t)$  with

$$B_t = \sum_{s=t}^T \left( \prod_{k=t}^s \frac{a_k}{1 + a_k} \right) \left\{ \begin{array}{l} -\ln \beta + \frac{1}{2\chi_s} \left( \frac{\nu S}{(1+r)a_s} + \frac{(1-\gamma)\phi\chi_s \xi_s^*}{1 + \xi_s^*} \right)^2 \\ + \frac{1}{a_s} \ln \left( \frac{1}{a_s} \right) - \frac{1+a_s}{a_s} \ln \left( \frac{1+a_s}{a_s} \right) \end{array} \right\} \quad \text{for } t = 0, \dots, T$$

and  $B_{T+1} = 0$ , and OF investor's value function is  $\tilde{V}_t(\tilde{W}_{i,t}) = -\exp(-A_t \tilde{W}_{i,t} - \tilde{B}_t)$  with

$$\tilde{B}_t = \sum_{s=t}^T \left( \prod_{k=t}^s \frac{a_k}{1 + a_k} \right) \left\{ \begin{array}{l} -\ln \beta + \frac{1}{2\chi_s} \left( \frac{\nu S}{(1+r)a_s} - \frac{\gamma\phi\chi_s \xi_s^*}{1 + \xi_s^*} \right)^2 \\ + \frac{1}{a_s} \ln \left( \frac{1}{a_s} \right) - \frac{1+a_s}{a_s} \ln \left( \frac{1+a_s}{a_s} \right) \end{array} \right\} \quad \text{for } t = 0, \dots, T$$

and  $\tilde{B}_{T+1} = 0$ ;

6. each HF investor's consumption is

$$c_{i,t} = \frac{W_{i,t}}{1 + a_t} + \frac{1}{\nu} \left( \frac{a_t}{1 + a_t} \right) \left( -\ln \beta + \frac{1}{2\chi_t} \left( \frac{\nu S}{(1+r)a_t} + \frac{(1-\gamma)\phi\chi_t \xi_t^*}{1 + \xi_t^*} \right)^2 + B_{t+1} + \ln a_t \right)$$



for  $t = 0, \dots, T$  and  $c_{i,T+1} = W_{i,T+1}$ , and each OF investor's consumption is

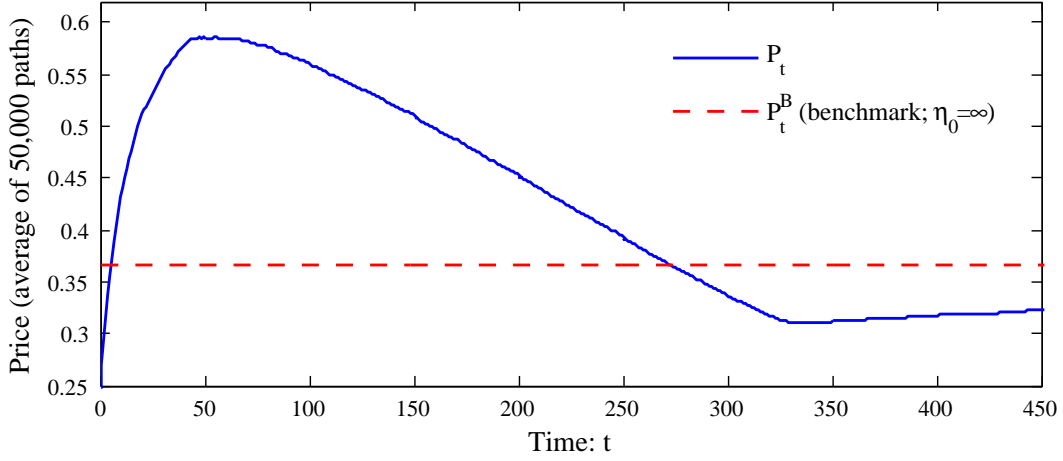
$$\tilde{c}_{i,t} = \frac{\tilde{W}_{i,t}}{1 + a_t} + \frac{1}{\nu} \left( \frac{a_t}{1 + a_t} \right) \left( -\ln \beta + \frac{1}{2\chi_t} \left( \frac{\nu S}{(1+r)a_t} - \frac{\gamma \phi \chi_t \xi_t^*}{1 + \xi_t^*} \right)^2 + \tilde{B}_{t+1} + \ln a_t \right)$$

for  $t = 0, \dots, T$  and  $\tilde{c}_{i,T+1} = \tilde{W}_{i,T+1}$ .

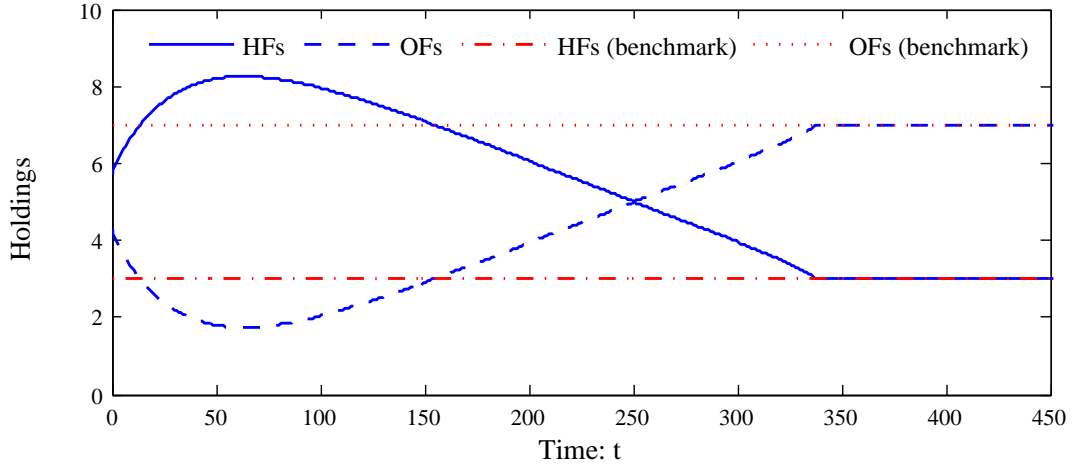
Proposition 2 characterizes the equilibrium in closed form, nesting Proposition 1 as a special case with  $\gamma = 1$ . Panel (a) of Figure 5 plots the average of 50,000 paths of  $P_t$ . It exhibits a bubble-like pattern similar to that of Figure 2(b) of Section 4. This is not surprising, as there is a significant fraction ( $\gamma = 0.3$ ) of HFs, whose learning effect and leverage effect jointly shape the inverse-U dynamics of  $P_t$  as in Section 4. The primary purpose of this section is to study how the funds' holdings—and the associated trading volume—evolve over time behind such swings in  $P_t$ . The key to understanding it is the relationship between the HF leverage  $\xi_t^*$  and the purchase orders,  $y_{i,t}$  and  $\tilde{y}_{i,t}$ . Statement 4 of Proposition 2 implies that not only the HF investors'  $y_{i,t}$  but also the OF investors'  $\tilde{y}_{i,t}$  depend on  $\xi_t^*$  because these investors' decisions depend on  $P_t$  that reflects  $\xi_t^*$  in equilibrium. Thus, both funds' holdings evolve over time depending on  $\xi_t^*$ , as presented in the following corollary.

**Corollary 1.** *The HFs' aggregate holding is  $\Theta_t \equiv \int_0^\gamma (1 + \xi_{i,t}) y_{i,t} di = \gamma S + \Pi_t$ , and the OFs' aggregate holding is  $\tilde{\Theta}_t \equiv \int_\gamma^1 \tilde{y}_{i,t} di = (1 - \gamma)S - \Pi_t$ , where  $\Pi_t \equiv \frac{\gamma(1-\gamma)\phi\chi_t\xi_t^*}{A_{t+1}(1+\xi_t^*)} \geq 0$ .*

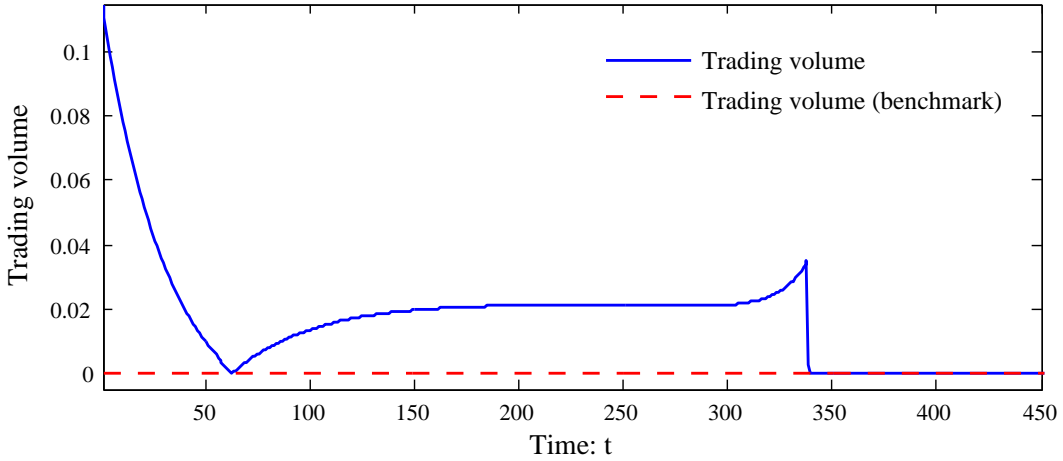
Corollary 1 states that the HFs' holding  $\Theta_t$  is the sum of the “stationary” level  $\gamma S$  and a time-varying component  $\Pi_t$ , whereas the OFs' holding  $\tilde{\Theta}_t$  is the stationary level  $(1 - \gamma)S$  subtracted by  $\Pi_t$ . Here,  $\Pi_t$  increases with  $\chi_t$  and  $\xi_t^*$ , reflecting the HFs' learning effect and leverage effect, respectively. Panel (b) of Figure 5 presents the dynamics of  $\Theta_t$  and  $\tilde{\Theta}_t$  for the same parameter values as panel (a). For comparison, we also plot the benchmark case with a noninnovative asset (i.e.,  $\eta_0 = \infty$ ), where  $\Pi_t = 0$  because  $\xi_t^* = 0$  for all  $t$ . The following two points are worth noting.



(a) Price (average of 50,000 simulated paths)



(b) Aggregate holding: HF's ( $\Theta_t$ ) and OF's ( $\tilde{\Theta}_t$ )



(c) Trading volume ( $|\Delta\Theta_t| = |\Delta\tilde{\Theta}_t|$ )

Figure 5: Dynamics of price, holdings, and trading volume. The parameter values are  $\gamma = 0.3$ ,  $r = 0.04$ ,  $\phi = 0.15$ ,  $\kappa = 1$ ,  $\nu = 0.2$ ,  $\beta = 1/(1+r)$ ,  $S = 10$ ,  $\eta_u = 5$ ,  $\eta_0 = 100$ ,  $\bar{\delta} = 0.18$ ,  $\hat{\delta}_0 = 0.18$ , and  $T = 1,000$ .

First, the evolution of the HFs' holding  $\Theta_t$  is positively related to that of the average price  $P_t$  of panel (a). That is, the HFs tend to increase their asset holdings together with the price that is growing beyond its benchmark level, and reduce them with the price downturn. This result is consistent with Brunnermeier and Nagel (2004), who find that hedge funds were “riding” the 1998–2000 technology bubble: their stock holdings were heavily tilted toward the technology stocks when their prices were rising, but they cut back their holdings before the prices collapsed. Brunnermeier and Nagel (2004) argue that their empirical finding is consistent with the model of Abreu and Brunnermeier (2003), in which rational arbitragers such as hedge funds ride a bubble that emerges and grows exogenously due to “irrationally exuberant” behavioral traders. Our result complements their argument and offers an additional insight: hedge funds may not only ride/avoid the upturn/downturn of asset prices but also *cause* these price swings. Indeed, in our model, the up-and-down dynamics of  $P_t$  reflect the evolution of the HFs' demand. In early periods, the HFs increase holdings as their investors increase  $y_{i,t}$  together with the return precision  $\chi_t$ . But, as the investor learning progresses, they lower  $\xi_t^*$  and decrease their holdings. In Section 4.1 where there are only HFs, the learning and leverage effects are entirely absorbed by the movement of  $P_t$  and are not reflected on their holdings, as market clearing requires the equilibrium holdings to equal  $S$  for all  $t$ . In contrast, in this section, those two effects are also reflected on the HFs' holdings because they are partly absorbed by the holdings of the OFs who act as the HFs' trading counterparties.

Second, the evolution of the HFs' holding  $\Theta_t$  is negatively related to that of the OFs' holding  $\tilde{\Theta}_t$ . While the HFs adjust their holdings to the same direction of the average  $P_t$ , the OFs alter them to the opposite direction. The OFs' holding is largest when the HFs have finished unloading the asset and  $\xi_t^*$  hits zero (around  $t = 330$ ). This result is consistent with Ang, Gorovyy, and van Inwegen (2011), who report that hedge funds' leverage was counter-cyclical to that of other financial intermediaries during the 2007–

2009 crisis: hedge fund leverage decreased before the crisis and was lowest in early 2009 when financial sector leverage was highest. Mathematically, the result is trivial because market clearing requires  $\Theta_t + \tilde{\Theta}_t = S$  and thus  $\Delta\Theta_t = -\Delta\tilde{\Theta}_t$ . Economically, the result follows because the OFs serve as trading counterparties to the HFs that alter holdings over time due to their agency frictions. In early periods, each HF expert is bidding up the price to cater to growing purchase orders  $y_{i,t}$  as well as the leveraged purchase  $\xi_t^* y_{i,t}$ . For such a high price, the OFs—which are not subject to agency frictions—optimally sell the asset to the HFs, reducing their holdings over time. At some point, the HFs’ leverage effect surpasses the learning effect and the HFs start to unload the asset. They deleverage and push down the price over time, to which the OFs respond by increasing their holdings. After the HF leverage  $\xi_t^*$  reaches zero, their holdings stay at the stationary levels.

Panel (c) of Figure 5 shows the dynamics of trading volume, defined as the number of shares traded in the market, i.e.,  $|\Delta\Theta_t|$  (which equals  $|\Delta\tilde{\Theta}_t|$ ). In the benchmark case with a noninnovative asset (the red dashed line), the volume is zero for all  $t \geq 1$  because all the funds trade only in  $t = 0$  and keep the stationary levels of holdings for the rest of the time horizon. In the innovative asset case (the blue solid line), there is a “trading frenzy” right after the asset is introduced to the market: the trading volume is large when  $t$  is very small where the investors are highly uncertain about the asset, as the HFs aggressively buy it from the OFs. Intuitively, the volume is largest in the beginning because the speed of investor learning is fastest in the beginning in the sense that the return precision  $\chi_t$  is concave in  $t$  (see Figure 3(a)), and thus the learning effect is strongest in the beginning. The HFs continue increasing the holdings (at a diminishing speed) until around  $t = 60$ , where they switch to selling the asset to the OFs.<sup>13</sup> Afterwards, the trading volume increases over time because the HFs unload the asset more and more aggressively, reflecting the fact that the learning effect is fading out over time and hence

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<sup>13</sup>In the figure, it looks like the volume hits zero at the turning point around  $t = 60$ ; but it actually stays positive at a low level.

the leverage effect becomes pronounced relatively. After the HF leverage  $\xi_t^*$  reaches zero, the trading volume is zero since the funds keep their stationary levels of holdings.

## 6 Conclusion

This paper develops a dynamic asset-market equilibrium model in which (1) a new and innovative asset with as-yet-unknown average payoff is traded (e.g., Internet stocks, biotech stocks, or sophisticated structured products), and (2) investors delegate investment to experts. Over time, investors learn about the asset's average payoff from fund returns. Experts can secretly renege on investors' purchase orders and take on leveraged positions in the asset in an attempt to manipulate investors' beliefs, thereby attracting more orders and thus more fees. Despite full rationality of long-lived agents, the asset's equilibrium price exhibits bubble-like dynamics on average: gradual upswing, overshoot, and eventual reversal. The up-and-down swings are caused by the combination of (1) the learning effect—an upward pressure on the price as the investors' learning unravels the asset's uncertainty over time, and (2) the leverage effect—a downward pressure on the price as the experts deleverage over time. The price tends to overshoot because the experts' use of leverage pushes up the asset's aggregate demand and thus its market-clearing price. The model predicts that swings and overshooting of prices are more pronounced for new and innovative assets with highly uncertain payoff characteristics than for old-economy assets already familiar in the market. Consistent with empirical evidence, hedge funds increase holdings during the bubble-like price upturn but decrease them in the downturn, counter-cyclically to other market participants' holdings. Innovative assets have high trading volume when investors have large uncertainty about the assets.

For future research, it would be interesting to extend this model to settings with imperfectly competitive and/or illiquid markets, because in reality a significant amount of

innovative financial assets are traded in OTC markets (e.g., Duffie, Gârleanu, and Pedersen 2005) or thin markets with price impact (e.g., Kyle 1989). Studying the equilibrium relation between delegated investment and price dynamics in such settings may yield further economic insights and policy implications.

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# Appendix

## A Evolution of Estimates

Let  $\eta_t \equiv 1/\text{Var}[\bar{\delta}|\mathcal{H}_t]$  be the precision of the experts' period- $t$  estimate of  $\bar{\delta}$ . By standard Kalman filtering, a new observation of  $\delta_t$  will update the estimate of  $\bar{\delta}$  and its precision as follows:

$$\hat{\delta}_t = \lambda_t \hat{\delta}_{t-1} + (1 - \lambda_t) \delta_t \quad \text{for } \lambda_t \equiv \frac{\eta_{t-1}}{\eta_t}, \quad (\text{A.1})$$

where

$$\eta_t = \eta_{t-1} + \eta_u. \quad (\text{A.2})$$

The initial value of  $\hat{\delta}_t$ ,  $\hat{\delta}_0 > 0$ , and the initial value of  $\eta_t$ ,  $\eta_0 > 0$ , are exogenously given. The initial value of  $\lambda_t$  is  $\lambda_1 = \eta_0/\eta_1 = \eta_0/(\eta_0 + \eta_u)$ . From (A.1) and (A.2), we have

$$\lambda_{t+1} = \frac{\eta_t}{\eta_{t+1}} = \frac{\eta_t}{\eta_t + \eta_u} \quad (\text{A.3})$$

and

$$\lambda_t = \frac{\eta_{t-1}}{\eta_t} = \frac{\eta_t - \eta_u}{\eta_t} \iff \eta_t = \frac{\eta_u}{1 - \lambda_t}. \quad (\text{A.4})$$

Plugging (A.4) into (A.3) yields

$$\lambda_{t+1} = \frac{\frac{\eta_u}{1 - \lambda_t}}{\frac{\eta_u}{1 - \lambda_t} + \eta_u} = \frac{1}{2 - \lambda_t}. \quad (\text{A.5})$$

The updating factor  $\lambda_t$  is the same for (3.1) and (3.2). This is shown as follows. Let  $\eta_{i,t} \equiv 1/\text{Var}[\bar{\delta}|\mathcal{H}_{i,t}^I]$  be the precision of investor  $i$ 's estimate of  $\bar{\delta}$ . As in (A.2),  $\eta_{i,t}$  evolves as

$$\eta_{i,t} = \eta_{i,t-1} + \eta_u. \quad (\text{A.6})$$

Since  $\mathcal{H}_0$  and  $\mathcal{H}_{i,0}^I$  are both empty sets,  $\eta_{i,0} = \eta_0$  for all  $i$ . Thus, (A.2) and (A.6) imply that  $\eta_{i,t} = \eta_t$  for all  $i$  and  $t$ . So we have  $\eta_{i,t-1}/\eta_{i,t} = \eta_{t-1}/\eta_t = \lambda_t$ , as required.

## B Proof of Lemmas 1 and 2

First, we derive the period- $t$  precision of asset return,  $\chi_t \equiv 1/\text{Var}_t[R_{t+1}]$ . To do so, it is useful to compute the conditional volatility of  $\delta_{t+1}$ :

$$\text{Var}_t[\delta_{t+1}] = \text{Var}_t[\bar{\delta} + u_{t+1}] = \frac{1}{\eta_t} + \frac{1}{\eta_u} = \frac{1}{\eta_u} \left( \frac{1}{\lambda_{t+1}} - 1 \right) + \frac{1}{\eta_u} = \frac{1}{\eta_u \lambda_{t+1}}. \quad (\text{B.1})$$

Plugging the price conjecture (3.3) into the definition of  $R_{t+1}$  and using the investor's learning rule (3.2), the excess asset return is

$$\begin{aligned} R_{t+1} &\equiv \delta_{t+1} + P_{t+1} - (1+r)P_t \\ &= \delta_{t+1} + a_{t+1} \hat{\delta}_{t+1}^I - b_{t+1} - (1+r)P_t \\ &= \delta_{t+1} + a_{t+1} \int_0^1 \hat{\delta}_{i,t+1}^I di - b_{t+1} - (1+r)P_t \\ &= \delta_{t+1} + a_{t+1} \int_0^1 \left( \lambda_{t+1} \hat{\delta}_{i,t} + (1-\lambda_{t+1}) \delta_{t+1} \right) di - b_{t+1} - (1+r)P_t \\ &= \delta_{t+1} + a_{t+1} \lambda_{t+1} \hat{\delta}_t^I + a_{t+1} (1-\lambda_{t+1}) \delta_{t+1} - b_{t+1} - (1+r)P_t \\ &= (1 + a_{t+1}(1-\lambda_{t+1})) \delta_{t+1} + a_{t+1} \lambda_{t+1} \hat{\delta}_t^I - b_{t+1} - (1+r)P_t. \end{aligned} \quad (\text{B.2})$$

From (B.2) and (B.1), we have

$$\begin{aligned} \text{Var}_t[R_{t+1}] &= (1 + a_{t+1}(1-\lambda_{t+1}))^2 \text{Var}_t[\delta_{t+1}] \\ &= \frac{(1 + a_{t+1}(1-\lambda_{t+1}))^2}{\eta_u \lambda_{t+1}}, \end{aligned} \quad (\text{B.3})$$

which yields  $\chi_t$  as in (3.10).

Now, we derive the investor's value function and investment policy. In the final period  $t = T + 1$ , the investors do not have optimization problems. Each of them just consumes her entire wealth, i.e.,  $c_{i,T+1} = W_{i,T+1}$ . Thus,  $A_{T+1} = \nu$  and  $B_{T+1} = 0$ . The investor's problem in period  $t = 0, \dots, T$  is solved as follows. Using dynamic budget constraint (2.5) and conjectured value function (3.6), we have

$$\text{E} [V_{t+1}(W_{i,t+1}) | \mathcal{F}_{i,t}^I] = -\exp \left( \begin{array}{c} -A_{t+1} \left( \hat{R}_{i,t+1}^I (1 + \xi_t^*) y_{i,t} - \phi y_{i,t} + (1+r)(W_{i,t} - c_{i,t}) \right) \\ -\frac{1}{2} A_{t+1} (1 + \xi_t^*)^2 y_{i,t}^2 \frac{1}{\chi_t} \end{array} \right). \quad (\text{B.4})$$

From (B.4) and Bellman equation (3.7), the first-order condition (FOC) for  $y_{i,t}$  is

$$\hat{R}_{i,t+1}^I(1 + \xi_t^*) - \phi - A_{t+1}(1 + \xi_t^*)^2 y_{i,t} \frac{1}{\chi_t} = 0 \iff y_{i,t} = \frac{\chi_t(\hat{R}_{i,t+1}^I(1 + \xi_t^*) - \phi)}{A_{t+1}(1 + \xi_t^*)^2}. \quad (\text{B.5})$$

From (B.4) and (B.5),

$$\text{E} [V_{t+1}(W_{i,t+1}) | \mathcal{F}_{i,t}^I] = -\exp\left(-\frac{1}{2}A_{t+1}^2(1 + \xi_t^*)^2 y_{i,t}^2 \frac{1}{\chi_t} - A_{t+1}(1 + r)(W_{i,t} - c_{i,t}) - B_{t+1}\right). \quad (\text{B.6})$$

Market clearing implies  $y_{i,t} = S/(1 + \xi_t^*)$  in equilibrium. Plugging this into (B.6),

$$\text{E} [V_{t+1}(W_{i,t+1}) | \mathcal{F}_{i,t}^I] = -\exp\left(-\frac{1}{2}A_{t+1}^2 S^2 \frac{1}{\chi_t} - A_{t+1}(1 + r)(W_{i,t} - c_{i,t}) - B_{t+1}\right). \quad (\text{B.7})$$

Thus, Bellman equation (3.7) is rewritten as

$$V_t(W_{i,t}) = \max_{c_{i,t}} \{-\exp(-\nu c_{i,t}) - \exp(-\psi_t - A_{t+1}(1 + r)(W_{i,t} - c_{i,t}))\}, \quad (\text{B.8})$$

$$\text{where } \psi_t \equiv -\ln \beta + \frac{1}{2}A_{t+1}^2 S^2 \frac{1}{\chi_t} + B_{t+1}. \quad (\text{B.9})$$

The FOC for  $c_{i,t}$  is

$$\begin{aligned} \nu \exp(-\nu c_{i,t}) - \exp(-\psi_t) A_{t+1}(1 + r) \exp(-A_{t+1}(1 + r)(W_{i,t} - c_{i,t})) &= 0 \\ \iff \ln \nu - \nu c_{i,t} &= -\psi_t + \ln(A_{t+1}(1 + r)) - A_{t+1}(1 + r)(W_{i,t} - c_{i,t}) \\ \iff \psi_t + \ln\left(\frac{\nu}{A_{t+1}(1 + r)}\right) + A_{t+1}(1 + r)W_{i,t} &= (\alpha + A_{t+1}(1 + r))c_{i,t} \\ \iff c_{i,t} &= \frac{A_{t+1}(1 + r)}{\nu + A_{t+1}(1 + r)}W_{i,t} + \frac{1}{\nu + A_{t+1}(1 + r)}\left(\psi_t + \ln\left(\frac{\nu}{A_{t+1}(1 + r)}\right)\right). \end{aligned} \quad (\text{B.10})$$

Plugging (B.10) back into (B.8), we have

$$\begin{aligned} V_t(W_{i,t}) &= -\exp\left(-\frac{\nu A_{t+1}(1 + r)}{\nu + A_{t+1}(1 + r)}W_{i,t} - \frac{\nu}{\nu + A_{t+1}(1 + r)}\left(\psi_t + \ln\left(\frac{\nu}{A_{t+1}(1 + r)}\right)\right)\right) \\ &= -\exp\left(-\frac{\nu A_{t+1}(1 + r)}{\nu + A_{t+1}(1 + r)}W_{i,t} - \frac{\nu}{\nu + A_{t+1}(1 + r)}\psi_t\right) \\ &\quad \times \left(\left(\frac{A_{t+1}(1 + r)}{\nu}\right)^{-\frac{A_{t+1}(1 + r)}{\nu + A_{t+1}(1 + r)}}\left(1 + \frac{A_{t+1}(1 + r)}{\nu}\right)\right). \end{aligned} \quad (\text{B.11})$$

Taking log to (B.11),

$$-A_t W_{i,t} - B_t = -\frac{\nu A_{t+1}(1+r)}{\nu + A_{t+1}(1+r)} W_{i,t} - \frac{\nu}{\nu + A_{t+1}(1+r)} \psi_t - \frac{A_{t+1}(1+r)}{\nu + A_{t+1}(1+r)} \ln \left( \frac{A_{t+1}(1+r)}{\nu} \right) + \ln \left( 1 + \frac{A_{t+1}(1+r)}{\nu} \right). \quad (\text{B.12})$$

From (B.12) we have

$$A_t = \frac{\nu A_{t+1}(1+r)}{\nu + A_{t+1}(1+r)} \quad (\text{B.13})$$

and

$$B_t = \frac{\nu}{\nu + A_{t+1}(1+r)} \psi_t + \frac{A_{t+1}(1+r)}{\nu + A_{t+1}(1+r)} \ln \left( \frac{A_{t+1}(1+r)}{\nu} \right) - \ln \left( 1 + \frac{A_{t+1}(1+r)}{\nu} \right). \quad (\text{B.14})$$

Using (B.9), (B.14) is rearranged as

$$B_t = \frac{\nu}{\nu + A_{t+1}(1+r)} \left( \begin{array}{c} B_{t+1} - \ln \beta + \frac{1}{2} A_{t+1}^2 S^2 \frac{1}{\chi_t} \\ + \frac{A_{t+1}(1+r)}{\nu} \ln \left( \frac{A_{t+1}(1+r)}{\nu} \right) - \frac{\nu + A_{t+1}(1+r)}{\nu} \ln \left( \frac{\nu + A_{t+1}(1+r)}{\nu} \right) \end{array} \right). \quad (\text{B.15})$$

Solving (B.13) backward from the terminal value  $A_{T+1} = \nu$ , we have

$$A_t = \nu \left( 1 + \left( \frac{1}{1+r} \right) + \left( \frac{1}{1+r} \right)^2 + \cdots + \left( \frac{1}{1+r} \right)^{T+1-t} \right)^{-1} = \frac{\nu}{1 + a_t}. \quad (\text{B.16})$$

Using (B.16), (B.15) is rewritten as

$$B_t = m_t (B_{t+1} + n_t), \quad (\text{B.17})$$

$$\text{where } m_t \equiv \frac{a_t}{1 + a_t}, \quad (\text{B.18})$$

$$n_t \equiv -\ln \beta + \frac{1}{2} \left( \frac{\nu S}{(1+r)a_t} \right)^2 \frac{1}{\chi_t} + \frac{1}{a_t} \ln \left( \frac{1}{a_t} \right) - \frac{1 + a_t}{a_t} \ln \left( \frac{1 + a_t}{a_t} \right). \quad (\text{B.19})$$

Solving (B.17) backward from the terminal value  $B_{T+1} = 0$ , we have

$$\begin{aligned} B_t &= m_t n_t + m_t m_{t+1} n_{t+1} + m_t m_{t+1} m_{t+2} n_{t+2} + \cdots + m_t m_{t+1} \cdots m_T n_T \\ &= \sum_{s=t}^T \left( \prod_{k=t}^s m_k \right) n_s, \end{aligned} \tag{B.20}$$

which is equivalent to (3.9) in the main text.  $\square$

## C Proof of Lemma 3

First, we determine investor  $i$ 's purchase order on an arbitrary off-the-equilibrium path where expert  $i$  is deviating from his equilibrium strategy.

**Lemma 4.** *If expert  $i$  plays  $(\xi_{i,0}, \dots, \xi_{i,T})$  when investor  $i$  believes that he plays  $(\xi_0^*, \dots, \xi_T^*)$ , then investor  $i$ 's purchase order in period  $t = 1, \dots, T$  is*

$$y_{i,t} = y_t + y_{i,t}^+, \tag{C.1}$$

$$\text{where } y_t \equiv \frac{\chi_t (\hat{R}_{t+1} (1 + \xi_t^*) - \phi)}{A_{t+1} (1 + \xi_t^*)^2} \tag{C.2}$$

$$\text{and } y_{i,t}^+ \equiv \frac{\chi_t (1 + a_{t+1} (1 - \lambda_{t+1}))}{A_{t+1} (1 + \xi_t^*)} \sum_{s=1}^t \left( \prod_{k=s+1}^t \lambda_k \right) (1 - \lambda_s) \left( \frac{\xi_{i,s-1} - \xi_{s-1}^*}{1 + \xi_{s-1}^*} \right) R_s. \tag{C.3}$$

*Proof of Lemma 4:* In order to prove Lemma 4, we prove the following two claims.

**Claim C.1.** *If investor  $i$  believes that the payoff history up to period  $t$  is  $\mathcal{H}_{i,t}^I = (\delta_{i,1}^I, \dots, \delta_{i,t}^I)$ , her estimate of  $\bar{\delta}$  in an arbitrary period  $t$  is*

$$\hat{\delta}_{i,t}^I = \hat{\delta}_t + \sum_{s=1}^t \left( \prod_{k=s+1}^t \lambda_k \right) (1 - \lambda_s) (\delta_{i,s}^I - \delta_s). \tag{C.4}$$

*Proof of Claim C.1:* See Appendix C of Sato (2014). (*End of proof of Claim C.1.*)

**Claim C.2.** *Investor  $i$ 's expected excess asset return is*

$$\hat{R}_{i,t+1}^I = \hat{R}_{t+1} + (1 + a_{t+1} (1 - \lambda_{t+1})) (\hat{\delta}_{i,t}^I - \hat{\delta}_t). \tag{C.5}$$

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<sup>14</sup>In (C.3), we abuse notation and set  $\prod_{k=t+1}^t \lambda_k \equiv 1$ .

*Proof of Claim C.2:* From (B.2), the expected excess return conditional on the true history  $\mathcal{H}_t$  is

$$\hat{R}_{t+1} \equiv E[R_{t+1}|\mathcal{H}_t] = (1 + a_{t+1}(1 - \lambda_{t+1}))\hat{\delta}_t + a_{t+1}\lambda_{t+1}\hat{\delta}_t^I - b_{t+1} - (1 + r)P_t. \quad (\text{C.6})$$

The expected excess return from investor  $i$ 's perspective (conditional on her inferred history  $\mathcal{H}_{i,t}^I$ ) is

$$\hat{R}_{i,t+1}^I \equiv E[R_{t+1}|\mathcal{F}_{i,t}^I] = (1 + a_{t+1}(1 - \lambda_{t+1}))\hat{\delta}_{i,t}^I + a_{t+1}\lambda_{t+1}\hat{\delta}_t^I - b_{t+1} - (1 + r)P_t. \quad (\text{C.7})$$

From (C.6) and (C.7) we obtain (C.5). (*End of proof of Claim C.2.*)

Now Claims C.1 and C.2 can be used to rearrange the investor's order (3.11) as follows:

$$\begin{aligned} y_{i,t} &= \frac{\chi_t(\hat{R}_{i,t+1}^I(1 + \xi_t^*) - \phi)}{A_{t+1}(1 + \xi_t^*)^2} \\ &= \frac{\chi_t(\hat{R}_{t+1}(1 + \xi_t^*) - \phi)}{A_{t+1}(1 + \xi_t^*)^2} + \frac{\chi_t}{A_{t+1}(1 + \xi_t^*)}(\hat{R}_{i,t+1}^I - \hat{R}_{t+1}) \\ &= \frac{\chi_t(\hat{R}_{t+1}(1 + \xi_t^*) - \phi)}{A_{t+1}(1 + \xi_t^*)^2} + \frac{\chi_t(1 + a_{t+1}(1 - \lambda_{t+1}))}{A_{t+1}(1 + \xi_t^*)} \sum_{s=1}^t \left( \prod_{k=s+1}^t \lambda_k \right) (1 - \lambda_s)(\delta_{i,s}^I - \delta_s). \end{aligned} \quad (\text{C.8})$$

Substituting (3.13) into (C.8) then yields

$$\begin{aligned} y_{i,t} &= \frac{\chi_t(\hat{R}_{t+1}(1 + \xi_t^*) - \phi)}{A_{t+1}(1 + \xi_t^*)^2} + \frac{\chi_t(1 + a_{t+1}(1 - \lambda_{t+1}))}{A_{t+1}(1 + \xi_t^*)} \sum_{s=1}^t \left( \prod_{k=s+1}^t \lambda_k \right) (1 - \lambda_s) \left( \frac{\xi_{i,s-1} - \xi_{s-1}^*}{1 + \xi_{s-1}^*} \right) R_s \\ &= y_t + y_{i,t}^+, \end{aligned}$$

as required. (*End of proof of Lemma 4.*)

Now we prove Lemma 3. To simplify the expert's period- $t$  objective, note the following points.

- The fee on the current order,  $\phi y_{i,t}$ , can be omitted from the original objective function (2.3) because, from (3.11),  $y_{i,t}$  is independent of the expert's actual choice of  $\xi_{i,t}$ .
- By Lemma 4, the future order  $y_{i,t+\tau}$  ( $\tau = 1, 2, \dots, T - t$ ) is linear in  $y_{t+\tau}$  and  $y_{i,t+\tau}^+$ . Since (2.3) is linear in  $y_{i,t+\tau}$ , it follows that (2.3) is linear in  $y_{t+\tau}$  and  $y_{i,t+\tau}^+$ . This implies that  $y_{t+\tau}$  can be omitted from (2.3) because the expert cannot influence  $y_{t+\tau}$  by his choice of  $\xi_{i,t}$ . That is, only  $y_{i,t+\tau}^+$  is relevant for his choice of  $\xi_{i,t}$ .
- Conjecture 2—which is verified later—implies that the expert's current action ( $\xi_{i,t}$ ) does not affect his own future actions ( $\xi_{i,t+1}, \dots, \xi_{i,T}$ ) both on and off the equilibrium path. Thus, his costs of

renewing in future periods can be omitted from (2.3).

Taking these points into account, the expert's period- $t$  maximization problem reduces to

$$\max_{\xi_{i,t} \in [0, \infty)} -\kappa \xi_{i,t} + \mathbb{E} \left[ \sum_{\tau=1}^{T-t} \beta^\tau \phi y_{i,t+\tau}^+ \middle| \mathcal{F}_{i,t}^E \right],$$

$$\text{where } y_{i,t+\tau}^+ \equiv \frac{\chi_{t+\tau} (1 + a_{t+\tau+1} (1 - \lambda_{t+\tau+1}))}{A_{t+\tau+1} (1 + \xi_{t+\tau}^*)} \sum_{s=1}^{t+\tau} \left( \prod_{k=s+1}^{t+\tau} \lambda_k \right) (1 - \lambda_s) \left( \frac{\xi_{i,s-1} - \xi_{s-1}^*}{1 + \xi_{s-1}^*} \right) R_s. \quad (\text{C.9})$$

Since  $y_{i,t+\tau}^+$  is a linear function of  $(\xi_{i,0}, \dots, \xi_{i,t+\tau-1})$ , the marginal effect of the expert's current action  $\xi_{i,t}$  on  $y_{i,t+\tau}^+$  is independent of his actions in other periods,  $(\xi_{i,0}, \dots, \xi_{i,t-1}, \xi_{i,t+1}, \dots, \xi_{i,t+\tau-1})$ . Hence, in  $y_{i,t+\tau}^+$  given by (C.9), only the term corresponding to  $s = t + 1$  is relevant for the choice of  $\xi_{i,t}$ . Thus, an equivalent problem is

$$\max_{\xi_{i,t} \in [0, \infty)} -\kappa \xi_{i,t} + \mathbb{E} \left[ \sum_{\tau=1}^{T-t} \beta^\tau \phi \frac{\chi_{t+\tau} (1 + a_{t+\tau+1} (1 - \lambda_{t+\tau+1}))}{A_{t+\tau+1} (1 + \xi_{t+\tau}^*)} \left( \prod_{k=t+2}^{t+\tau} \lambda_k \right) (1 - \lambda_{t+1}) \left( \frac{\xi_{i,t} - \xi_t^*}{1 + \xi_t^*} \right) R_{t+1} \middle| \mathcal{F}_{i,t}^E \right].$$

This is rewritten as

$$\max_{\xi_{i,t} \in [0, \infty)} -\kappa \xi_{i,t} + \phi \left( \frac{\xi_{i,t} - \xi_t^*}{1 + \xi_t^*} \right) \Omega_t \hat{R}_{t+1}, \quad (\text{C.10})$$

$$\text{where } \Omega_t \equiv (1 - \lambda_{t+1}) \sum_{\tau=1}^{T-t} \beta^\tau \frac{\chi_{t+\tau} (1 + a_{t+\tau+1} (1 - \lambda_{t+\tau+1}))}{A_{t+\tau+1} (1 + \xi_{t+\tau}^*)} \left( \prod_{k=t+2}^{t+\tau} \lambda_k \right) \text{ for } t = 0, \dots, T-1$$

and  $\Omega_T \equiv 0$ . Choosing  $\xi_{i,t} \geq 0$  to maximize (C.10), the FOC is given by (3.14) in the main text.  $\square$

## D Proof of Proposition 1

Statements 1–5 are proved in the main text. Statement 6 follows by rearranging (B.10) with (B.16).  $\square$

## E Dynamics of $\Omega_t$

For  $t = T$ , we have  $\Omega_T = 0$ . For  $t = 0, \dots, T-1$ , from (3.15) we have

$$\Omega_t = (1 - \lambda_{t+1}) \sum_{\tau=1}^{T-t} \beta^\tau \left( \prod_{k=t+2}^{t+\tau} \lambda_k \right) M_{t+\tau}, \quad \text{where } M_{t+\tau} \equiv \frac{\chi_{t+\tau} (1 + a_{t+\tau+1} (1 - \lambda_{t+\tau+1}))}{A_{t+\tau+1} (1 + \xi_{t+\tau}^*)}.$$

Let us denote  $\prod_{k=t+2}^{t+1} \lambda_k \equiv 1$  (by abuse of notation). Then we have

$$\begin{aligned}\frac{\Omega_t}{1 - \lambda_{t+1}} &= \beta M_{t+1} + \beta^2 \lambda_{t+2} M_{t+2} + \beta^3 \lambda_{t+2} \lambda_{t+3} M_{t+3} + \cdots + \beta^{T-t} \lambda_{t+2} \lambda_{t+3} \cdots \lambda_T M_T, \\ \frac{\Omega_{t+1}}{1 - \lambda_{t+2}} &= \beta M_{t+2} + \beta^2 \lambda_{t+3} M_{t+3} + \beta^3 \lambda_{t+3} \lambda_{t+4} M_{t+4} + \cdots + \beta^{T-t-1} \lambda_{t+3} \lambda_{t+4} \cdots \lambda_T M_T.\end{aligned}$$

These two equations yields the difference equation of  $\Omega_t$  for  $t = 0, \dots, T - 1$ :

$$\frac{\Omega_t}{1 - \lambda_{t+1}} = \beta M_{t+1} + \beta \frac{\lambda_{t+2}}{1 - \lambda_{t+2}} \Omega_{t+1}. \quad (\text{E.1})$$

Note that (A.5) implies  $1 - \lambda_{t+1} = (1 - \lambda_{t+2})/\lambda_{t+2}$ . Using this, (E.1) is rewritten as

$$\Omega_t = \beta \left( \Omega_{t+1} + \left( \frac{1 - \lambda_{t+2}}{\lambda_{t+2}} \right) \frac{\chi_{t+1} (1 + a_{t+2} (1 - \lambda_{t+2}))}{A_{t+2} (1 + \xi_{t+1}^*)} \right). \quad (\text{E.2})$$

From (E.2) and the terminal value  $\Omega_T = 0$ , we obtain  $\{\Omega_t\}_{t=0}^T$  by backward induction.

## F Proof of Proposition 2

The conjectures about the equilibrium price and the HF expert's strategy remain the same as Conjecture 1 and Conjecture 2 of Section 3, respectively. The investors' out-of-equilibrium belief is still (3.5).

HF investor  $i$  chooses  $y_{i,t}$  and  $c_{i,t}$  to maximize  $-\mathbb{E}[\sum_{\tau=0}^{T-t} \beta^\tau \exp(-\nu c_{i,t+\tau}) | \mathcal{F}_{i,t}^I]$ , subject to the dynamic budget constraint  $W_{i,t+1} = Q_{i,t+1} - \phi y_{i,t} + (1+r)(W_{i,t} - c_{i,t} - P_t y_{i,t})$ . Guess and later verify that her value function is  $V_t(W_{i,t}) = -\exp(-A_t W_{i,t} - B_t)$ , where  $A_t$  is given by (3.8) and  $B_t$  is a deterministic variable. Using the dynamic budget constraint and conjectured value function, we have

$$\mathbb{E}[V_{t+1}(W_{i,t+1}) | \mathcal{F}_{i,t}^I] = -\exp \left( \begin{array}{l} -A_{t+1} \left( \hat{R}_{i,t+1}^I (1 + \xi_t^*) y_{i,t} - \phi y_{i,t} + (1+r)(W_{i,t} - c_{i,t}) \right) \\ -\frac{1}{2} A_{t+1} (1 + \xi_t^*)^2 y_{i,t}^2 \frac{1}{\chi_t} \end{array} \right) - B_{t+1}. \quad (\text{F.1})$$

From (F.1) and the Bellman equation, the FOC for  $y_{i,t}$  is

$$\hat{R}_{i,t+1}^I (1 + \xi_t^*) - \phi - A_{t+1} (1 + \xi_t^*)^2 y_{i,t} \frac{1}{\chi_t} = 0 \quad \iff \quad y_{i,t} = \frac{\chi_t (\hat{R}_{i,t+1}^I (1 + \xi_t^*) - \phi)}{A_{t+1} (1 + \xi_t^*)^2}. \quad (\text{F.2})$$



From (F.1) and (F.2),

$$\mathbb{E} [V_{t+1}(W_{i,t+1})|\mathcal{F}_{i,t}^I] = -\exp\left(-\frac{1}{2}A_{t+1}^2(1+\xi_t^*)^2\tilde{y}_{i,t}^2\frac{1}{\chi_t} - A_{t+1}(1+r)(W_{i,t} - c_{i,t}) - B_{t+1}\right). \quad (\text{F.3})$$

OF investor  $i$  chooses  $\tilde{y}_{i,t}$  and  $\tilde{c}_{i,t}$  to maximize  $-\mathbb{E}[\sum_{\tau=0}^{T-t}\beta^\tau \exp(-\nu\tilde{c}_{i,t+\tau})|\tilde{\mathcal{F}}_{i,t}^I]$ , subject to the dynamic budget constraint  $\tilde{W}_{i,t+1} = R_{t+1}\tilde{y}_{i,t} - \phi\tilde{y}_{i,t} + (1+r)(\tilde{W}_{i,t} - \tilde{c}_{i,t})$ . Guess and later verify that her value function is  $\tilde{V}_t(\tilde{W}_{i,t}) = -\exp(-A_t\tilde{W}_{i,t} - \tilde{B}_t)$ , where  $\tilde{B}_t$  is a deterministic variable. Since the OF expert never reneges on the investor's order, the investor correctly infers  $\delta_t$  always, both on and off the equilibrium path. Thus,  $\hat{R}_{i,t+1}^I = \hat{R}_{t+1}$  holds always. Hence,

$$\mathbb{E}[\tilde{V}_{t+1}(\tilde{W}_{i,t+1})|\tilde{\mathcal{F}}_{i,t}^I] = -\exp\left(-A_{t+1}\left(\hat{R}_{t+1}\tilde{y}_{i,t} - \phi\tilde{y}_{i,t} + (1+r)(\tilde{W}_{i,t} - \tilde{c}_{i,t}) - \frac{1}{2}A_{t+1}\tilde{y}_{i,t}^2\frac{1}{\chi_t}\right) - \tilde{B}_{t+1}\right). \quad (\text{F.4})$$

From (F.4) and the Bellman equation, the FOC for  $\tilde{y}_{i,t}$  is

$$\hat{R}_{t+1} - \phi - A_{t+1}\tilde{y}_{i,t}\frac{1}{\chi_t} = 0 \quad \iff \quad \tilde{y}_{i,t} = \frac{\chi_t(\hat{R}_{t+1} - \phi)}{A_{t+1}}. \quad (\text{F.5})$$

From (F.4) and (F.5),

$$\mathbb{E}[\tilde{V}_{t+1}(\tilde{W}_{i,t+1})|\tilde{\mathcal{F}}_{i,t}^I] = -\exp\left(-\frac{1}{2}A_{t+1}^2\tilde{y}_{i,t}^2\frac{1}{\chi_t} - A_{t+1}(1+r)(\tilde{W}_{i,t} - \tilde{c}_{i,t}) - \tilde{B}_{t+1}\right). \quad (\text{F.6})$$

In equilibrium,  $\hat{R}_{i,t+1}^I = \hat{R}_{t+1}$  for all  $i$ . So plugging (F.2) and (F.5) into the market clearing condition,

$$\begin{aligned} & \int_0^\gamma (1+\xi_{i,t})y_{i,t}di + \int_\gamma^1 \tilde{y}_{i,t}di = S \\ \iff & \frac{\gamma\chi_t(\hat{R}_{t+1}(1+\xi_t^*) - \phi)}{A_{t+1}(1+\xi_t^*)} + \frac{(1-\gamma)\chi_t(\hat{R}_{t+1} - \phi)}{A_{t+1}} = S \\ \iff & \hat{R}_{t+1} = \frac{A_{t+1}S}{\chi_t} + \frac{\gamma\phi}{1+\xi_t^*} + (1-\gamma)\phi. \end{aligned} \quad (\text{F.7})$$

Plugging (F.7) into (F.2) and (F.5), we have

$$y_{i,t} = \frac{S}{1+\xi_t^*} + \frac{\phi(1-\gamma)\chi_t\xi_t^*}{A_{t+1}(1+\xi_t^*)^2} \quad (\text{F.8})$$

$$\text{and } \tilde{y}_{i,t} = S - \frac{\phi\gamma\chi_t\xi_t^*}{A_{t+1}(1+\xi_t^*)}. \quad (\text{F.9})$$

Plugging (F.8) into (F.3), and (F.9) into (F.6), we have

$$\mathbb{E}[V_{t+1}(W_{i,t+1})|\mathcal{F}_{i,t}^I] = -\exp\left(-\frac{1}{2\chi_t}\left(A_{t+1}S + \frac{\phi(1-\gamma)\chi_t\xi_t^*}{1+\xi_t^*}\right)^2 - A_{t+1}(1+r)(W_{i,t} - c_{i,t}) - B_{t+1}\right) \quad (\text{F.10})$$

$$\text{and } \mathbb{E}[\tilde{V}_{t+1}(\tilde{W}_{i,t+1})|\tilde{\mathcal{F}}_{i,t}^I] = -\exp\left(-\frac{1}{2\chi_t}\left(A_{t+1}S - \frac{\phi\gamma\chi_t\xi_t^*}{1+\xi_t^*}\right)^2 - A_{t+1}(1+r)(\tilde{W}_{i,t} - \tilde{c}_{i,t}) - \tilde{B}_{t+1}\right). \quad (\text{F.11})$$

Thus, (F.10) implies that the Bellman equation for each HF investor is

$$V_t(W_{i,t}) = \max_{c_{i,t}} \{-\exp(-\nu c_{i,t}) - \exp(-\psi_t - A_{t+1}(1+r)(W_{i,t} - c_{i,t}))\} \quad (\text{F.12})$$

$$\text{with } \psi_t \equiv -\ln\beta + \frac{1}{2\chi_t}\left(A_{t+1}S + \frac{\phi(1-\gamma)\chi_t\xi_t^*}{1+\xi_t^*}\right)^2 + B_{t+1}, \quad (\text{F.13})$$

and (F.11) implies that the one for each OF investor is

$$\tilde{V}_t(\tilde{W}_{i,t}) = \max_{\tilde{c}_{i,t}} \{-\exp(-\nu\tilde{c}_{i,t}) - \exp(-\tilde{\psi}_t - A_{t+1}(1+r)(\tilde{W}_{i,t} - \tilde{c}_{i,t}))\} \quad (\text{F.14})$$

$$\text{with } \tilde{\psi}_t \equiv -\ln\beta + \frac{1}{2\chi_t}\left(A_{t+1}S - \frac{\phi\gamma\chi_t\xi_t^*}{1+\xi_t^*}\right)^2 + \tilde{B}_{t+1}. \quad (\text{F.15})$$

Now, following the same steps in Appendix B (from (B.8) to (B.20)), we obtain the values of  $B_t$  and  $\tilde{B}_t$  as well as consumptions  $c_{i,t}$  and  $\tilde{c}_{i,t}$  as presented in Proposition 2.

Each HF expert's optimization problem is identical to Section 3.5. The equilibrium  $\xi_t^*$  given  $\hat{R}_{t+1}$  is given by (3.16). Solving the system of equations (F.7) and (3.16) for two unknown  $\hat{R}_{t+1}$  and  $\xi_t^*$  yields their values presented in Proposition 2. Having obtained  $\hat{R}_{t+1}$ , the price  $P_t$  is readily obtained following the same steps in Section 3.6.  $\square$