Public bads, heterogeneous beliefs, and the value of information*

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Abstract

This paper considers an economy where players have heterogeneous beliefs about the uncertain consequences of their collective actions. The roles of beliefs and preferences are examined, followed by a detailed investigation of the impacts of information in the presence of belief heterogeneity and ambiguity. In particular, it is shown that new information can worsen the free-riding problem, even when it better reflects the correct risk than the players' beliefs. When beliefs are highly heterogeneous, adding information noise can be Pareto-improving, for which the degree of risk and ambiguity aversion play asymmetric roles.

Keywords: externality; uncertainty; heterogeneous beliefs; information

JEL classification: H40; D80; Q54

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1 Introduction

In this paper, we develop a simple model of public bads where players have heterogeneous beliefs about the consequences of their collective actions. In our model, players can reduce the negative impact of public bads at a private cost. While the private cost is certain, the damage from public bads is subject to deep uncertainty. Uncertainty exists not only in the sense that the damage has a probability distribution, but also in the sense that the distribution itself is unknown. In other words, players are faced with ambiguity in terms of what would happen in the absence of action against public bads. The difficulty in estimating the true distribution of damage leads to disagreements among players about the risk of not taking action. Such heterogeneous beliefs result in the uncoordinated actions of players, which might be a source of additional inefficiency. The availability of public information and the partial resolution of ambiguity that follows could then mitigate the inefficiency by facilitating the convergence of beliefs.

Our model encompasses various problems of public-bad nature. Perhaps the most relevant application would be global environmental problems, such as climate change. Although recent decades have shown considerable progress in the scientific basis of climate change (IPCC, 2007), state-of-the-art knowledge has yet to provide a clear picture of the possible consequences of increasing carbon concentration in the atmosphere. For instance, an important metric called climate sensitivity, which measures the change in temperature due to a doubling of carbon concentration, is known to be inherently uncertain (Roe and Baker, 2007). While a number of scientific studies have estimated the possible values of this important parameter, the proposed risks are not necessarily in agreement with each other (Meinshausen et al., 2009). On one hand, this implies that addressing climate change involves decision-making under ambiguity.

The lack of clear-cut consensus among scientists, on the other hand, allows people to have different beliefs. It is left to the subjective interpretation of individuals as to how credible each of the proposed risk estimates is. Some people choose to be optimistic about the impacts

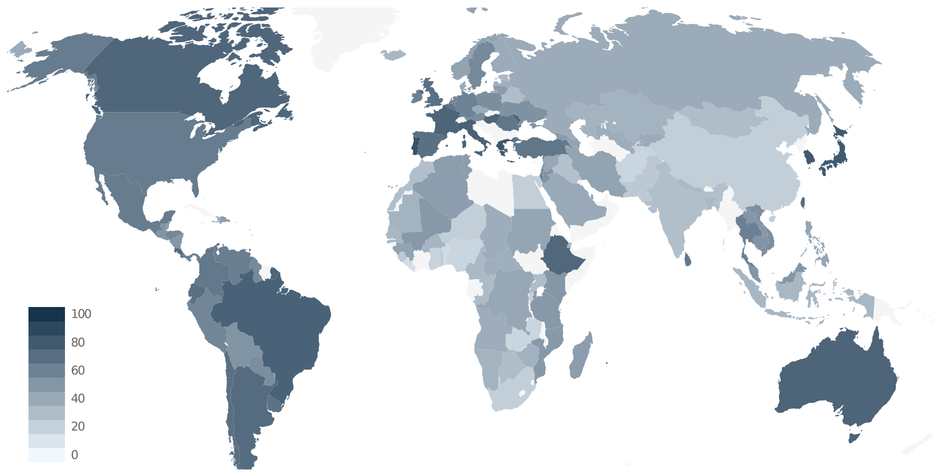

Figure 1: Percentage of respondents who consider climate change to be a threat (Pelham, 2009).

of climate change, arguing that the climate system is not as sensitive to human-induced carbon emission as is predicted by some scientific studies. Others are decidedly pessimistic, believing that catastrophic scenarios are more likely than predicted by optimistic risk estimates. In fact, according to a survey conducted in 127 countries, the public's perception of climate change significantly varies across countries (Pelham, 2009). For example, more than 90% of the respondents in Japan believe that climate change is caused by human activities. In the United States, on the other hand, less than one in two people think that the problem is human-induced. In France, 75% of people perceive climate change as a serious threat, whereas the percentage sharply drops to only 21% in China. Figure 1 illustrates the diversity of climate-related risk perception.

The discrepancy in beliefs consequently creates an obstacle to collective risk prevention. Since the cause and consequence of public bads stretch across different players, the actions of independent players should be coordinated if the problem is to be efficiently addressed. However, when facing the threat of climate change, individual countries react in quite different ways. Countries in the European Union, for example, are relatively more willing to curb carbon dioxide emission. In the United States, on the other hand, the consequential value of lowering carbon emissions is less appreciated. Some developing countries, such as China and India, are even more reluctant to engage in mitigation activities. These uncoordinated actions at least partly reflect the heterogeneity in their beliefs, since the expected benefit of carbon mitigation is affected by their subjective beliefs. Therefore, heterogeneity of beliefs adds inefficiency of a different kind on top of the externality associated with public bads.

A question of interest is whether new information can mitigate the inefficiency by encouraging an update of subjective beliefs of players. In the context of climate change, continuous efforts have been made to resolve the ambiguity in climate science and, as a result, new findings about the true risks of climate change become available from time to time. These occasional findings, if integrated into players' beliefs, could have a significant influence on the formation of domestic and international climate policy, as exemplified by the series of influential reports of the Intergovernmental Panel on Climate Change (IPCC). The impact of new information is, of course, dependent upon a number of factors. It depends on how players initially perceive the risks and what kind of information becomes newly available. The players' preference with respect to risk and ambiguity also plays a role. With our framework, these issues can be investigated in a tractable way.

We do not explicitly model how the subjective and possibly incorrect beliefs emerge from ambiguity. Mounting evidence, however, identifies a set of psychological biases that distort people's beliefs in various economic situations. The experimental evidence summarized by DellaVigna (2009), for instance, suggests that people have systematically incorrect beliefs and most people underestimate the probability of negative events. More recently, Hommes (2012) reported the persistent emergence of irrational and heterogeneous beliefs in laboratory experiments. Despite the existing evidence and its potentially important implications, the role of heterogeneous beliefs has only been investigated in a limited number of economic models[1]. In particular, the consideration of strategic incentives is largely absent in the analysis of heterogeneous beliefs.

[1] There is literature that studies the implications of heterogeneous beliefs in a financial market (Harrison and Kreps, 1978; Varian, 1985; Detemple and Murthy, 1994). The role of ambiguity has also been investigated in this literature (Condie, 2008).

Following the seminal work of Samuelson (1954), a myriad of papers have studied the issue of strategic incentives associated with public goods and public bads. While the implication of uncertainty to public good provision is not generally straightforward, the existence of uncertainty is known to affect the free-riding incentive of players under certain circumstances (Gradstein et al., 1992, 1993). Sandler et al. (1987), for instance, showed that the players' voluntary contribution to public good would increase in the presence of uncertainty if the utility function has a certain property regarding its third derivative. In relation to ambiguity, Eichberger and Kelsey (2002) examined the effect of ambiguity in symmetric games with externalities and found that ambiguity will either increase or decrease the equilibrium strategy, depending on the nature of the strategic interaction. More recently, Bramoullé and Treich (2009) examined the effect of uncertainty on pollution emissions and welfare in a strategic context. They found that emissions are always lower under uncertainty, which is a demonstration of risk-reducing considerations. In this strand of literature, however, the possibility of heterogeneous beliefs has not been taken into account.

Our analysis also complements the growing literature on the value of public information. Based on the model of a beauty contest, Morris and Shin (2002) showed that disseminating public information can decrease social welfare when players receive a private signal in addition to publicly observable information. Since this pioneering work, the welfare implications of public information have been vigorously examined by Angeletos and Pavan (2004), Cornand and Heinemann (2008), and James and Lawler (2011), among others. Since our model does not involve private information, the analysis of the present paper is not directly comparable to these other studies. Unlike the existing studies, however, we clarify how heterogeneous priors are translated into equilibrium behavior and identify in what condition the value of public information becomes negative under ambiguity. In this regard, our paper is related to the recent contribution of Koufopoulos and Kozhan (2014), who presents an example where an increase in ambiguity leads to a strict Pareto improvement in insurance markets.

The structure of the paper is as follows. Section 2 is devoted to the description of the model. Based on a fairly general framework, Section 3 examines characteristics of the equilibrium. Section 4 demonstrates how the framework presented in this paper can be used to investigate problems of interest, such as the value of information. To this end, we focus on a particular class of models where risk and beliefs are both represented by normal distributions. This class of models, together with the exponential specification of utility function, allows us to solve the equilibrium in a closed form.

We then show that the arrival of new information can worsen the free-riding problem, both in terms of the amount of public bads and the level of individual welfare. This happens even if the newly available information more accurately reflects the correct risk of public bads than the players' beliefs. We also consider the situation where an authoritative scientific community can add some information noise before the news (i.e., scientific findings) becomes available to players. It is shown that adding information noise will never mitigate the public-bad nature of the problem if the heterogeneity only exists in the mean of priors. When the beliefs are highly heterogeneous, however, a certain amount of information noise can be Pareto-improving. Section 5 concludes.

2 Model

This section explains the structure of the model and introduces its basic assumptions. To establish the context, we interpret the model as representing a global environmental problem such as climate change.

2.1 Basic game

Our stylized economy consists of $n \geq 2$ identical players. They interact with each other only through a negative production externality. Let $y_i \in \mathbb{R}_+$ be the amount of output produced by player i and $e(y_i) \in \mathbb{R}_+$ be the level of pollution associated with output y_i. For the sake of simplicity, we abstract the production process and assume that the output $\bar{y} > 0$ is exogenously given and identical across players. Accordingly, we take the baseline level $\bar{e} := e(\bar{y})$ of pollution as given.

The amount of pollution is reduced by abatement $a_i \in \mathbb{R}_+$, which is chosen independently by each player. The abatement effort requires a cost $C(a_i)$ at a local level. The cost function satisfies $C' := \partial C/\partial a_i > 0$, $C'' := \partial^2 C/\partial a_i^2 > 0$, and $C(0) = 0$. The net emission E at the aggregate level is then given by $E = \sum_{i=1}^{n}(\bar{e} - a_i) = n\bar{e} - A$, where $A := \sum_{i=1}^{n} a_i$. The aggregate net emission determines the damage $D(E;\beta)$ from pollution, for which we assume $D' := \partial D/\partial E > 0$ and $D'' := \partial^2 D/\partial E^2 \geq 0$. Notice that D is influenced by the parameter β. This parameter is meant to be a proxy of climate sensitivity. The damage $D(E;\beta)$ and marginal damage $D'(E;\beta)$ of pollution are both increasing in β. The damage and the abatement costs are subtracted from output $\bar{y}$, the remainder of which is consumed by the players. Consumption x_i of player i is therefore determined by

$$x_i = \bar{y} - D(E;\beta) - C(a_i). \tag{2.1}$$

We assume $\bar{y}$ is sufficiently large so that $x_i > 0$ and $E > 0$ are satisfied at equilibrium. To ensure an interior solution, it is also assumed that $D'(n\bar{e};\beta) > C'(0)$ and $n^{-1}C'(\bar{e}) > D'(0;\beta)$. When there is no uncertainty, the utility of player i is then determined by $u(x_i)$ for some strictly increasing and strictly concave function $u : \mathbb{R}_+ \to \mathbb{R}$.

2.2 Uncertainty and decision making

The true value of β is unknown. Let $B \subset \mathbb{R}$ be the set of all possible values of β and $\Delta(B)$ be the set of all probability density functions defined over B. If the density function of β is known to be $f \in \Delta(B)$, then the expected utility of player i is given by

$$\mathbb{E}[u(x_i)] = \int_B u(\bar{y} - D(E;\beta) - C(a_i))f(\beta)d\beta. \tag{2.2}$$

As mentioned in the introduction, however, the value of β is uncertain, not only in the sense that the parameter has a probability distribution, but also because the distribution itself is not known. To be more specific, we restrict ourselves to a particular case of ambiguity where the value of β has been estimated by several scientific studies and a variety of possible distributions of β have been proposed. Let $\Theta \subset \mathbb{R}$ be the set of all such studies. We denote by $f(\cdot|\theta) \in \Delta(B)$ the probability density function proposed by a particular scientific study $\theta \in \Theta$.

To players, there is no a priori information available. Then, players subjectively form beliefs about the relative credibility of each of the possible distributions. Denote by $g_i \in \Delta(\Theta)$ the subjective prior of player i defined over the set Θ of all proposed distributions. Notice that here we divert from the standard assumption of common prior and allow for the possibility of priors being heterogeneous. Moreover, we assume that the profile $\{g_i\}_{i=1}^n \in \times_{i=1}^n \Delta(\Theta)$ of subjective priors is common knowledge. In other words, players are assumed to agree to disagree on the reliability of each scientific study. The heterogeneity does not come from asymmetric information, but rather from intrinsic differences in how to view the world. Otherwise, the priors would be necessarily identical, due to the combination of the common knowledge assumption and the rationality of players.

In the absence of additional information, players choose their abatement level based on their own beliefs, given the knowledge of the set $\{f(\cdot|\theta)\}_{\theta\in\Theta}$ of distributions and the profile $\{g_i\}_{i=1}^n$ of subjective priors. To formalize this process, we follow Klibanoff et al. (2005) and assume that players' decision utility V_i under ambiguity is given by

$$V_i := \int_{\Theta} \phi\left(\mathbb{E}\left[u_i|\theta\right]\right) g_i(\theta) d\theta \quad \text{with} \quad \mathbb{E}\left[u_i|\theta\right] := \int_B u(x_i) f(\beta|\theta) d\beta, \tag{2.3}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing and concave function. With this representation, players' attitudes towards risk and ambiguity can be separately incorporated. Just as in the case of the standard expected utility model, the strength of risk aversion is measured by the concavity of function u. Similarly, the strength of ambiguity aversion is measured by the concavity of function ϕ.

2.3 Information structure

The true value of parameter β is inherently unknown and will continue to be so in the foreseeable future, such as in the case of climate sensitivity. We assume, on the other hand, that there is a 'correct' risk assessment of β. In other words, there is the unique scientific study $\theta_* \in \Theta$, such that the corresponding risk estimate $f(\cdot|\theta_*)$ correctly captures the inherent risk of β. Although it is unknown which scientific study provides the correct risk estimate, new information about index θ_* becomes available through occasional scientific discoveries. This new information is modeled as a signal $\mu_* \in \Theta$, the value of which is realized according to

$$\mu_* = \theta_* + \eta, \quad \text{where} \quad \eta \sim N(0, \sigma_*^2). \tag{2.4}$$

The variance $\sigma_*^2 \geq 0$ represents the uncertainty remaining in the state-of-the-art scientific knowledge in pinning down the index θ_*.

Suppose, for the moment, that the signal-generating process (2.4) is entirely known to players. Once the signal μ_* is observed, players can update their belief based on the Bayes' rule. The posterior $g_i(\cdot|\mu_*) \in \Delta(\Theta)$ is then given by $g_i(\theta|\mu_*) \propto L(\mu_*|\theta)g_i(\theta)$, where L is the likelihood function of normal distribution with mean μ_* and variance σ_*^2. Notice that the posterior $g_i(\cdot|\mu_*)$ is irrational in the sense that it is influenced by the purely subjective priors, even after the objectively reliable information becomes available. This reflects the behavioral evidence that players have systematically biased beliefs (DellaVigna, 2009).

2.4 Equilibrium and welfare

Since both the priors and posteriors are common knowledge, the model is essentially a game with complete information. Thus, the standard Nash equilibrium is sufficient as the solution concept. To be more precise, we define the equilibrium by the action profile $a := (a_i)_{i=1}^n$, such that

$$a_i \in \operatorname*{argmax}_{a_i} V_i(a_i, a_{-i}) \quad \text{given} \quad a_{-i} := (a_j)_{j \neq i} \tag{2.5}$$

for all i.

The objective function V_i is defined as in (2.3), whose dependence on the action profile is now made explicit. The prior g_i is replaced by the posterior $g_i(\cdot|\mu_*)$ when the signal μ_* is received by players. To distinguish the equilibria before and after the information becomes available, we denote by $\tilde{a} := (\tilde{a}_i)_{i=1}^n$ the equilibrium action profile corresponding to signal μ_*.

Since the correct risk of β is represented by $f(\cdot|\theta_*)$, the players' welfare (as opposed to decision utility) is given by

$$W_i^c(a) := \phi(\mathbb{E}[u_i|\theta_*]) \quad \text{with} \quad \mathbb{E}[u_i|\theta_*] := \int_B u(x_i) f(\beta|\theta_*) d\beta. \tag{2.6}$$

The index θ_*, however, is not known. The only reliable information about θ_* is the realized value of μ_*. Thus, we evaluate the players' welfare based on the objectively-determined expected value of W_i, namely,

$$W_i(a) := \mathbb{E}[W_i^c(a)|\mu_*] = \int_\Theta \phi(\mathbb{E}[u_i|\theta])\, g_*(\theta) d\theta, \tag{2.7}$$

where $g_* \in \Delta(\Theta)$ is the density of θ_*, conditional on μ_*. Notice that g_* is the density function of a normal distribution whose mean and variance are given by μ_* and σ_*^2, respectively. We call g_* the *rational belief* in the sense that it purely represents the objective information about the value of θ_*. Also worth noting is that the welfare function is identical across the different players. Since the cost function is strictly convex, efficiency therefore requires that the abatement level be the same for all players. Concavity of u and ϕ implies that there exists a unique level of efficient aggregate abatement, which we denote by A_*. The existence and uniqueness of such A_* is discussed in Appendix B.1. The efficient level of individual abatement is given by $a_* = A_*/n$.

3 General characteristics of equilibrium

Let us first focus on the case where the new information is not yet available to players. At equilibrium, the first-order condition implies

$$C'(a_i) = \int_B D'(E;\beta) f_i(\beta) d\beta. \tag{3.1}$$

Here, f_i is the density function defined by

$$f_i(\beta) := \int_\Theta \hat{f}_i(\beta|\theta)\hat{g}_i(\theta)d\theta, \tag{3.2}$$

where

$$\hat{f}_i(\beta|\theta) \propto u'(x_i)f(\beta|\theta), \quad \hat{g}_i(\theta) \propto \phi'(\mathbb{E}[u(x_i)|\theta])\mathbb{E}[u'(x_i)|\theta]g_i(\theta). \tag{3.3}$$

Notice first that the left-hand side of (3.1) is the marginal abatement cost. The right-hand side is a weighted average of the marginal abatement benefit. If the players' preference is neutral, both in terms of risk and ambiguity, then the density f_i in (3.2) coincides with the pure subjective risk $f_i^c := \int_\Theta f(\beta|\theta)g_i(\theta)d\theta$. In this case, (3.1) simply means that players choose their abatement effort so that the marginal abatement cost and the purely subjective expected marginal benefit are equalized.

However, when players are not risk or ambiguity neutral, the expected marginal benefit on the right-hand side of (3.1) is 'distorted'. It is distorted in the sense that the expectation is not taken based on the pure subjective risk f_i^c, but instead based on some other density f_i. The density f_i reflects the players' subjective risk assessment, just like the pure subjective risk f_i^c. However, it is adjusted according to their preference regarding risk and ambiguity, as is seen in (3.3). This suggests that in order to characterize the equilibrium, we should clarify how beliefs and preferences are translated into the adjusted subjective risk f_i.

To further characterize the equilibrium, we impose a certain structure to the set of scientific risk estimates.

Assumption 1. The family $\{f(\cdot|\theta)\}_{\theta\in\Theta}$ of probability density functions has a strict monotone-likelihood-ratio property. Namely,

$$f(\beta'|\theta')f(\beta|\theta) - f(\beta'|\theta)f(\beta|\theta') > 0 \quad \forall\beta' > \beta, \forall\theta' > \theta. \tag{3.4}$$

To interpret this assumption, notice that under Assumption 1, $\theta' > \theta$ implies that $f(\cdot|\theta')$ strictly dominates $f(\cdot|\theta)$ in the sense of first-degree stochastic dominance. In particular, since $D(E;\beta)$ is strictly increasing in β, $\int_B D(E;\beta)f(\beta|\theta')d\beta > \int_B D(E;\beta)f(\beta|\theta)d\beta$ for any E. In other words, scientific study θ' is unambiguously more pessimistic than θ in terms of the expected damage from pollution. Thus, what is required by Assumption 1 is that the set of available scientific risk estimates can be ranked from the most optimistic to the most pessimistic one.

With this interpretation in mind, we can then characterize a belief as being more optimistic when it puts relatively heavier weight on the scientific studies with smaller index numbers. Our first proposition shows that optimistic subjective beliefs are, quite intuitively, translated into a weaker willingness to abate pollution.

Proposition 1. *If*

$$g_i(\theta)g_j(\theta') - g_i(\theta')g_j(\theta) > 0 \tag{3.5}$$

for all $\theta' > \theta$, then f_j strictly first-degree stochastically dominates f_i and therefore, player i abates less than player $j \neq i$ at equilibrium.

Proof. See Appendix A.2. □

If condition (3.5) is satisfied, then the expected damage and the expected marginal damage of pollution for a given level of abatement effort are smaller for player i than for player j. In other words, player i is unambiguously more optimistic than player j. We should mention that this is a sufficient condition, but not a necessary condition, for one player to be less willing to abate pollution than the other. In fact, once the functional forms are specified, the relationship between the equilibrium abatement effort and beliefs can be characterized based on a much less restrictive condition.

Since there exists the production externality in the economy, the equilibrium abatement effort is likely to be insufficient relative to the efficient level A_*. This might not be the case, of course, when some or all of the players have highly pessimistic priors. Such an unrealistic case, however, is not of interest in this study. To exclude such cases, we restrict our analysis to the set of 'realistic' beliefs. To be formal, denote the collection of all belief profiles by $\mathcal{G}(g_*) \subset \times_{i=1}^{n} \Delta(\Theta)$, such that the corresponding equilibrium outcome is insufficient.

Proposition 2. *The collection $\mathcal{G}(g_*)$ is nonempty for any $g_* \in \Delta(\Theta)$. In particular, if $g_i = g_*$ for all i, then the equilibrium abatement corresponding* *to this belief profile is insufficient in the sense that $A < A_*$.*

Proof. See Appendix A.3. $\square$

Proposition 2 shows that even if every player has the rational belief, the equilibrium outcome is still insufficient (and thus inefficient). This is due to the existence of externality. As a result, $\mathcal{G}(g_*)$ is always nonempty. Hence, it makes sense to restrict our attention to only the belief profiles in $\mathcal{G}(g_*)$.

Proposition 2 indicates that inefficiency arises at equilibrium as long as the profile of beliefs is contained in a neighborhood of g_*. In particular, when the risk of pollution-induced damage is underestimated relative to the rational belief, the outcome is even less efficient than in the case of the rational belief being shared by every player. Consider, for example, a hypothetical scenario where all players have an identical belief represented by some $g \in \Delta(\Theta)$. Combining Propositions 1 and 2 yields the following result.

Proposition 3. *If*

$$g(\theta)g_*(\theta') - g(\theta')g_*(\theta) > 0 \tag{3.6}$$

for all $\theta' > \theta$, then the equilibrium outcome is Pareto-dominated by the case where every player has the correct belief as their prior.

Proof. See Appendix A.4. $\square$

In light of Proposition 1, condition (3.6) means that players underestimate the risk in the sense that their homogeneous belief g puts heavier weight on relatively optimistic risk estimates than the rational belief g_* does. In such a case, the equilibrium abatement effort will be far from sufficient and the players will end up with a lower-than-possible level of welfare.

When beliefs are heterogeneous, on the other hand, inefficiency of a different kind arises, in addition to the existence of externality and the underestimation of risk. As Proposition 1 indicates, heterogeneity in beliefs is likely to be translated into heterogeneity of behaviors at equilibrium. Such uncoordinated behaviors, combined with the convexity of cost function, lead to inefficient abatement efforts at the aggregate level. To see this, let $(a_i)_{i=1}^n$ be the equilibrium abatement, such that $a_j \neq a_i$ for some $j \neq i$. Then the Jensen's inequality shows $n^{-1} \sum_{i=1}^{n} C(a_i) - C(A/n) =: \Delta C > 0$ and for each i

$$x_i = \bar{y} - D(E;\beta) - C(a_i) < \bar{y} - D(E;\beta) - C(A/n) + \Delta x_i \tag{3.7}$$

for any realization of β, where $\Delta x_i := C(A/n) - C(a_i) + \Delta C$. Notice that $(\Delta x_i)_{i=1}^{n}$ is a feasible reallocation scheme because $\sum_{i=1}^{n} \Delta x_i = 0$. Hence, by choosing the average abatement level A/n instead of a_i and reallocating consumption according to $(\Delta x_i)_{i=1}^{n}$, all players will be better off.

Assuming that the equilibrium abatement is insufficient, the question of importance is whether or not new scientific discoveries can facilitate the players' abatement efforts. The observations above suggest that the new information could play a positive role in reducing the existing inefficiency. In particular, when the risk is underestimated and/or there is a discrepancy in priors, then a public signal containing some information of the correct risk estimate would have a desirable consequence by encouraging the update of otherwise optimistic priors and expediting the convergence of heterogeneous beliefs.

As we will see in the next section, however, the story is not that simple. Even if players underestimate the risk of pollution-induced damage and there exists heterogeneity in their priors, there can still be a case where new information unambiguously worsens the situation. This is largely due to the fact that once new information becomes available, the situation becomes less ambiguous, which in turn weakens the incentive of risk/ambiguity-averse players to reduce pollution. Therefore, what plays a key role here is the preference for risk and ambiguity aversion.

To understand how the players' preference is translated into their equilibrium behavior, let us first focus on $\hat{f}_i$ in (3.3). Recall that $f(\cdot|\theta)$ is the objective probability density proposed by a particular scientific study θ. The expression (3.3) indicates that this objective risk estimate is not directly used in evaluating the expected marginal benefit. Before being applied to the final evaluation of expected damage, it is 'reinterpreted' by players as $\hat{f}_i(\cdot|\theta)$ based on their risk preference. The following lemma clarifies how $\hat{f}_i$ and f are related to each other.

Lemma 1. *For each $\theta \in \Theta$, $\hat{f}_i(\cdot|\theta)$ strictly dominates $f(\cdot|\theta)$ in the sense of first-degree stochastic dominance if and only if u is strictly concave.*

Proof. See Appendix A.5. □

Lemma 1 states that in choosing their abatement efforts, risk averters reinterpret the scientific risk estimates in a pessimistic way. The reinterpretation is pessimistic in the sense that $\int D(E;\beta)\hat{f}_i(\beta|\theta)d\beta > \int D(E;\beta)f(\beta|\theta)d\beta$, namely, the expected damage is conceived as larger than it was originally meant to be for each θ. The converse is true for players with a risk-loving preference.

Similarly, the expression $\hat{g}_i$ in (3.3) indicates that when players aggregate the set of reinterpreted risks $\{\hat{f}_i(\cdot|\theta)\}_{\theta\in\Theta}$, they do not directly use their own belief g_i, but instead use their preference-adjusted belief $\hat{g}_i$. In other words, they 'update' their belief g_i into $\hat{g}_i$ in accordance with their risk and ambiguity attitude. How this update is done is clarified by the following lemma.

Lemma 2. *If u and ϕ are concave and at least one of the concavities is strict, then $\hat{g}_i$ strictly dominates g_i in the sense of first-degree stochastic dominance.*

Proof. See Appendix A.6. □

If $\hat{g}_i$ first-degree stochastically dominates g_i, it roughly means that the former gives larger weight to relatively more pessimistic risk estimates than the latter does. Hence, what is indicated by Lemma 2 is that players behave as if they were more pessimistic than they actually are when their preference is risk or ambiguity-averse.

Combining these lemmas yields the following proposition.

Proposition 4. *Suppose u is strictly concave and ϕ is concave. Then, the preference-adjusted subjective risk f_i strictly first-degree-stochastically dominates the pure subjective risk f_i^c. As a result, the aggregate abatement at equilibrium is greater than in the case of risk- and ambiguity-neutral preference.*

Proof. See Appendix A.7. □

Moreover, as the next proposition shows, the players' preference for stronger ambiguity aversion is translated into a greater abatement incentive.

Proposition 5. *The more ambiguity averse that players are, the larger the aggregate abatement is at equilibrium.*

Proof. See Appendix A.8. □

What is suggested by Propositions 4 and 5 is that risk- and ambiguity-averse players have an extra incentive to engage in pollution abatement as long as the situation is ambiguous. Then, reducing the existing ambiguity in any way weakens the players' abatement incentive. If the publication of new scientific information significantly reduces the existing ambiguity, then the weakening of abatement incentive that follows will at least partially offset the positive effects of the scientific discovery. When the degree of ambiguity aversion is sufficiently large, this side effect of public information might even outweigh all of its positive impacts combined. As a consequence, the society could end up with lower welfare than in the absence of the new scientific information.

When and in what condition does such a paradoxical consequence follow from newly available information? Clarifying these conditions would have profound policy implications and it is this task to which we turn in the next section.

4 Value of information

In this section, we focus on a class of models where risk and beliefs are both represented by normal distributions. This class of models, together with exponential specification of utility functions, allows us to solve the equilibrium in a closed form.

4.1 Specifications

We henceforth specify the functional forms of u and ϕ as

$$u(x) := -\frac{1}{\alpha}e^{-\alpha x} \quad \text{and} \quad \phi(u) := -\frac{1}{1+\xi}(-u)^{1+\xi} \tag{4.1}$$

for some $\alpha > 0$ and $\xi > -1$. Notice that α is the index of constant absolute risk aversion and ξ corresponds to the index of constant relative ambiguity aversion. Also, for analytical tractability, we assume that the damage and cost functions are of the forms

$$D(E;\beta) := \beta\delta E \quad \text{and} \quad C(a_i) := \frac{\nu}{2}a_i^2 \tag{4.2}$$

for some constants $\delta, \nu > 0$.

Furthermore, we focus our attention to the case where the proposed risks and the players' priors are both well represented by normal distributions. To be more precise, $f(\cdot|\theta)$ is the density of normal distribution $N(\theta, \sigma_u^2)$ for some $\sigma_u^2 > 0$. Note that this satisfies Assumption 1. With this specification, θ can be regarded as the point estimate of β provided by scientific study θ. The variance σ_u^2 reflects an inevitable inaccuracy associated with the estimation procedure commonly used in the scientific literature. Prior g_i is represented by the density of normal distribution $N(\mu_i, \sigma_i^2)$ for some $\mu_i > 0$ and $\sigma_i^2 > 0$. The mean $\mu_i \in \Theta$ can be interpreted as the index of the most reliable scientific study for player i. The variance σ_i^2 captures the lack of confidence in player i's prior. A profile of priors is represented by $\Gamma := \{\mu_i, \sigma_i^2\}_{i=1}^n$.

A bit of tedious computation then yields

$$V_i(a) = -\frac{\alpha^{-(1+\xi)}}{1+\xi}e^{-\alpha(1+\xi)v_i(a)}, \quad v_i(a) := \bar{y} - \delta\mu_i E - \frac{\delta^2}{2}\gamma_i E^2 - \frac{\nu}{2}a_i^2, \tag{4.3}$$

where $\gamma_i := \alpha\left[\sigma_u^2 + (1+\xi)\sigma_i^2\right]$. The derivation is given in Appendix B.2. Note that γ_i summarizes player i's attitude towards uncertainty (σ_u^2) and ambiguity (σ_i^2), the latter of which is magnified by the index of ambiguity aversion, ξ.

The first-order condition then boils down to

$$a_i = \rho\mu_i + E\rho\delta\gamma_i, \tag{4.4}$$

where $\rho := \delta/\nu > 0$. This implies that

$$A = \frac{n^{-1}}{n^{-1}+\rho\delta\bar{\gamma}}n\rho\bar{\mu} + \frac{\rho\delta\bar{\gamma}}{n^{-1}+\rho\delta\bar{\gamma}}n\bar{e}, \quad E = \frac{\bar{e}-\rho\bar{\mu}}{n^{-1}+\rho\delta\bar{\gamma}}, \tag{4.5}$$

where $\bar{\mu} := n^{-1}\sum_i \mu_i$ and $\bar{\gamma} := n^{-1}\sum_i \gamma_i$. The equilibrium level of abatement is therefore

$$a_i = \rho\mu_i + \frac{\bar{e} - \rho\bar{\mu}}{n^{-1} + \rho\delta\bar{\gamma}}\rho\delta\gamma_i. \tag{4.6}$$

The right-hand side of (4.6) is an increasing function of μ_i. Hence, just as expected from Proposition 1, the more pessimistic players are, the more stringent their abatement efforts would be. Also, as predicted by Propositions 4 and 5, the right-hand side of (4.6) is an increasing function of α, ξ, and σ_i^2. Players become more willing to reduce pollution in the presence of risk and ambiguity. A relatively less confident player consequently bears a relatively large share of the global effort to reduce pollution.

A similar computation yields

$$W_i(a) = -\frac{\alpha^{-(1+\xi)}}{1+\xi}e^{-\alpha(1+\xi)w_i(a)}, \quad w_i(a) := \bar{y} - \delta\mu_* E - \frac{\delta^2}{2}\gamma_* E^2 - \frac{\nu}{2}a_i^2, \tag{4.7}$$

where $\gamma_* := \alpha\left[\sigma_u^2 + (1+\xi)\sigma_*^2\right]$. Hence, the efficient level A_* of aggregate abatement is uniquely determined by

$$A_* = \frac{n^{-2}}{n^{-2} + \rho\delta\gamma_*}n^2\rho\mu_* + \frac{\rho\delta\gamma_*}{n^{-2} + \rho\delta\gamma_*}n\bar{e} \tag{4.8}$$

and the corresponding individual abatement effort is $a_* = A_*/n$. A brief inspection reveals that $A < A_*$, even if $\mu_i = \mu_*$ and $\sigma_i^2 = \sigma_*^2$ for all i. However, without any restriction on the set of possible priors, every outcome, including the efficient one, can be supported as an equilibrium. In what follows, we restrict our analysis to a set of reasonable priors. In particular, let us assume that the risk of climate change is underestimated, in the sense that $\mu_i < \mu_*$ and $\sigma_i^2 < n\sigma_*^2$ for all i. This ensures that the equilibrium abatement level satisfies $A < A_*$. In this case, publishing new scientific findings apparently makes sense.

4.2 Impact of new information

The question of particular interest is whether the new information μ_* mitigates or amplifies the free-riding problem. More importantly, what is the welfare implication of the new information? The observation in the preceding section suggests that there might be a case where new information actually harms the society. We say that the value of information is negative if every player is worse off after players obtain the information. Similarly, we say that the value of information is positive if every player is better off under the updated beliefs. Below, we clarify a condition in which the value of information is unambiguously negative.

Before presenting the result, we note that once the signal μ_* is observed, the players' posterior $g_i(\cdot|\mu_*)$ is given by a normal distribution whose mean $\tilde{\mu}_i$ and variance $\tilde{\sigma}_i^2$ are given by

$$\tilde{\mu}_i = \frac{\sigma_*^2}{\sigma_*^2 + \sigma_i^2}\mu_i + \frac{\sigma_i^2}{\sigma_*^2 + \sigma_i^2}\mu_* \quad \text{and} \quad \tilde{\sigma}_i^2 = \frac{\sigma_*^2}{\sigma_*^2 + \sigma_i^2}\sigma_i^2, \tag{4.9}$$

respectively. These expressions already indicate that the new information has three distinct effects. First, the mean of the posterior gets closer to the mean of the rational belief, in the sense that $|\tilde{\mu}_i - \mu_*| < |\mu_i - \mu_*|$. This *rationalization effect* helps improve efficiency because the risk is underestimated in the priors. The second and closely related effect is the *convergence effect*. Since the beliefs are updated based on the common public information, the posteriors are less heterogeneous than the priors. For instance, in the extreme case where the precision $1/\sigma_*^2$ of the new information is infinite, the players' posteriors completely coincide with one another. This achieves a welfare gain by eliminating the inefficiency associated with uncoordinated actions. The last effect, which we call the *confidence effect*, works in the opposite direction. Having obtained the additional information, players become more confident about their beliefs. In fact, (4.9) shows that $\tilde{\sigma}_i^2 \leq \min\{\sigma_i^2, \sigma_*^2\}$ for all i, which results in the players' weaker willingness to abate pollution.

The value of information is determined by these three effects, which in turn depend on the priors and preference of the players. Let $\tilde{A} := \sum_{i=1}^n \tilde{a}_i$ be the equilibrium aggregate abatement after the information μ_* becomes available and $\tilde{W}_i$ be the corresponding welfare of player i. The following proposition gives a sufficient condition in which the confidence effect outweighs the combined effects of rationalization and convergence.

Proposition 6. *For each (α, ξ), there exist $\Delta\mu > 0$ and $\Delta\sigma^2 > 0$ such that*

(i) if $\sum_i |\mu_* - \mu_i| < \Delta\mu$, then $\tilde{A} < A$, and

(ii) if furthermore $\sum_i |\sigma_*^2 - \sigma_i^2| < \Delta\sigma^2$, then $\tilde{W}_i < W_i$ for all i.

Moreover, $\Delta\mu$ is increasing in α and ξ.

Proof. See Appendix A.9. □

Proposition 6 first shows that if the underestimation of the risk in the priors is not very significant, then the total abatement will decline as a result of new information. Even if the underestimation is significant, the abatement will still decline when players are highly risk- and/or ambiguity-averse. Moreover, if the heterogeneity in the priors is not very significant, in the sense that σ_i^2 is close to σ_*^2 for all i, then the equilibrium outcome under new information is strictly Pareto-dominated by the original outcome. This can be the case even if the risk is underestimated in the priors.

These results have a profound implication on information policy under ambiguity and heterogeneous beliefs. Suppose, for instance, that there is an authoritative community of scientists whose role is to make the recent scientific findings accessible to the general public. A real-world example of such a community is IPCC in the context of climate change. The science behind climate change is so complex that it is not easy to convey the precise message of the recent findings to those who are not familiar with the scientific literature. This implies that if new scientific findings are to be well understood by the general public, they need to be summarized and endorsed by a credible scientific authority. This is why IPCC publishes assessment reports concerning the risk of climate change and updates the information on a regular basis. Interpreted in line with our model, the information contained in the assessment report is signal μ_*.

One important implication of our results is that regularly publishing assessment reports with minor updates might do more harm than good. Once an assessment report is published, the updated mean of the players' beliefs become closer to that of rational belief, which is likely to increase the willingness to reduce pollution when the risk is initially underestimated. Also, since heterogeneity in beliefs always causes inefficiency, facilitating belief convergence by publishing the information appears to be a good idea. The first assessment report will most likely work as desired because, in many cases, the risk is significantly underestimated or the risk is even unknown by the general public when the public-bad problem first emerges. The second assessment report might also work if there remains a wide gap between the correct risk and people's beliefs. At some point, however, as the gap becomes narrower, publishing new assessment reports will eventually end up with a weaker incentive to abate pollution. This is especially the case when players are highly ambiguity-averse. Moreover, as the beliefs become less heterogeneous, the resulting outcome can in fact be Pareto-dominated by the status quo. Therefore, instead of routinely summarizing the recent developments in scientific literature, the assessment reports should be published only when significantly novel findings are available, relative to the already well-publicized knowledge.

4.3 Pareto-improving ambiguity

To further investigate the consequence of new information, let us modify the information structure and now suppose that after signal μ_* materializes, some information noise can be credibly added to the signal before it becomes available to players. In other words, while the signal-generating process (2.4) itself is known to the players, the variance σ_*^2 is unknown. The authoritative scientific community can then at least partially manipulate the variance of the signal; this could be done by choosing unclear phrasing or ambiguous wording in their assessment report. Accordingly, instead of μ_*, players receive a noisy signal μ_*^ε, such that

$$\mu_*^\varepsilon = \mu_* + \varepsilon, \quad \varepsilon \sim N(0, \sigma_\varepsilon^2). \tag{4.10}$$

The variance $\sigma_\varepsilon^2 \geq 0$ captures the strength of information noise.

Given the possibility of information being manipulated, the most satisfactory model needs to incorporate the strategic interaction between players and the scientific community. However, for the sake of simplicity, we assume that players are naive in the sense that they do not consider the possibility of noise being added to the signal. Simple algebra then tells us that the posterior $g_i(\cdot|\mu_*^\varepsilon)$ is represented by $N(\tilde{\mu}_i, \tilde{\sigma}_i^2)$, where

$$\tilde{\mu}_i = \frac{\sigma_i^2}{\sigma_*^2 + \sigma_\varepsilon^2 + \sigma_i^2}\mu_* + \frac{\sigma_*^2 + \sigma_\varepsilon^2}{\sigma_*^2 + \sigma_\varepsilon^2 + \sigma_i^2}\mu_i, \quad \tilde{\sigma}_i^2 = \frac{\sigma_*^2 + \sigma_\varepsilon^2}{\sigma_*^2 + \sigma_\varepsilon^2 + \sigma_i^2}\sigma_i^2. \tag{4.11}$$

The information noise affects players' behavior in two ways. On one hand, the accuracy of the newly available information might be underestimated by the players. Therefore, information noise potentially allows optimistic players to remain more optimistic than they should be. On the other hand, it provides ambiguity averters with an additional incentive to abate pollution by making the situation more ambiguous.

Notice that the analysis of the preceding section can be nested as a special case of this information structure. When $\sigma_\varepsilon^2 = 0$, the information structure boils down to the one in the preceding section. At the opposite extreme is the infinite amount of information noise, $\sigma_\varepsilon^2 = \infty$, which corresponds to the case where the information is not published in the first place. We are interested in whether adding a positive and finite amount of information noise can be Pareto-improving. To be more precise, we say that *Pareto-improving ambiguity* is possible if there exists $\sigma_\varepsilon^2 \in (0, \infty)$, such that

$$\tilde{W}_i > \tilde{W}_i\big|_{\sigma_\varepsilon^2 = 0} > \tilde{W}_i\big|_{\sigma_\varepsilon^2 = \infty} \tag{4.12}$$

for all i. The second inequality requires that the value of information be positive. In this case, publishing new information without any noise subsequently makes players better off. When Pareto-improving ambiguity is possible, it is even better to add a certain amount of information noise upon the publication of the information.

4.3.1 Heterogeneity only in μ_i

To see the impact of information noise on the equilibrium, observe that

$$\frac{\partial \tilde{a}_i}{\partial \sigma_\varepsilon^2} = \rho \frac{\partial \tilde{\mu}_i}{\partial \sigma_\varepsilon^2} + (n\bar{e} - \tilde{A})\delta\rho \frac{\partial \tilde{\gamma}_i}{\partial \sigma_\varepsilon^2} - \delta\rho\tilde{\gamma}_i \frac{\partial \tilde{A}}{\partial \sigma_\varepsilon^2}. \tag{4.13}$$

The first and second terms on the right-hand side of (4.13) represent the direct impacts of information noise. Since $\partial\tilde{\mu}_i/\partial\sigma_\varepsilon^2$ is negative, the first term represents the fact that noisy signals weaken the rationalization effect of the new information. On the other hand, $\partial\tilde{\gamma}_i/\partial\sigma_\varepsilon^2$ in the second term is positive, reflecting the fact that the confidence effect is also weakened. What is captured by the third term is the free-riding or the strategic substitution effect. The larger the value of $\delta\rho\tilde{\gamma}_i > 0$, the stronger the substitution effect among the players' abatement will be. The impact on the total abatement is

$$\frac{\partial \tilde{A}}{\partial \sigma_\varepsilon^2} = \left(1 + \delta\rho \sum_i \tilde{\gamma}_i\right)^{-1} \left\{\rho \sum_i \frac{\partial \tilde{\mu}_i}{\partial \sigma_\varepsilon^2} + (n\bar{e} - \tilde{A})\delta\rho \sum_i \frac{\partial \tilde{\gamma}_i}{\partial \sigma_\varepsilon^2}\right\}. \tag{4.14}$$

Notice that this impact would be smaller when the strategic substitution effect, $\delta\rho\sum_i \tilde{\gamma}_i$, is larger. The welfare implication of information noise can be seen in

$$\frac{\partial \tilde{w}_i}{\partial \sigma_\varepsilon^2} = \delta\left\{\mu_* + \tilde{E}\delta\gamma_*\right\} \frac{\partial \tilde{A}}{\partial \sigma_\varepsilon^2} - \delta\left\{\tilde{\mu}_i + \tilde{E}\delta\tilde{\gamma}_i\right\} \frac{\partial \tilde{a}_i}{\partial \sigma_\varepsilon^2}. \tag{4.15}$$

This expression suggests that if adding information noise is to be Pareto-improving, it must increase $\tilde{A}$ to a sufficiently large extent relative to the corresponding changes of $\tilde{a}_i$. This would be difficult when the strategic substitution effect is significantly large.

To examine the possibility of Pareto-improving ambiguity, let us first consider the case where heterogeneity only exists in the means of priors. In this case, we have the following clear-cut result.

Proposition 7. *If there is no heterogeneity in $\{\sigma_i^2\}_{i=1}^n$, then Pareto-improving ambiguity is impossible.*

Proof. See Appendix A.10 □

Proposition 7 states that as long as players are equally confident about their priors, adding information noise to the public signal would never be a good idea. Since the equilibrium abatement is insufficient in the absence of new information, information noise is Pareto-improving only if $\partial\tilde{A}/\partial\sigma_\varepsilon^2 > 0$. On the other hand, the value of information is positive only if $\tilde{A} > A$. This suggests that Pareto-improving ambiguity is possible only when $\partial\tilde{A}/\partial\sigma_\varepsilon^2 > 0$ and $\tilde{A} > A$ are simultaneously satisfied for some $\sigma_\varepsilon^2 \geq 0$. However, when there is no heterogeneity in σ_i^2, Appendix A.10 shows that for any $\sigma_\varepsilon^2 \geq 0$, $\partial\tilde{A}/\partial\sigma_\varepsilon^2 > 0$ is equivalent to $\tilde{A} < A$. This means that the impact of information noise on the total abatement is monotonic. In other words, information noise can improve welfare only if the value of information is negative. However, if the value of information is negative, then the information should not be published in the first place. Conversely, whenever the value of new information is positive, the information should be publicized as clearly as possible if there is no heterogeneity in the players' confidence.

4.3.2 Heterogeneity both in μ_i and σ_i^2

When the priors are highly heterogeneous, in the sense that not only the means, but also the variances are different across the players, there does exist a case in which Pareto-improving ambiguity is possible. Before providing the main proposition, we present a couple of preliminary results.

Proposition 8. *If*

$$\frac{\mu_* - \bar{\mu}}{\bar{\sigma}^2} > \frac{\bar{e} - \rho\mu_*}{n^{-1} + \delta\rho\alpha\sigma_u^2}\delta\alpha(1+\xi) > \frac{1}{n}\sum_i \frac{\mu_* - \mu_i}{\sigma_i^2}, \tag{4.16}$$

there then exists $\bar{s} > 0$ such that for any $\sigma_*^2 < \bar{s}$,

$$\left.\frac{\partial\tilde{A}}{\partial\sigma_\varepsilon^2}\right|_{\sigma_\varepsilon^2=0} > 0 \text{ and } \tilde{A}\big|_{\sigma_\varepsilon^2=0} > A. \tag{4.17}$$

Proof. See Appendix A.11. □

Proposition 8 shows that unlike the case with homogeneous confi-

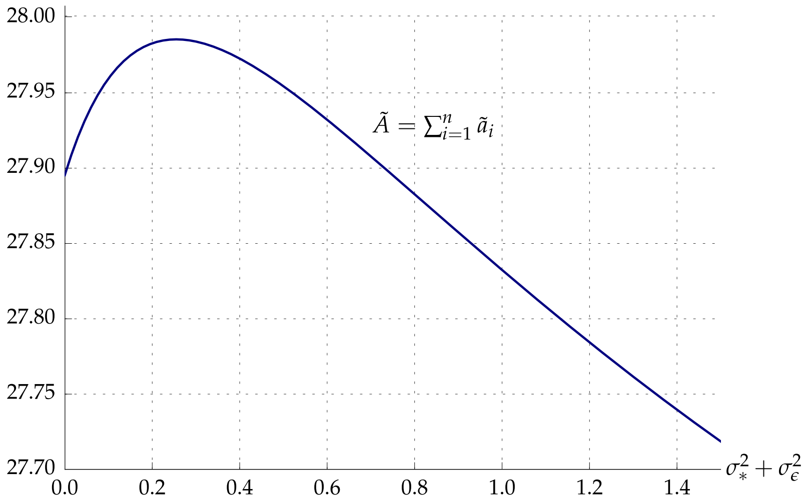

Figure 2: Non-monotonic relationship between equilibrium total abatement and information noise. ($n = 20$, $\bar{e} = 40$, $\delta = 0.0005$, $\nu = 0.0020$, $\sigma_u^2 = 1.0$, $\alpha = 1.5$, $\xi = 2.1$) Numerical specification of priors is provided in Figure 3.

dence, the impact of information noise on the total abatement can be non-monotonic as long as the true precision of new information is sufficiently high. A sufficient condition for such a non-monotonicity is given by (4.16). Notice that the inequalities (4.16) never hold when there is no heterogeneity in σ_i^2 because, in that case, the very left- and right-hand sides of the inequalities coincide. When condition (4.16) holds, (4.17) shows that the total abatement increases for a small amount of information noise and decreases for a large amount of noise. A numerical example of this non-monotonic relationship is provided in Figure 2.

Proposition 8 is only meaningful if there exists a reasonable set of parameter values that satisfy (4.16). The purpose of the following two propositions is to clarify a necessary and sufficient condition for the existence of such parameters. As it turns out, the parametric condition implied by (4.16) is not as restrictive as it might appear.

Proposition 9. *For a given profile Γ of priors, define $R_\Gamma \subset \mathbb{R}^2$ by*

$$R_\Gamma := \{(\alpha, \xi) \in (0, \infty) \times (-1, \infty) \mid (4.16) \text{ holds}\}. \tag{4.18}$$

R_Γ *is nonempty if and only if* Γ *satisfies*

$$\frac{\mu_* - \bar{\mu}}{\bar{\sigma}^2} > \frac{1}{n} \sum_i \frac{\mu_* - \mu_i}{\sigma_i^2}. \tag{4.19}$$

Proof. See Appendix A.12 $\square$

Proposition 10. *For any* $\mu_* > 0$*, there exists a prior profile such that (a)* $\mu_* > \mu_i > 0$ *for all* i*, (b)* $\sigma_i^2 > 0$ *for all* i*, and (c) (4.19) is satisfied.*

Proof. See Appendix B.3. $\square$

Proposition 9 shows that there always exists a pair (α, ξ) that is consistent with (4.16) if and only if the inequality (4.19) is satisfied. Proposition 10 then shows that there always exists a profile Γ of priors that satisfies (4.19), as well as a reasonable set of requirements.

Condition (4.19) is crucial for the non-monotonic relationship between the aggregate abatement and information noise. If (4.19) is to be satisfied, then there must exist heterogeneity both in μ_i and σ_i^2.

Proposition 11. *If prior profile* Γ *satisfies (4.19), then it must be the case that* $\mu_i \neq \mu_j$ *for some* i, j*,* $\sigma_i^2 \neq \sigma_j^2$ *for some* i, j*, and*

$$\sum_{i=1}^{n} \left\{ 1 - \frac{\mu_* - \mu_i}{\mu_* - \bar{\mu}} \right\} \frac{1}{\sigma_i^2} > 0. \tag{4.20}$$

Proof. See Appendix B.4. $\square$

The inequality (4.20) means that σ_i^2 must be large if $\mu_* - \mu_i$ is large. This requires that relatively more optimistic players must be relatively less confident, while relatively more pessimistic players must be relatively more confident.

The profile of priors depicted in Figure 3, for instance, satisfies (4.19). In this example, there are two groups of players. The first group consists of those who have beliefs with larger μ_i and smaller σ_i^2. One could label those players as being *confident pessimists*. On the other hand, the second group consists of players whose beliefs have smaller μ_i and larger σ_i^2. They could be referred to as *less confident optimists*. Those groups are affected differently by the newly available information in two distinct

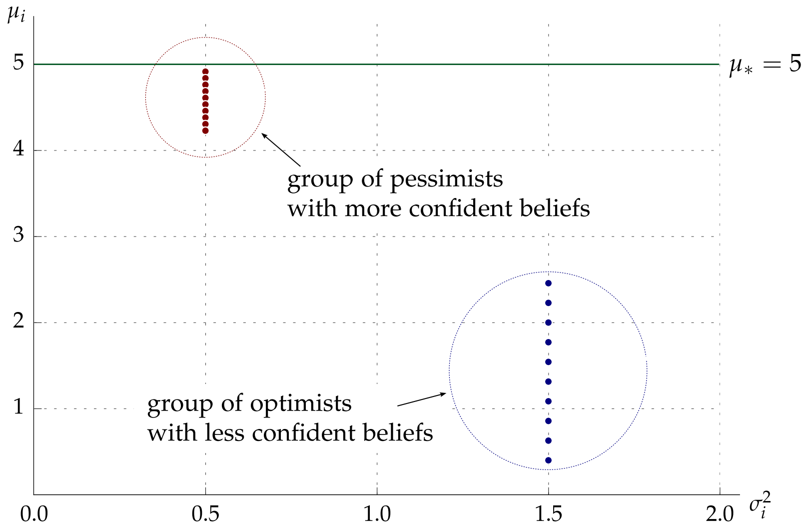

Figure 3: An illustration of heterogeneous priors ($n = 20$)

ways. First, the rationalization effect is smaller for the pessimists and is likely to be outweighed by the confidence effect. Adding information noise, which weakens the confidence effect, will therefore increase the equilibrium abatement of pessimists. An analogous argument suggests that the optimists will decrease their abatement as the information becomes noisy. Second, since the pessimists are more confident about their original priors, their beliefs are effectively updated only if the precision of new information is sufficiently high. This implies that when the information is noisy, the impact on the optimists' priors dominates, creating a downward slope, as in Figure 2. As the information becomes precise, the impact on the pessimists' priors kicks in and as a result, the upward slope in Figure 2 appears for small values of $\sigma_*^2 + \sigma_\varepsilon^2$.

For the numerical example of Γ in Figure 3, the corresponding set R_Γ of (α, ξ) is illustrated in Figure 4. It is worth noting here that set R_Γ occupies a non-negligible part of the α-ξ plane. Hence, the non-monotonic relationship between the total abatement and information noise identified in Proposition 8 is not an exceptional case. What is also clear from the figure is that such a non-monotonic relationship emerges only when the degrees of risk and ambiguity aversion are not simultaneously large. If players are highly risk- and ambiguity-averse, then the existence of ambiguity provides a strong incentive to pollution mitigation. In such a case, information noise, which adds extra ambiguity, always works

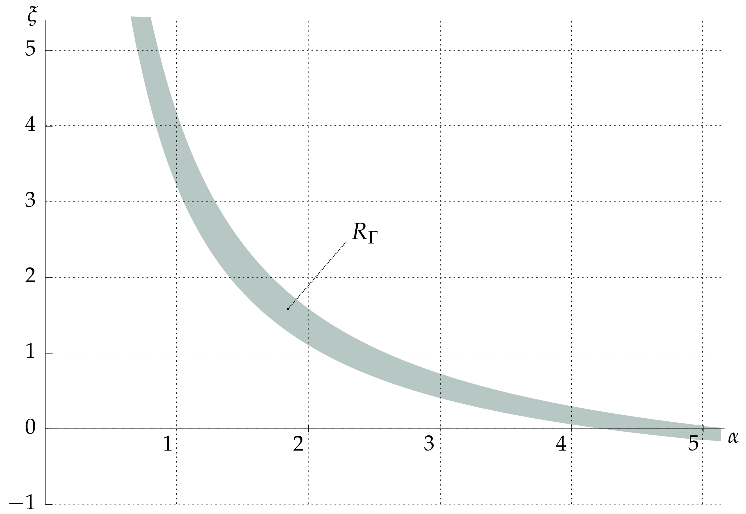

Figure 4: An illustration of set R_Γ in the α-ξ plane. Numerical specification is the same as in Figures 2 and 3.

in favor of increasing total abatement. As a result, the equilibrium total abatement would be a monotonically increasing function of information noise.

We now turn to the main result of this section.

Proposition 12. *Suppose the number n of players is sufficiently large and the prior profile Γ satisfies (4.19) so that R_Γ is nonempty. There exists a nonempty open subset $R'_\Gamma \subset R_\Gamma$ and for each pair $(\alpha, \xi) \in R'_\Gamma$, there exists $\bar{s} > 0$ such that for any $\sigma_*^2 < \bar{s}$, Pareto-improving ambiguity is possible.*

Proof. See Appendix A.13. □

We note that even if adding some information noise increases the total abatement, it does not necessarily imply that every player is better off. A larger total abatement can be achieved by extra ambiguity at the expense of the welfare of some highly ambiguity-averse players. What is shown by Proposition 12 is that as long as the prior profile Γ satisfies (4.19), then there are cases in which players are in fact all better off due to a small amount of information noise. Figure 5 provides a numerical example of such a case. Notice that in this example, $\tilde{W}_i|_{\sigma_\varepsilon^2=0} > \tilde{W}_i|_{\sigma_\varepsilon^2=\infty}$ for all i. This means that the value of information is positive in the absence of information noise. When σ_*^2 is sufficiently small, there is

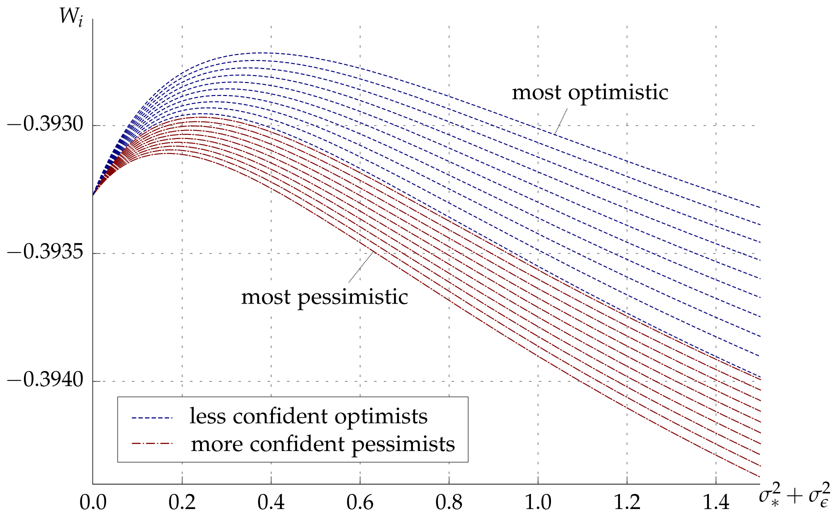

Figure 5: Welfare implication of information noise with $\alpha = 1.5$ and $\xi = 2.1$. Numerical specification is the same as in Figures 2 and 3.

a positive and finite level $\sigma_\varepsilon^2 \in (0, \infty)$ of information noise, such that $\tilde{W}_i > \tilde{W}_i|_{\sigma_\varepsilon^2 = 0} > \tilde{W}_i|_{\sigma_\varepsilon^2 = \infty}$ for all i.

We should emphasize that the roles played by the degrees of risk- and ambiguity-aversion are not symmetric here. For Pareto-improving ambiguity to be possible, the degree of ambiguity aversion can be arbitrarily large, while the degree of risk aversion has an upper bound. The next corollary formalizes this point.

Corollary 1. *The set of ξ included in R'_Γ is not bounded above, while the set of α included in R'_Γ is bounded above.*

Proof. See Appendix A.14. □

The asymmetric roles of risk and ambiguity aversions emerge due to the strategic substitution effect we discussed earlier. Higher degrees of risk and ambiguity aversion both amplify the strategic substitution effect through the corresponding increases of $\delta\rho \sum_i \tilde{\gamma}_i$ in (4.14). As is seen in (4.14) and (4.15), intensification of the strategic substitution effect in turn makes it difficult for information noise to be Pareto-improving. However, since $\tilde{\gamma}_i = \alpha \left[\sigma_u^2 + (1 + \xi)\tilde{\sigma}_i^2\right]$, the influence of ambiguity aversion diminishes when $\tilde{\sigma}_i^2$ is small. In other words, preference about ambiguity only matters when there remains a sufficiently large ambiguity. This is why an arbitrarily larger degree of ambiguity aversion is consistent with Pareto-improving information noise, as long as the remaining ambiguity is very small. On the other hand, the influence of risk aversion remains even if $\tilde{\sigma}_i^2 = 0$. Accordingly, when the degree of risk aversion is sufficiently large, the strategic substitution effect dominates, making Pareto-improving ambiguity impossible. It is important to note that this result partly depends on our assumption of $\sigma_u^2 > 0$, which means that the risk of β remains, even after the ambiguity is completely resolved. Although in a rather narrow context, the present analysis provides an interesting example where the preferences for risk and ambiguity aversion play asymmetric roles.

5 Conclusions

This paper developed a model of public bads where players have heterogeneous beliefs about the consequences of their collective actions. Based on a simple analysis of the model, we were able to shed light on an important trade-off associated with information policies. In the presence of belief heterogeneity and ambiguity, the value of information depends on the rationalization effect, the convergence effect, and the confidence effect. Depending on the players' preference about risk and ambiguity, as well as on the players' subjective beliefs, one effect dominates the other.

Among the most interesting implications is that regularly publishing new information with minor updates might do more harm than good, especially if the players are highly ambiguity-averse. Instead, the new information should be published only when significantly novel findings are available. Moreover, as long as players are equally confident about their beliefs, adding noise to the public information is never a good idea. When the players' beliefs are highly heterogeneous, on the other hand, Pareto improvement can be achieved by choosing unclear phrasing or ambiguous wording in the published information.

There are several directions for future research that appear fruitful. First, it would be of interest to investigate the implications of heterogeneous beliefs and ambiguity to the possible cooperation among players.

This line of analysis will be straightforward, given the simplicity and the tractability of our model. Another interesting direction would be the consideration of strategic interaction between the players and the policy makers. While the additional layer of strategic interplay might compromise the tractability of the model, such an extension will surely be realistic and would provide economically useful insights.

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A Proofs

A.1 Lemma for propositions

To prove the propositions in the main text, it is useful to sumarize the following result as a lemma, which is reminiscent of the classical result of Milgrom (1981).

Lemma 3. *Let $Z \subset \mathbb{R}$. For any pair $\psi_k : Z \to \mathbb{R}_{++}, k \in \{0,1\}$ of functions, the following are equivalent:*

(a) For any probability density h with support $\bar{Z} \subset Z$,

$$\int_{s \leq z} \hat{h}_1(s) ds < \int_{s \leq z} \hat{h}_0(s) ds \quad \forall z < \sup \bar{Z}, \tag{A.1}$$

where $\hat{h}_k(z) \propto \psi_k(z) h(z)$ for $k = 0, 1$.

(b) For any $z \in Z$, $\psi_1(z')\psi_0(z) - \psi_1(z)\psi_0(z') > 0$ for all $z' > z$.

Proof. Suppose (a) is true. Fix $z \in Z$. For each $z' > z$, consider a density function h with support $\bar{Z} = \{z, z'\}$ such that $h(z) = h(z') = 1/2$. Then (a) implies

$$\frac{\psi_1(z)}{\psi_1(z) + \psi_1(z')} < \frac{\psi_0(z)}{\psi_0(z) + \psi_0(z')} \tag{A.2}$$

and hence $\psi_1(z')\psi_0(z) - \psi_1(z)\psi_0(z') > 0$ for all $z' > z$.

Conversely, suppose (b) is true. Choose an arbitrary density function h with support $\bar{Z} \subset Z$. If $\bar{Z}$ is a singleton, the claim of (a) is vacuously true. Assume that $\bar{Z}$ contains more than two elements. Then choose $z^* \in \bar{Z}$ such that $z^* < \sup \bar{Z}$. Then (b) implies

$$\frac{\psi_1(z')h(z')}{\psi_1(z)} > \frac{\psi_0(z')h(z')}{\psi_0(z)} \tag{A.3}$$

if $z' > z^* \geq z$. Hence,

$$\frac{1}{\psi_1(z)} \int_{z' > z^*} \psi_1(z')h(z')dz' > \frac{1}{\psi_0(z)} \int_{z' > z^*} \psi_0(z')h(z')dz', \tag{A.4}$$

for all $z \leq z^*$. It follows that

$$\frac{\int_{z \leq z^*} \hat{h}_1(z)dz}{1 - \int_{z \leq z^*} \hat{h}_1(z)dz} = \frac{\int_{z \leq z^*} \hat{h}_1(z)dz}{\int_{z' > z^*} \hat{h}_1(z')dz'} < \frac{\int_{z \leq z^*} \hat{h}_0(z)dz}{\int_{z' > z^*} \hat{h}_0(z')dz'} = \frac{\int_{z \leq z^*} \hat{h}_0(z)dz}{1 - \int_{z \leq z^*} \hat{h}_0(z)dz},$$

which in turn implies

$$\int_{z \leq z^*} \hat{h}_1(z)dz < \int_{z \leq z^*} \hat{h}_0(z)dz. \tag{A.5}$$

Since $z^* < \sup \bar{Z}$ is arbitrarily chosen, (a) follows. $\square$

A.2 Proof of Proposition 1

Since the cost functions are identical among players, it suffices to show that the expected marginal abatement benefit is larger for player j than i when both players choose the same level of abatement. Fix arbitrary levels of total and individual abatement as A and $a = a_i = a_j$, respectively. Then clearly $x_i = x_j$ and thus $\hat{f}_i(\cdot|\theta) = \hat{f}_j(\cdot|\theta)$ for all $\theta \in \Theta$. To see the relationship between $\hat{g}_i$ and $\hat{g}_j$, consider a special case of Lemma 3 in Appendix A.1 where $Z = \Theta$, $\psi_1 := g_j$, $\psi_0 := g_i$, and specify the density h by $h(\theta) \propto \phi'(\mathbb{E}[u|\theta])\mathbb{E}[u'|\theta]$. Then $\psi_1(\theta')\psi_0(\theta) - \psi_1(\theta)\psi_0(\theta') = g_j(\theta')g_i(\theta) - g_i(\theta)g_i(\theta') > 0$ for any $\theta' > \theta$. Lemma 3 shows that $\hat{g}_j$ strictly first-degree stochastically dominates $\hat{g}_i$. Hence, for any $\beta \in B$,

$$\begin{aligned}\int_{\beta' \leq \beta} f_i(\beta')d\beta' &= \int_{\theta \in \Theta} \left[\int_{\beta' \leq \beta} \hat{f}_j(\beta'|\theta)d\beta' \right] \hat{g}_i(\theta)d\theta \\ &> \int_{\theta \in \Theta} \left[\int_{\beta' \leq \beta} \hat{f}_j(\beta'|\theta)d\beta' \right] \hat{g}_j(\theta)d\theta = \int_{\beta' \leq \beta} f_j(\beta')d\beta',\end{aligned} \tag{A.6}$$

where the first equality follows from $\hat{f}_i = \hat{f}_j$ and the strict inequality follows from the stochastic dominance of $\hat{g}_j$ against $\hat{g}_i$. This shows that f_j strictly first-degree stochastically dominates f_i. Therefore, we conclude $\int_B D'(E;\beta)f_j(\beta)d\beta > \int_B D'(E;\beta)f_i(\beta)d\beta$, which proves our claim.

A.3 Proof of Proposition 2

For each $A \geq 0$, define the expected marginal benefit by $EMB(A) := \int_B D'(n\bar{e} - A;\beta)f_A(\beta)d\beta$, where $f_A(\beta) := \int_\Theta \hat{f}_A(\beta|\theta)\hat{g}_A(\theta)d\theta$, $\hat{f}_A(\beta|\theta) \propto u'(x_A)f(\beta|\theta)$, $\hat{g}_A(\theta) \propto \phi'(\mathbb{E}[u(x_A)|\theta])\mathbb{E}[u'(x_A)|\theta]g_*(\theta)$, and $x_A := \bar{y} - D(n\bar{e} - A;\beta) - C(A/n)$. Since A_* is the efficient level of aggregate abatement, it must be the case that $n^{-1}C'(A_*/n) = EMB(A_*)$. On the other hand, the equilibrium level A of aggregate abatement is determined by $C'(A/n) = EMB(A)$. Since C' is increasing, this implies $A < A_*$.

A.4 Proof of Proposition 3

A similar argument as in Proposition 1 shows that that the equilibrium total abatement is smaller than in the case of rational belief being shared by all players. Since the abatement level is insufficient even in the latter case and since W_i is a strictly concave function of total abatement, the result immediately follows.

A.5 Proof of Lemma 1

Notice that Lemma 1 is a special case of Lemma 3 in Appendix A.1 where $Z = B$, $h = f$, $\psi_0(\beta) := 1$, and $\psi_1(\beta) := u'(\bar{y} - D(E;\beta) - C(a_i))$. By statement (b) of Lemma 3, $u'(\bar{y} - D(E;\beta') - C(a_i)) > u'(\bar{y} - D(E;\beta) - C(a_i))$ for all $\beta' > \beta$, meaning u' is strictly increasing in β. Observe

$$\frac{\partial}{\partial\beta}u'(\bar{y} - D(E;\beta) - C(a_i)) = -u''(x_i)\frac{\partial D(E;\beta)}{\partial\beta}, \tag{A.7}$$

which is strictly positive if and only if u is strictly concave. Hence, by Lemma 3, $\hat{f}_i(\cdot|\theta) \propto u'(x_i)f(\cdot|\theta)$ strictly first-degree stochastically dominates $f(\cdot|\theta)$ if and only if u is strictly concave.

A.6 Proof of Lemma 2

Consider a special case of Lemma 3 in Appendix A.1 where $Z = \Theta$, $h = g_i$, $\psi_0(\theta) := 1$, and $\psi_1(\theta) := \phi'(\mathbb{E}[u|\theta])\mathbb{E}[u'|\theta]$. By Lemma 3, $\hat{g}_i \propto \phi'(\mathbb{E}[u|\theta])\mathbb{E}[u'|\theta]g_i$ strictly first-order stochastically dominates g_i if and only if $\phi'(\mathbb{E}[u|\theta])\mathbb{E}[u'|\theta]$ is strictly increasing in θ.

It then suffices to show that $\phi'(\mathbb{E}[u|\theta])\mathbb{E}[u'|\theta]$ is strictly increasing in θ when u and v are both concave and at least one of the concavities is strict. First notice that under Assumption 1, $\mathbb{E}[u|\theta]$ is strictly decreasing in θ because u is strictly decreasing in β. This means that $\phi'(\mathbb{E}[u|\theta])$ is (strictly) increasing in θ if ϕ is (strictly) concave. Similarly, $\mathbb{E}[u'|\theta]$ is (strictly) increasing in θ if u is (strictly) concave. Therefore, $\phi'(\mathbb{E}[u|\theta])\mathbb{E}[u'|\theta]$ is strictly increasing in θ if ϕ and u are both concave and at least one of them is strictly concave.

A.7 Proof of Proposition 4

Notice that Assumption 1 implies that for each $\beta \in B$, $\int_{\beta' \leq \beta} f(\beta'|\theta) d\beta'$ is strictly decreasing in θ. Hence for any $\beta \in B$,

$$\begin{aligned}
\int_{\beta' \leq \beta} f_i(\beta') d\beta' &= \int_{\theta \in \Theta} \int_{\beta' \leq \beta} \hat{f}_i(\beta'|\theta) d\beta' \hat{g}_i(\theta) d\theta && \text{(A.8)} \\
&< \int_{\theta \in \Theta} \int_{\beta' \leq \beta} f(\beta'|\theta) d\beta' \hat{g}_i(\theta) d\theta && \text{(A.9)} \\
&< \int_{\theta \in \Theta} \int_{\beta' \leq \beta} f(\beta'|\theta) d\beta' g_i(\theta) d\theta = \int_{\beta' \leq \beta} f_i^c(\beta') d\beta',
\end{aligned}$$

where the first and second inequalities follow from Lemma 1 and Lemma 2, respectively. Therefore, f_i strictly first-degree stochastically dominates f_i^c. Since $D'(E;\beta)$ is strictly increasing in β, this means for each level A of aggregate abatement, the subjective expected marginal benefit is strictly larger under f_i than under f_i^c for all i. Then the claim of the proposition follows from the first-order condition (3.1).

A.8 Proof of Proposition 5

Suppose players become more ambiguity averse and their ambiguity attitude is represented by ϕ_M instead of ϕ. This means there exists an increasing and strictly concave function $M : \mathbb{R} \to \mathbb{R}$ such that $\phi_M(u) = M(\phi(u))$. Let $\hat{g}_i^M(\cdot)$ and $\hat{g}_i(\cdot)$ be the preference-adjusted prior of players with ϕ_M and ϕ, respectively. Then for any $\theta' > \theta$

$$\begin{aligned}
\frac{\hat{g}_i^M(\theta)}{\hat{g}_i^M(\theta')} &= \frac{M'(\phi(\mathbb{E}[u(x_i)|\theta]))}{M'(\phi(\mathbb{E}[u(x_i)|\theta']))} \frac{\phi'(\mathbb{E}[u(x_i)|\theta])\mathbb{E}[u'(x_i)|\theta] g_i(\theta)}{\phi'(\mathbb{E}[u(x_i)|\theta'])\mathbb{E}[u'(x_i)|\theta'] g_i(\theta')} && \text{(A.10)} \\
&< \frac{\phi'(\mathbb{E}[u(x_i)|\theta])\mathbb{E}[u'(x_i)|\theta] g_i(\theta)}{\phi'(\mathbb{E}[u(x_i)|\theta'])\mathbb{E}[u'(x_i)|\theta'] g_i(\theta')} = \frac{\hat{g}_i(\theta)}{\hat{g}_i(\theta')}, && \text{(A.11)}
\end{aligned}$$

which means that relatively pessimistic study θ' obtains a larger weight when individuals become more ambiguity averse. In particular, $\hat{g}_i^M$ strictly first-order stochastically dominates $\hat{g}_i$. Then the statement of the proposition follows from the same argument as in Proposition 4.

A.9 Proof of Proposition 6

Observe

$$\tilde{A} - A = \frac{n^{-1}\rho}{n^{-1} + \rho\delta\tilde{\gamma}} \sum_{i=1}^{n} \frac{\sigma_i^2}{\sigma_*^2 + \sigma_i^2} \left\{ (\mu_* - \mu_i) - \frac{\bar{e} - \rho\bar{\mu}}{n^{-1} + \rho\delta\bar{\gamma}} \delta\alpha(1+\xi)\sigma_i^2 \right\}.$$

Notice that the right-hand side is continuous and strictly increasing in $(\mu_* - \mu_i)$ for each i. Since the entire term is strictly negative when $\mu_* - \mu_i = 0$ for all i, the first assertion of the proposition follows. Observe that the second term in the brace is increasing in α and ξ, which shows that $\Delta\mu$ is larger for larger values of α and ξ.

To prove the second assertion, notice

$$\frac{\tilde{w}_i - w_i}{\nu 2^{-1}} = (A - \tilde{A})(a_i + \tilde{a}_i) \left[\frac{a_i - \tilde{a}_i}{A - \tilde{A}} - \frac{\mu_* + \mu_* + (n\bar{e} - A)\delta\gamma_* + (n\bar{e} - \tilde{A})\delta\gamma_*}{\mu_i + \tilde{\mu}_i + (n\bar{e} - A)\delta\gamma_i + (n\bar{e} - \tilde{A})\delta\tilde{\gamma}_i} \right].$$

Since $\mu_* > \tilde{\mu}_i > \mu_i$ and $\gamma_* > \tilde{\gamma}_i$, the second term in the square bracket is greater than one when $\gamma_i = \gamma_*$, which is implied by $\sigma_i^2 = \sigma_*^2$. If μ_i is close to μ_*, the result (i) shows $A > \tilde{A}$. If furthermore σ_i^2 is sufficiently close to σ_*^2, then $\tilde{a}_i$ is close to $\tilde{A}/n$ and thus $a_i > \tilde{a}_i$ for all i. This in turn implies $A - \tilde{A} > a_i - \tilde{a}_i > 0$ for all i. Therefore, the first term in the square bracket is strictly smaller than one, which completes the proof.

A.10 Proof of Proposition 7

Notice first

$$\begin{aligned} \tilde{w}_i - w_i = \frac{\delta}{2} &\left\{ \mu_* + \frac{\bar{e} - \rho\tilde{\mu}}{n^{-1} + \delta\rho\tilde{\gamma}} \delta\gamma_* + \mu_* + \frac{\bar{e} - \rho\bar{\mu}}{n^{-1} + \delta\rho\bar{\gamma}} \delta\gamma_* \right\} (\tilde{A} - A) \\ - \frac{\delta}{2} &\left\{ \tilde{\mu}_i + \frac{\bar{e} - \rho\tilde{\mu}}{n^{-1} + \delta\rho\tilde{\gamma}} \delta\tilde{\gamma}_i + \mu_i + \frac{\bar{e} - \rho\bar{\mu}}{n^{-1} + \delta\rho\bar{\gamma}} \delta\gamma_i \right\} (\tilde{a}_i - a_i), \end{aligned} \tag{A.12}$$

where

$$\tilde{A} - A = \frac{n^{-1}}{n^{-1} + \delta\rho\tilde{\gamma}} \sum_i \frac{\sigma_i^2}{\sigma_*^2 + \sigma_\varepsilon^2 + \sigma_i^2} \rho \left\{ (\mu_* - \mu_i) - \frac{\bar{e} - \rho\bar{\mu}}{n^{-1} + \delta\rho\bar{\gamma}} \delta\alpha(1+\xi)\sigma_i^2 \right\}, \tag{A.13}$$

and

$$\tilde{a}_i - a_i = \frac{\sigma_i^2}{\sigma_*^2 + \sigma_\varepsilon^2 + \sigma_i^2}\rho\left\{(\mu_* - \mu_i) - \frac{\bar{e} - \rho\bar{\mu}}{n^{-1} + \delta\rho\bar{\gamma}}\delta\alpha(1+\xi)\sigma_i^2\right\}$$
$$- \frac{n^{-1}\delta\rho\tilde{\gamma}_i}{n^{-1} + \delta\rho\tilde{\gamma}}\sum_j \frac{\sigma_j^2}{\sigma_*^2 + \sigma_\varepsilon^2 + \sigma_j^2}\rho\left\{(\mu_* - \mu_j) - \frac{\bar{e} - \rho\bar{\mu}}{n^{-1} + \delta\rho\bar{\gamma}}\delta\alpha(1+\xi)\sigma_j^2\right\}. \tag{A.14}$$

We prove the proposition by combining the following two lemmas.

Lemma 4. *For a given level of $\sigma_\varepsilon^2 \geq 0$,*

1. *if $\partial\tilde{w}_i/\partial\sigma_\varepsilon^2 > 0$ for all i, then it must be the case that $\partial\tilde{A}/\partial\sigma_\varepsilon^2 > 0$;*
2. *if $\tilde{w}_i > w_i$ for all i, then it must be the case that $\tilde{A} > A$.*

Proof. Suppose, by way of contradiction, $\partial\tilde{w}_i/\partial\sigma_\varepsilon^2 > 0$ for all i and $\partial\tilde{A}/\partial\sigma_\varepsilon^2 \leq 0$. Then (4.15) implies $\partial\tilde{a}_i/\partial\sigma_\varepsilon^2 < 0$ for all i and $\partial\tilde{A}/\partial\sigma_\varepsilon^2 = \sum_i \partial\tilde{a}_i/\partial\sigma_\varepsilon^2 < 0$. Hence $\partial\tilde{w}_i/\partial\sigma_\varepsilon^2 > 0$ and (4.15) imply

$$\frac{\mu_* + \frac{\bar{e}-\rho\tilde{\mu}}{n^{-1}+\delta\rho\tilde{\gamma}}\delta\gamma_*}{\tilde{\mu}_i + \frac{\bar{e}-\rho\tilde{\mu}}{n^{-1}+\delta\rho\tilde{\gamma}}\delta\tilde{\gamma}_i} < \frac{\partial\tilde{a}_i/\partial\sigma_\varepsilon^2}{\partial\tilde{A}/\partial\sigma_\varepsilon^2} < 1. \tag{A.15}$$

But this is impossible because $\mu_* > \tilde{\mu}_i$ and $\sigma_*^2 > \tilde{\sigma}_i^2$ for all i.

To see the latter part of the proposition suppose, by way of contradiction, $\tilde{w}_i > w_i$ for all i and $\tilde{A} \leq A$. Then (A.12) implies $\tilde{a}_i < a_i$ for all i and $\tilde{A} < A$. Then it follows from $\tilde{w}_i > w_i$ and (A.12) that for each i

$$\frac{\mu_* + \frac{\bar{e}-\rho\tilde{\mu}}{n^{-1}+\delta\rho\tilde{\gamma}}\delta\gamma_* + \mu_* + \frac{\bar{e}-\rho\bar{\mu}}{n^{-1}+\delta\rho\bar{\gamma}}\delta\gamma_*}{\tilde{\mu}_i + \frac{\bar{e}-\rho\tilde{\mu}}{n^{-1}+\delta\rho\tilde{\gamma}}\delta\tilde{\gamma}_i + \mu_i + \frac{\bar{e}-\rho\bar{\mu}}{n^{-1}+\delta\rho\bar{\gamma}}\delta\gamma_i} < \frac{\tilde{a}_i - a_i}{\tilde{A} - A} < 1, \tag{A.16}$$

which is impossible since $\mu_* > \tilde{\mu}_i$, $\sigma_*^2 > \tilde{\sigma}_i^2$ for all i and $A < A_*$. □

Lemma 5. *Suppose $\sigma_i^2 = \bar{\sigma}^2 > 0$ for all i. For a given level of $\sigma_\varepsilon^2 \geq 0$, $\partial\tilde{A}/\partial\sigma_\varepsilon^2 > 0$ if and only if $\tilde{A} < A$.*

Proof. Since $\sigma_i^2 = \bar{\sigma}^2 > 0$ for all i, combining (4.14) and (A.13) yields

$$\tilde{A} - A = -\left(1 + \frac{\rho\delta\alpha(1+\xi)}{n^{-1} + \delta\rho\bar{\gamma}}\left[\frac{\bar{\sigma}^2}{\sigma_*^2 + \sigma_\varepsilon^2 + \bar{\sigma}^2}\right]\bar{\sigma}^2\right)^{-1}(\sigma_*^2 + \sigma_\varepsilon^2 + \bar{\sigma}^2)\frac{\partial\tilde{A}}{\partial\sigma_\varepsilon^2}, \tag{A.17}$$

from which the result follows. $\square$

A.11 Proof of Proposition 8

We note $\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\tilde{\mu}_i = \mu_*$, $\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\tilde{\sigma}_i^2 = 0$, $\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\tilde{\gamma}_i = \alpha\sigma_u^2$, and

$$\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\frac{\partial\tilde{\mu}_i}{\partial\sigma_\varepsilon^2} = -\frac{\mu_* - \mu_i}{\sigma_i^2}, \quad \lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\frac{\partial\tilde{\gamma}_i}{\partial\sigma_\varepsilon^2} = \alpha(1+\xi). \tag{A.18}$$

Hence,

$$\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\frac{\partial\tilde{A}}{\partial\sigma_\varepsilon^2} = \frac{\rho}{n^{-1} + \delta\rho\alpha\sigma_u^2}\left\{\frac{\bar{e} - \rho\mu_*}{n^{-1} + \delta\rho\alpha\sigma_u^2}\delta\alpha(1+\xi) - \frac{1}{n}\sum_i\frac{\mu_* - \mu_i}{\sigma_i^2}\right\}, \tag{A.19}$$

$$\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\frac{\partial\tilde{a}_i}{\partial\sigma_\varepsilon^2} = \frac{1}{n}\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\frac{\partial\tilde{A}}{\partial\sigma_\varepsilon^2} + \rho\left(\frac{1}{n}\sum_j\frac{\mu_* - \mu_j}{\sigma_j^2} - \frac{\mu_* - \mu_i}{\sigma_i^2}\right), \tag{A.20}$$

$$\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\tilde{A} - A = \frac{\rho}{n^{-1} + \delta\rho\alpha\sigma_u^2}\left\{\frac{\mu_* - \bar{\mu}}{\bar{\sigma}^2} - \frac{\bar{e} - \rho\bar{\mu}}{n^{-1} + \delta\rho\bar{\gamma}}\delta\alpha(1+\xi)\right\}\bar{\sigma}^2, \tag{A.21}$$

$$\begin{aligned}\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\tilde{a}_i - a_i = \rho&\left\{\frac{\mu_* - \mu_i}{\sigma_i^2} - \frac{\bar{e} - \rho\bar{\mu}}{n^{-1} + \delta\rho\bar{\gamma}}\delta\alpha(1+\xi)\right\}\sigma_i^2\\ &- \rho\frac{\delta\rho\alpha\sigma_u^2}{n^{-1} + \delta\rho\alpha\sigma_u^2}\left\{\frac{\mu_* - \bar{\mu}}{\bar{\sigma}^2} - \frac{\bar{e} - \rho\bar{\mu}}{n^{-1} + \delta\rho\bar{\gamma}}\delta\alpha(1+\xi)\right\}\bar{\sigma}^2.\end{aligned} \tag{A.22}$$

Observe that (A.19) implies

$$\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\frac{\partial\tilde{A}}{\partial\sigma_\varepsilon^2} > 0 \iff \frac{\bar{e} - \rho\mu_*}{n^{-1} + \delta\rho\alpha\sigma_u^2}\delta\alpha(1+\xi) > \frac{1}{n}\sum_i\frac{\mu_* - \mu_i}{\sigma_i^2}. \tag{A.23}$$

On the other hand, (A.21) implies

$$\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\tilde{A} > A \iff \frac{\mu_*-\bar{\mu}}{\bar{\sigma}^2} > \frac{\bar{e}-\rho\bar{\mu}}{n^{-1}+\delta\rho\bar{\gamma}}\delta\alpha(1+\xi) \tag{A.24}$$

$$\iff \frac{\mu_*-\bar{\mu}}{\bar{\sigma}^2} > \frac{\bar{e}-\rho\mu_*}{n^{-1}+\delta\rho\alpha\sigma_u^2}\delta\alpha(1+\xi). \tag{A.25}$$

By (A.23) and (A.25), $\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\tilde{A} > A$ and $\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\partial\tilde{A}/\partial\sigma_\varepsilon^2 > 0$ if and only if (4.16) is satisfied. Hence, if (4.16) is satisfied, there exists $\bar{s} > 0$ such that $\lim_{\sigma_\varepsilon^2\to 0}\tilde{A} > A$ and $\lim_{\sigma_\varepsilon^2\to 0}\partial\tilde{A}/\partial\sigma_\varepsilon^2 > 0$ for any $\sigma_*^2 < \bar{s}$, which completes the proof.

A.12 Proof of Proposition 9

Define for each $\alpha \in \mathbb{R}_{++}$

$$\overline{\xi}(\alpha) := \left(\frac{\mu_*-\bar{\mu}}{\bar{\sigma}^2}\right)\frac{n^{-1}+\delta\rho\alpha\sigma_u^2}{\delta\alpha(\bar{e}-\rho\mu_*)} - 1 > -1, \tag{A.26}$$

$$\underline{\xi}(\alpha) := \left(\frac{1}{n}\sum_i \frac{\mu_*-\mu_i}{\sigma_i^2}\right)\frac{n^{-1}+\delta\rho\alpha\sigma_u^2}{\delta\alpha(\bar{e}-\rho\mu_*)} - 1 > -1 \tag{A.27}$$

so that for each $\alpha \in \mathbb{R}_{++}$,

$$\frac{\mu_*-\bar{\mu}}{\bar{\sigma}^2} > \frac{\bar{e}-\rho\mu_*}{n^{-1}+\delta\rho\alpha\sigma_u^2}\delta\alpha(1+\xi) > \frac{1}{n}\sum_i \frac{\mu_*-\mu_i}{\sigma_i^2} \tag{A.28}$$

if and only if $\overline{\xi}(\alpha) > \xi > \underline{\xi}(\alpha)$. Then $R = \cup_{\alpha\in\mathbb{R}_{++}}\{\{\alpha\}\times(\underline{\xi}(\alpha),\overline{\xi}(\alpha))\}$. Since $\underline{\xi}(\alpha) < \overline{\xi}(\alpha)$ if and only if (4.19) is satisfied, the set R is nonempty if and only if (4.19) is satisfied.

A.13 Proof of Proposition 12

Suppose (4.19) is satisfied. Note that

$$\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\frac{\partial\tilde{w}_i}{\partial\sigma_\varepsilon^2} = \delta\left\{\mu_* + \frac{\bar{e}-\rho\mu_*}{n^{-1}+\delta\rho\alpha\sigma_u^2}\delta\alpha\sigma_u^2\right\}\left(\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\frac{\partial\tilde{A}}{\partial\sigma_\varepsilon^2} - \lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\frac{\partial\tilde{a}_i}{\partial\sigma_\varepsilon^2}\right), \tag{A.29}$$

which with (A.19) and (A.20) implies

$$\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\frac{\partial\tilde{w}_i}{\partial\sigma_\varepsilon^2}>0 \iff \lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\frac{\partial\tilde{A}}{\partial\sigma_\varepsilon^2}>\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\frac{\partial\tilde{a}_i}{\partial\sigma_\varepsilon^2} \iff \frac{\mu_*-\mu_i}{\sigma_i^2}>\underline{m}(\alpha,\xi), \tag{A.30}$$

where

$$\underline{m}(\alpha,\xi):=\left(\frac{1+\delta\rho\alpha\sigma_u^2}{n^{-1}+\delta\rho\alpha\sigma_u^2}\right)\frac{1}{n}\sum_i\frac{\mu_*-\mu_i}{\sigma_i^2} + \left(1-\frac{1+\delta\rho\alpha\sigma_u^2}{n^{-1}+\delta\rho\alpha\sigma_u^2}\right)\frac{\bar{e}-\rho\mu_*}{n^{-1}+\delta\rho\alpha\sigma_u^2}\delta\alpha(1+\xi). \tag{A.31}$$

On the other hand, $\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\tilde{w}_i-w_i>0$ if and only if

$$\frac{\mu_*+\frac{\bar{e}-\rho\mu_*}{n^{-1}+\delta\rho\alpha\sigma_u^2}\delta\alpha\sigma_u^2+\mu_*+\frac{\bar{e}-\rho\bar{\mu}}{n^{-1}+\delta\rho\bar{\gamma}}\delta\alpha\sigma_u^2}{\mu_*+\frac{\bar{e}-\rho\mu_*}{n^{-1}+\delta\rho\alpha\sigma_u^2}\delta\alpha\sigma_u^2+\mu_i+\frac{\bar{e}-\rho\bar{\mu}}{n^{-1}+\delta\rho\bar{\gamma}}\delta\gamma_i}>\frac{\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\tilde{a}_i-a_i}{\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\tilde{A}-A}. \tag{A.32}$$

Notice that the left-hand side of (A.32) is greater than 1 if and only if

$$\frac{\bar{e}-\rho\bar{\mu}}{n^{-1}+\delta\rho\bar{\gamma}}\delta\alpha(1+\xi)<\frac{\mu_*-\mu_i}{\sigma_i^2}. \tag{A.33}$$

If (A.33) does not hold, then the right-hand side of (A.32) is negative and (A.32) is satisfied. If (A.33) does hold, (A.32) is satisfied if the right-hand side of (A.32) is less than or equal to 1, which is equivalent to

$$\frac{\mu_*-\mu_i}{\sigma_i^2}\le\overline{m}_i(\alpha,\xi)=:\left(\frac{1+\delta\rho\alpha\sigma_u^2}{n^{-1}+\delta\rho\alpha\sigma_u^2}\frac{\bar{\sigma}^2}{\sigma_i^2}\right)\frac{\mu_*-\bar{\mu}}{\bar{\sigma}^2} + \left(1-\frac{1+\delta\rho\alpha\sigma_u^2}{n^{-1}+\delta\rho\alpha\sigma_u^2}\frac{\bar{\sigma}^2}{\sigma_i^2}\right)\frac{\bar{e}-\rho\bar{\mu}}{n^{-1}+\delta\rho\bar{\gamma}}\delta\alpha(1+\xi). \tag{A.34}$$

Note (A.34) is a sufficient condition for (A.32). By combining (A.30) and (A.34), we conclude that $\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\tilde{w}_i>w_i$ and $\lim_{\sigma_*^2,\sigma_\varepsilon^2\to 0}\partial\tilde{w}_i/\partial\sigma_\varepsilon^2>0$ if $(\alpha,\xi)\in R$ and

$$\underline{m}(\alpha,\xi)<\frac{\mu_*-\mu_i}{\sigma_i^2}\le\overline{m}_i(\alpha,\xi). \tag{A.35}$$

We shall prove the set $R'':=\{(\alpha,\xi)\in R\,|\,\text{(A.35) holds for all } i\}$ is nonempty.

For each $\alpha \in \mathbb{R}_{++}$, define $\hat{\xi}(\alpha) := 2^{-1}[\underline{\xi}(\alpha) + \overline{\xi}(\alpha)] \in (\underline{\xi}(\alpha), \overline{\xi}(\alpha))$ so that $(\alpha, \hat{\xi}(\alpha)) \in R$ for all $\alpha \in \mathbb{R}_{++}$. Notice

$$\lim_{\alpha \to 0} \underline{m}(\alpha, \hat{\xi}(\alpha)) < \frac{\mu_* - \mu_i}{\sigma_i^2} < \lim_{\alpha \to 0} \overline{m}_i(\alpha, \hat{\xi}(\alpha)) \tag{A.36}$$

for all i if n is sufficiently large. Therefore, there must exist $\overline{\alpha} > 0$ such that $(\alpha, \hat{\xi}(\alpha)) \in \mathrm{Int}(R'')$ for all $\alpha < \overline{\alpha}$. Since $(\alpha, \hat{\xi}(\alpha))$ is an interior point of R'' for each $\alpha < \overline{\alpha}$, there exists a neighborhood $O(\alpha)$ such that $(\alpha, \hat{\xi}(\alpha)) \in O(\alpha) \subset R''$ for each $\alpha < \overline{\alpha}$. Then $R' := \bigcup_{\alpha < \overline{\alpha}} O(\alpha) \subset R'' \subset R$ is a nonempty open subset of R such that for each $(\alpha, \xi) \in R'$,

$$\lim_{\sigma_*^2, \sigma_\varepsilon^2 \to 0} \tilde{w}_i > w_i \text{ and } \lim_{\sigma_*^2, \sigma_\varepsilon^2 \to 0} \frac{\partial \tilde{w}_i}{\partial \sigma_\varepsilon^2} > 0, \tag{A.37}$$

which completes the proof.

A.14 Proof of Corollary 1

The proof of Proposition 12 shows that there exists $\overline{\alpha} > 0$ such that $(\alpha, \hat{\xi}(\alpha)) \in R'$ for all $\alpha < \overline{\alpha}$. Since $\lim_{\alpha \to 0} \hat{\xi}(\alpha) = \infty$, this implies that the set of ξ included in R' is not bounded above. On the other hand, let $(\alpha_k, \xi_k)_{k \in \mathbb{N}}$ be an arbitrary sequence in R such that $\lim_{k \to \infty} \alpha_k = \infty$. Since $\lim_{k \to \infty} \underline{m}(\alpha_k, \xi_k) = n^{-1} \sum_i (\mu_* - \mu_i)/\sigma_i^2$, (A.30) implies $(\alpha_k, \xi_k) \notin R'$ for sufficiently large k. Therefore, the set of α included in R' is bounded above.

B Supplements (for online publication)

B.1 Existence and uniqueness of A_*

Observe

$$\frac{dW_i(A/n)}{dA} = \int_\Theta \left(\phi'(\mathbb{E}[u(x)|\theta])\mathbb{E}\left[u'(x) \left\{ D'(E;\beta) - \frac{1}{n}C'(A/n) \right\} \middle| \theta \right] \right) g_*(\theta) d\theta. \tag{B.1}$$

Since $\lim_{A\to 0}\{D'(E;\beta) - \frac{1}{n}C'(A/n)\} > 0$ and $\lim_{A\to n\bar{e}}\{D'(E;\beta) - \frac{1}{n}C'(A/n)\} < 0$ for each $\beta \in B$, there exists $A_* \in (0, n\bar{e})$ such that $dW_i(A_*/n)/dA = 0$.
Also notice

$$\begin{aligned}\frac{d^2W_i(A/n)}{dA^2} = \int_\Theta \Bigg(& \phi''(\mathbb{E}[u(x)|\theta]) \left\{ \mathbb{E}\left[u'(x) \left\{ D'(E;\beta) - \frac{1}{n}C'(A/n) \right\} \middle| \theta \right] \right\}^2 \\ & + \phi'(\mathbb{E}[u(x)|\theta])\mathbb{E}\Bigg[u''(x) \left\{ D'(E;\beta) - \frac{1}{n}C'(A/n) \right\}^2 \\ & - u'(x) \left\{ D''(E;\beta) + \frac{1}{n^2}C''(A/n) \right\} \Bigg| \theta \Bigg] \Bigg) g_*(\theta) d\theta,\end{aligned} \tag{B.2}$$

which is strictly negative because

$$\begin{aligned}&\frac{\phi''(\mathbb{E}[u(x)|\theta])}{\phi'(\mathbb{E}[u(x)|\theta])} \left\{ \mathbb{E}\left[u'(x) \left\{ D'(E;\beta) - \frac{1}{n}C'(A/n) \right\} \middle| \theta \right] \right\}^2 \\ &< \mathbb{E}\left[u'(x) \left(D''(E;\beta) + \frac{1}{n^2}C''(A/n) - \frac{u''(x)}{u'(x)} \left\{ D'(E;\beta) - \frac{1}{n}C'(A/n) \right\}^2 \right) \right]\end{aligned} \tag{B.3}$$

for each $\theta \in \Theta$. The left-hand side is less than or equal to zero while the right-hand side is strictly positive. Hence, $W_i(A/n)$ as a function of A is strictly concave, which implies A_* must be unique.

B.2 Derivation of equation 4.3

Notice first that since $f(\cdot|\theta)$ is normal,

$$\mathbb{E}\left[u(x_i)|\theta\right] = -\frac{1}{\alpha}\int_B e^{-\alpha(\bar{y}-\beta\delta E - C(a_i))} f(\beta|\theta)d\beta \tag{B.4}$$

$$= -\frac{1}{\alpha}e^{-\alpha(\bar{y}-C(a_i))}e^{\alpha\delta E\theta + \frac{1}{2}\alpha^2\delta^2E^2\sigma_u^2}, \tag{B.5}$$

and thus

$$\phi(\mathbb{E}\left[u(x_i)|\theta\right]) = -\frac{\alpha^{-(1+\xi)}}{1+\xi}e^{-\alpha(1+\xi)(\bar{y}-C(a_i)-\frac{1}{2}\alpha\sigma_u^2\delta^2E^2)}e^{\alpha(1+\xi)\delta E\theta}. \tag{B.6}$$

Normality of g_i then implies

$$V_i(a) = -\frac{\alpha^{-(1+\xi)}}{1+\xi}e^{-\alpha(1+\xi)(\bar{y}-C(a_i)-\frac{1}{2}\alpha\sigma_u^2\delta^2E^2)}\int_\Theta e^{\alpha(1+\xi)\delta E\theta}g_i(\theta)d\theta \tag{B.7}$$

$$= -\frac{\alpha^{-(1+\xi)}}{1+\xi}e^{-\alpha(1+\xi)v_i(a)}, \tag{B.8}$$

where

$$v_i(a) := \bar{y} - \delta\mu_i E - \frac{\delta^2}{2}\gamma_i E^2 - \frac{\nu}{2}a_i^2 \tag{B.9}$$

and $\gamma_i := \alpha\left[\sigma_u^2 + (1+\xi)\sigma_i^2\right]$.

B.3 Proof of Proposition 10

Choose v_1 such that $0 < v_1 < n^{-1}\mu_*$ and define $\{\mu_i\}_{i=1}^n$ by $\mu_i := \mu_* - i \cdot v_1$ for each i. Then $\{\mu_i\}_{i=1}^n$ satisfies (a). Also, put $\sigma_i^2 := (i+v_2)v_1$ for some $v_2 > 0$. Clearly, $\{\sigma_i^2\}_{i=1}^n$ satisfies (b).

Observe then

$$\mu_* - \bar{\mu} = \frac{v_1}{n}\sum_{i=1}^n i, \quad \bar{\sigma}^2 = \frac{v_1}{n}\sum_{i=1}^n i + v_1v_2, \quad \frac{\mu_* - \mu_i}{\sigma_i^2} = \frac{i}{i+v_2}. \tag{B.10}$$

Hence

$$\frac{\mu_* - \bar{\mu}}{\bar{\sigma}^2} = \frac{\frac{1}{n}\sum_{i=1}^n i}{\frac{1}{n}\sum_{i=1}^n i + v_2}, \quad \frac{1}{n}\sum_{i=1}^n \frac{\mu_* - \mu_i}{\sigma_i^2} = \frac{1}{n}\sum_{i=1}^n \frac{i}{i+v_2}. \tag{B.11}$$

We note

$$\lim_{v_2\to 0}\frac{\mu_*-\bar{\mu}}{\bar{\sigma}^2}=1, \quad \lim_{v_2\to 0}\frac{1}{n}\sum_{i=1}^{n}\frac{\mu_*-\mu_i}{\sigma_i^2}=1, \tag{B.12}$$

and

$$\frac{\partial}{\partial v_2}\left\{\frac{\mu_*-\bar{\mu}}{\bar{\sigma}^2}\right\}\Bigg|_{v_2=0}=-\left(\frac{1}{n}\sum_{i=1}^{n}i\right)^{-1}>-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{i}=\frac{\partial}{\partial v_2}\left\{\frac{1}{n}\sum_{i=1}^{n}\frac{\mu_*-\mu_i}{\sigma_i^2}\right\}\Bigg|_{v_2=0}, \tag{B.13}$$

where the inequality follows from the fact that the Harmonic mean is always smaller than the Arithmetic mean. Therefore

$$\lim_{v_2\to 0}\left\{\frac{\mu_*-\bar{\mu}}{\bar{\sigma}^2}-\frac{1}{n}\sum_{i=1}^{n}\frac{\mu_*-\mu_i}{\sigma_i^2}\right\}=0 \tag{B.14}$$

and

$$\frac{\partial}{\partial v_2}\left\{\frac{\mu_*-\bar{\mu}}{\bar{\sigma}^2}-\frac{1}{n}\sum_{i=1}^{n}\frac{\mu_*-\mu_i}{\sigma_i^2}\right\}\Bigg|_{v_2=0}>0. \tag{B.15}$$

This means that there exists $\bar{v}_2>0$ such that for any $v_2\in(0,\bar{v}_2)$

$$\frac{\mu_*-\bar{\mu}}{\bar{\sigma}^2}-\frac{1}{n}\sum_{i=1}^{n}\frac{\mu_*-\mu_i}{\sigma_i^2}>0, \tag{B.16}$$

which completes the proof.

B.4 Proof of Proposition 11

The result that heterogeneity is required both in μ_i and σ_i^2 is immediate from contradiction argument. To see the last part of the proposition, notice (4.19) is equivalent to

$$\left(\frac{1}{n}\sum_{i=1}^{n}\frac{\mu_*-\mu_i}{\mu_*-\bar{\mu}}\frac{1}{\sigma_i^2}\right)^{-1}>\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2. \tag{B.17}$$

On the other hand, since the Harmonic mean is always smaller than the Arithmetic mean,

$$\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2\geq\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma_i^2}\right)^{-1}, \tag{B.18}$$

where the inequality must be strict because $\sigma_i^2 \neq \sigma_j^2$. Therefore, we have

$$\left(\frac{1}{n} \sum_{i=1}^{n} \frac{\mu_* - \mu_i}{\mu_* - \bar{\mu}} \frac{1}{\sigma_i^2} \right)^{-1} > \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \right)^{-1}, \tag{B.19}$$

from which the result follows.