Distortion Risk Measures in Action

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Abstract

The notion of risk measure is indispensable for financial risk management. Although the value-at-risk (VaR) is still used in practice, it is not coherent in the sense of Artzner et al. [2], so it is natural to seek for other risk measures fulfilling some desirable theoretical properties. According to Kusuoka [6], the class of distortion risk measures[a] is the one we should use if we stick to the law invariance and comonotonic additivity in addition to the four axioms of coherence, and is broad enough to fully express agents' subjective assessment of risk. If one is to apply this risk measure to practical financial risk management problems, it is necessary to pick one distortion function suitable for his/her purpose. One's choice from among these distortion risk measures should be based on his/her attitude towards risk, but we need some quantitative guidelines helping our understanding of the characteristics of those risk measures.

There are many known parametric families of distortion functions to construct distortion risk measures (including the renowned expected shortfall (ES), and the author's proposals in Tsukahara [12]). One of the main purpose of the present paper is to give a comprehensive comparative study of several families of distortion risk measures in several aspects. VaR is also included as a target for comparison in view of its practical popularity.

For the class of distortion risk measures, a natural estimator has the form of L-statistics. In Tsukahara [14], we have investigated the large sample properties of general L-statistics based on weakly dependent data and apply them to our estimator. Under certain regularity conditions, which are somewhat weaker than the ones found in the literature, we prove that the estimator is strongly consistent and asymptotically normal. Furthermore we give a consistent estimator for its asymptotic variance using spectral density estimators of a related stationary sequence. The behavior of the estimator is examined using simulation in a simple inverse-gamma autoregressive stochastic volatility model. It is found both theoretically and by simulation study that the estimator always suffers a negative bias. We will discuss bias correction methods in the i.i.d. case and the possibility of their extension to the dependent case using bootstrap methods with dependent data. Also, we will indicate how the asymptotic results for our estimator can be extended to the estimator for law invariant risk measures, which are obtained by dropping comonotonic additivity requirement.

Another issue we would like to address in this paper is backtesting. The purpose

[a]They are called spectral risk measures in Acerbi [1], and weighted V@R in Cherny [3].

of backtesting is usually twofold: to monitor the performance of the model and estimation methods for risk measurement, and to compare relative performance of the models and methods (see e.g., McNeil et al. [8]). We focus on the former in this paper. Statistically, it is just a form of cross validation; the ex ante risk measure forecast from the model is compared with the ex post realized portfolio loss. In the case of VaR, a popular procedure for backtesting VaR depends on the number of *VaR violations*; we say that a VaR violation occurred when the loss exceeds VaR. Extending a backtesting procedure for the renowned expected shortfall (ES) suggested in McNeil and Frey [7], we construct, based on a simple observation, a backtesting procedure for distortion risk measures, and check its effectiveness in a simulation study using ES and also proportional odds distortion risk measure. Some related issues such as (non-)elicitability of distortion risk measures and a consistency notion of Davis [5] will be discussed in passing.

Capital allocation is an important application of risk measure theory. The problem is how to allocate the overall risk capital to the individual investment opportunities *in a fair way*, and the Euler capital allocation emerged with certain economical justifications (see Tasche [9] and Denault [4]). The Euler capital allocation based on distortion risk measures are easy to compute and widely applicable (c.f. Tsanakas and Barnett [11], Tsanakas [10]), and we evaluate numerically how the Euler allocations based on different distortion risk measures (including proportional hazards distortion risk measure by Wang [15]) perform under several dependence scenarios using some copulas.

We remark that the portfolio optimization problem with distortion risk measures is investigated in Acerbi [1]. We conclude with some discussions on the multiperiod extension of distortion risk measures and its associated problems.

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