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Metzler Functions and the Shortest-Path Problem

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Abstract

Metzler functions arise in various kinds of system optimization problems such as the stable matching of two-sided markets and the α-cores of games with punishment-dominance relations. In this paper, we discuss the shortest-path problem as an application of the theory of Metzler functions. We derive the Moore-Bellman-Ford algorithm for the shortest-path problem.

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Metzler Functions and the Shortest-Path Problem

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Abstract

Metzler functions arise in various kinds of system optimization problems such as the stable matching of two-sided markets and the α-cores of games with punishment-dominance relations. In this paper, we discuss the shortest-path problem as an application of the theory of Metzler functions. We derive the Moore-Bellman-Ford algorithm for the shortest-path problem.

1 Introduction

This discussion is based on the theory of *Metzler functions*. A Metzler function on a finite set N is defined as an N-dimensional vector-valued function from N-dimensional space such that any increase in the i-th variable does not decrease the j-th coordinate of its value if i and j are distinct elements in N.

Metzler functions play an important role in algorithmic models. In the context of game theory, Masuzawa (2008) considered an inequality system described by a Metzler function and an N-vector $(a^i)_{i \in N}$ such that the i-th coordinate of the value of the function is larger than a^i. A fundamental result is that the maximum solution of the inequality system can be obtained by a simple algorithm, which decides whether or not $(a^i)_{i \in N}$ is in the

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α-core of a strategic game with punishment-dominance relations. Recently, it has been shown that inequality systems of the same kind arise in various system optimization problems, and the algorithms for determining the maximum solutions are well-known algorithms: the tâtonnement algorithm for determining the equilibrium price of markets with gross substitutability (Masuzawa 2012c), Meek's tax-cut algorithm for determining the winner of an election by single transferable voting (Masuzawa 2012a), and the deferred acceptance algorithm for determining the stable matching of two-sided markets (Masuzawa 2012b).

Metzler functions are related to super-modular functions. In fact, the differential of a super-modular function is a Metzler function. However, the converse does not hold because the partial differential equation described by a Metzler function is not necessarily integrable in the sense that $\partial f_i/\partial x_j \neq \partial f_j/\partial x_i$. In this sense, the theory of Metzler functions is applicable to a wider range of cases than the theory of the super-modular functions. Further, Metzler functions are sometimes more intuitive and easier to determine than super-modular functions. In such cases, a Metzler function is given as primitive data to us, while the corresponding super-modular function is constructed from the data.

In this paper, we analyze the Moore-Bellman-Ford algorithm for the shortest-path problem. We consider a potential assignment to all vertices of the network and define *the reduced length* of any edge by its original length plus the difference between the potentials of its head and tail. We define a function that assigns to any potential assignment a vector such that for any vertex, the corresponding coordinate is the minimum reduced length of outgoing edges of it. This function is a Metzler function because if the potential of a vertex increases, the value for another vertex does not decrease. We show that the shortest length from any vertex to the destination is characterized by the maximum potential assignment such that for any vertex, either its potential is $-\infty$ or the corresponding coordinate of the Metzler function is non-negative. Then, following the theory of Metzler functions, we introduce the Moore-Bellman-Ford algorithm as an instance of the algorithm for determining the maximum solution of the inequality system.

The remainder of this paper is organized as follows. We discuss the preliminary results in section 2. First, in subsection 2.1, we introduce Metzler functions. Next, in subsection 2.2, we briefly review the algorithmic theory of Metzler functions. In subsection 2.3 and 2.4, we discuss their relations to super-modular functions and Knaster-Tarski fixed-point theorem. In section 3, we discuss the shortest-path problem. First, in subsection 3.1, we formalize the shortest-path problems. Next, in subsection 3.2, we introduce the potential assignment and the associated Metzler function, and then we derive the Moore-Bellman-Ford algorithm. In section 4, we conclude this paper with comments on another theoretical approach to the algorithms for the shortest-path problem.

2 Metzler Function

2.1 Definition

Let N be a finite set. For all $i \in N$, let X^i and Y^i be sets with linear orders $\succsim_i$ and $\geq_i$[1], respectively. We write $x^i \succ_i w^i$ for all $x^i, w^i \in X^i$ iff it does

[1] A binary relation R on set X is a linear order iff it satisfies the following conditions.

Reflexivity : for all $x \in X$, xRx.

Transitivity : for all $x, y, z \in X$, xRz if xRy and yRz.

Antisymmetry : for all $x, y \in X$, $x = y$ if xRy and yRx.

Totality : for all $x, y \in X$, either xRy or yRx.

not hold that $w^i \gtrsim_i x^i$. Similarly, we write $y^i >_i z^i$ for all $y^i, z^i \in Y^i$. For any $S \subset N$, we abbreviate $\prod_{i \in S} X^i$ as X^S and $\prod_{i \in S} Y^i$ as Y^S. A typical element in X^S is denoted by x^S, and the i-th coordinate is denoted by x^i or $x(i)$. The projection of any given $x^S \in X^S$ onto X^T is denoted by x^T for $T \subset S$. On the other hand, given $x^S \in X^S$ and $x^T \in X^T$ such that $S \cap T = \emptyset$, by (x^S, x^T), we refer to an element of $X^{S \cup T}$ such that x^S and x^T are its projections. For all $w^S, x^S \in X^S$, we write $w^S \gtrsim x^S$ iff $w^i \gtrsim_i x^i$ for all $i \in S$.

Then, we can state the definition of the main concept.

Definition 1 *A function $f : X^N \to Y^N$ is called a Metzler function iff for all $i, j \in N$ and $x^N \in X^N$ and $z^i \in X^i$,*

$$f_j(x^N) \geq_j f_j(x^{N \setminus \{i\}}, z^i) \textit{ if } x^i \gtrsim_i z^i \textit{ and } i \neq j.$$

In other words, $f : X^N \to Y^N$ is a Metzler function if for all $i, j \in N$ $(i \neq j)$ and $x^{N \setminus \{i\}} \in X^{N \setminus \{i\}}$,

$$f_j(\cdot, x^{N \setminus \{i\}}) : x^i \mapsto f_j(x^i, x^{N \setminus \{i\}})$$

is order-preserving. Metzler functions are named after Lloyd Metzler[2] (1913-1980), who developed the theory of systems characterized by a square matrix $(a_{ij})_{i,j \in N}$ such that $a_{ij} \geq 0$ if $i \neq j$, which is called a *Metzler matrix*. When each of X^i and Y^i is the real line and f is differentiable, f is a Metzler function iff the Jacobian matrix of f, $(\partial f_i / \partial x^j)_{i,j \in N}$, is a Metzler matrix.

2.2 Algorithm

Throughout this paper, we assume that X^i is a finite set for all $i \in N$. For all nonempty $A^i \subset X^i$, by $\max A^i$ and $\min A^i$, we refer to the elements

[2]See Metzler (1973).

$z^i \in A^i$ and $w^i \in A^i$ such that $z_i \succsim_i x^i$ for all $x^i \in A^i$ and $x^i \succsim_i w^i$ for all $x^i \in A^i$, respectively. Further, for all $x^i \in X^i \setminus \{\max X^i\}$, by $up(x^i)$, we refer to $\min\{w^i | w^i \succ_i x^i\}$. Similarly, we use the notation, $down(x^i)$.

Problem 1

Input: a Metzler function $f : X^N \to Y^N$ *and* $y^N \in Y^N$,

Output: the maximum of $x^N \in X^N$ *such that*

for all $i \in N$, $f_i(x^N) \geq_i y^i$ *or* $x^i = \min X^i$.

We say that $x^N \in X^N$ is *feasible* for Problem 1 iff $f_i(x^N) \geq_i y^i$ or $x^i = \min X^i$ for all $i \in N$.

Algorithm 1 (Masuzawa 2008)

1. $\boldsymbol{x}^i := \max X^i$ for all $i \in N$;
2. $\boldsymbol{S} := \{i | \boldsymbol{x}^i \neq \min X^i, \text{ and } y^i >_i f_i(\boldsymbol{x}^N)\}$;
3. if $\boldsymbol{S} = \emptyset$, then stop;
   otherwise, for all $i \in \boldsymbol{S}$, choose $\boldsymbol{z}^i \in X^i$ such that $\boldsymbol{x}^i \succ_i \boldsymbol{z}^i$ and
   $\boldsymbol{z}^i \succsim_i \max\{x^i | \boldsymbol{x}^i \succ_i x^i \text{ and either } f_i(\boldsymbol{x}^{N\setminus\{i\}}, x^i) \geq_i y^i \text{ or } x^i = \min X^i\}$;
4. $\boldsymbol{x}^i := \boldsymbol{z}^i$ for all $i \in \boldsymbol{S}$;
5. Jump to 2.

First, in step 3, we can chose $\boldsymbol{z}^i$ by the following rule while some specific cases, more efficient rules are applicable:

Stepwise rule. $\boldsymbol{z}^i := down(\boldsymbol{x}^i)$ for all $i \in \boldsymbol{S}$.

Second, since $\boldsymbol{x}^i$ is monotonically decreasing, the algorithm above necessarily terminates. Further, for all feasible x^{*N}, $\boldsymbol{x}^N \succsim x^{*N}$ at any time during the computation. To see this, assume that $i \in \boldsymbol{S}$. If $\boldsymbol{x}^N \succsim x^{*N}$, then

$f_i(\boldsymbol{x}^{N\setminus\{i\}}, x^{*i}) \geq_i f_i(x^{*N})$, and hence, $\boldsymbol{z}^i \succsim x^{*i}$. Therefore, we have the following:

Proposition 1 (Masuzawa 2008) *Algorithm 1 terminates with the solution x^{*N} of Problem 1.*

2.3 Super-Modularity

Metzler functions are related to super-modularity. Let $\Re$ be the set of the real numbers. A function $g : X^N \to \Re$ is a *super-modular function* iff $g(x^N) + g(y^N) \leq g(x^N \vee y^N) + g(x^N \wedge y^N)$ for all $x^N, y^N \in X^N$, where $(x^N \vee y^N)^i := \max\{x^i, y^i\}$ and $(x^N \wedge y^N)^i := \min\{x^i, y^i\}$. It is well known that the differential of a super-modular function $g : \Re^N \to \Re$, denoted by $f_g : \Re^N \to \Re^N$, is a Metzler function. Conversely, given a continuously differentiable Metzler function, $f : \Re^N \to \Re^N$, there exists $g : \Re^N \to \Re$ such that $\partial g/\partial x^i = f_i$ for all $i \in N$ iff $\partial f_i/\partial x^j = \partial f_j/\partial x^i$ for all $i, j \in N$. In this sense, the theory of Metzler functions covers a wider range of cases than theory of super-modular functions.

In game theory, a strategic form game $\big(N, (X^i)_{i\in N}, (u^i)_{i\in N}\big)$ is called *a game with punishment-dominance relations* iff the payoff function $u^N : X^N \to Y^N$ is a Metzler function, where X^i is the set of strategies possible for player $i \in N$, and Y^i is the set of possible utility levels. On the other hand, a TU coalitional game $v : 2^N \to \Re$ with $v(\emptyset) = 0$ is called *convex* iff it is super-modular. For a convex TU coalitional game v, consider a strategic game defined by

$$N = \{1, 2, \ldots, n\}, \;\; X^i = \{c^i, d^i\} \text{ for all } i \in N,$$

$$u^i(x^N) := \begin{cases} v(\{j | x_j = c^j, i \geq j\}) - v(\{j | x_j = c^j, i > j\}) \\ v(\{i\}) \end{cases} \qquad \text{for all } i \in N.$$

Then, it is a game with punishment-dominance relations. Further, for any given coalitional game v, the strategic game constructed above satisfies the following:

$$\text{for all } S \subset N, \qquad \max_{x^S \in X^S} \min_{x^{N\setminus S} \in X^{N\setminus S}} \sum_{i \in S} u^i(x^S, x^{N\setminus S}) = v(S).$$

In other words, any TU convex game is constructed from a strategic form game with punishment-dominance relations. However, note that for any given strategic form game with punishment-dominance relations, the function v defined by for all $S \subset N$,

$$v(S) := \max_{x^S \in X^S} \min_{x^{N\setminus S} \in X^{N\setminus S}} \sum_{i \in S} u^i(x^S, x^{N\setminus S})$$

is not necessarily convex (Masuzawa, 2004).

2.4 Monotonicity of Knaster-Tarski Theorem

Consider the case when $X^N = Y^N$. A Metzler function $f : X^N \to X^N$ is a monotonic function in the sense of the Knaster-Tarski theorem if for all $i \in N$, $x^N \in X^N$, and $z^i \in X^i$,

$$f_i(x^N) \geq_i f_i(x^{N\setminus\{i\}}, z^i) \text{ if } x^i \succsim_i z^i.$$

Conversely, the solution of Problem 1 is characterized by the maximum fixed point of a monotonic function, $M : X^N \to X^N$, defined by

$$M^i(x^N) := \max\{w^i \in X^i | f_i(x^{N\setminus\{i\}}, w^i) \geq_i y^i \text{ or } w^i = \min X^i\} \text{ for all } i \in N.$$

Proposition 2 *$M : X^N \to X^N$ is a monotonic function and its maximum fixed point is the solution of Problem 1.*

Sometimes, $M^i(x^N)$ has a simple form and the application of the Knaster-Tarski fixed-point theorem is instructive for us. Adachi (2000) demonstrated the complete lattice theorem of two-sided markets via the Knaster-Tarski fixed-point theorem.

3 Shortest-Path Problem

3.1 Formalization of the Problem

First, we introduce *the shortest-path problem.* For a comprehensive discussion on this topic, see Korte and Vygen (2008, Chapter 7).

We define $\tilde{\Re}$ by $\tilde{\Re} := (\Re \cup \{\infty, -\infty\})^3$. Let V be a finite set of vertices, and let E be a finite set of directed edges, where an edge $e \in E$ links an ordered pair of vertices denoted by $(\psi_+(e), \psi_-(e))$. We assume that $\psi_+(e) \neq \psi_-(e)$ for all $e \in E$. A directed graph is identified by a list $G = (V, E, \psi_+, \psi_-)$. Further, we consider *the length* of an edge $e \in E$, which is given by the *length function* $c : E \to \Re$. For all $x, y \in V$, a sequence of edges

$$P : e_1, e_2, \dots, e_k$$

is called a *walk from x to y* iff (i) $v_0 := \psi_+(e_1) = x$ and $v_k := \psi_-(e_k) = y$, and (ii) $v_j := \psi_-(e_j) = \psi_+(e_{j+1})$ for $i = 1, 2, \dots, k-1$. We refer to k by $|P|$. The *length of the walk* is defined by $c(P) := \sum_{i=1}^{k} c(e_i)$. A walk, $e_1, e_2, \dots, e_k$, is *a path from x to y* iff it satisfies (i), (ii), and (iii) $v_i \neq v_j$ if $i \neq j$. A walk, $Q : e_1, e_2, \dots, e_k$, is called a *cycle* if $\psi_-(e_k) = \psi_+(e_1)$. It is a *negative cycle* iff $c(Q) < 0$. A *shortest walk from v to v^** is a walk from v to v^* that minimizes $c(P)$. A shortest walk is called a shortest path if it is a path. Note that if a shortest walk Q comprises a cycle, then $c(Q) = 0$. Thus, we can obtain a shortest path if there exists a shortest walk.

[3] We assume the following laws for $c \neq \infty, -\infty$:

$$(\infty + c) = \infty \qquad (-\infty + c) = -\infty \qquad -(-\infty) = \infty$$
$$-(\infty) = -\infty \qquad (\infty + \infty) = \infty \qquad (-\infty + \infty) = 0$$

The shortest-path problem is formalized as follows:

Problem 2

Input: A directed graph $G = (V, E, \psi_+, \psi_-)$*, a vertex* $v^* \in V$*, and a length function* $c : E \to \Re$*, where* $|V| = m$ *and* $|E| = n$

Output: A function $I : V \to 2^E$ *and a function* $d : V \to \Re$ *such that*

1. *for all* $v \in V$*,*

$$d(v) = \begin{cases} c(P) & \text{if there exists a shortest path } P \text{ from } v \text{ to } v^*, \\ +\infty & \text{if there exits no path from } v \text{ to } v^*, \\ -\infty & \text{if there exist a path from } v \text{ to } v^* \text{ and} \\ & \text{a path from } v \text{ to a vertex in a negative cycle,} \end{cases}$$

2. *for all* $e \in E$ *and all* $v \in V$ *such that* $d(v) \in \Re$ *,*

   $e \in I(v)$ *iff* $\psi_+(e) = v$ *and* e *is a part of the shortest-path from* v *to* v^**.*

3.2 Potential and the Algorithm

An element $\pi^V = (\pi(v))_{v \in V}$ of $\Re^V$ is called a *potential assignment* and $\pi(v)$ the potential of v for all $v \in V$. By a given potential assignment π^V, the *reduced length of edge* e *under* π^V is defined as

$$c(e, \pi^V) := \big(c(e) + \pi(\psi_-(e))\big) - \pi(\psi_+(e)).$$

By $L^v(\pi^V)$ we denote the minimum reduced length of $e \in E$ such that $\psi_+(e) = v$ under π^V. In other words,

$$L^v(\pi^V) := \quad \min\{c(e) + \pi(\psi_-(e)) | \psi_+(e) = v\} - \pi(v).$$

Obviously, we get the following theorem:

Theorem 1 $L : \tilde{\Re}^V \to \tilde{\Re}^V$ *is a Metzler function.*

Since our interest is in the shortest path, we can restrict the domain of $\pi(v)$ to $X^v := \{c(P)|P \text{ is a walk from } v, |P| \leq m\} \cup \{-\infty, +\infty\}$.

A potential assignment is useful to detect negative cycles from which v^* is accessible. Assume that (i) for all $e \in E$, $c(e, \pi^V) \geq 0$, and (ii) $\pi(v^*) \leq 0$. Define $V' := \{v|\pi(v) \neq -\infty\}$. For all $v \in V'$, if there exists a walk from v to v^*, then $\pi(v) \neq \infty$. In fact, if $\pi(v) = \infty$ and there exits a walk from v to v^*, then $\pi(\psi_-(e)) = \infty$ for all edges e of the walk and thus $\pi(v^*) = \infty$, which contradicts (ii). Similarly, for all $v \in V'$, v has no walk to v' such that $\pi(v') = -\infty$. It follows that, $\sum_{i=1}^{|P|} c(e_i, \pi^V) = -\pi(v) + \pi(v^*) + \sum_{i=1}^{|P|} c(e_i) \geq 0$ for any walk P from v to v^*. Thus, for all $v \in V'$, either v has the shortest path to v^* or it has no walk to v^*. If π^V is the solution of the shortest path problem, then $c(e, \pi^V) \geq 0$ for all $e \in E$ and V' is maximized. By definition, the maximum solution of the following problem also maximizes V'.

Problem 3

Input: A directed graph $G = (V, E, \psi_+, \psi_-)$, *a vertex* $v^* \in V$, *and a length function* $c : E \to \Re$.

Output: the maximum of $\pi^V \in X^V$ *such that*

$$
\begin{array}{llcl}
(i) & \pi(v^*) \leq 0, & \text{and} & \pi(v) \leq \infty \text{ for all } v \in V \setminus \{v^*\}, \text{and} \\
(ii) & L^v(\pi^V) \geq 0 & \text{or} & \pi(v) = -\infty \text{ for all } v \in V.
\end{array}
$$

Let d^V be the solution of the shortest-path problem and π_*^V be the maximum of Problem 3. Then, we have the following result.

Theorem 2 $d^V = \pi_*^V$.

Proof. Obviously, d^V is a feasible solution of Problem 3. By the definition of the maximum, we see that $d^V \leq \pi_*^V$. To see that $\pi_*^V \leq d^V$, first consider the case in which there exists a shortest path, P, from v to v^*. By induction on $|P|$, we can obtain that $\pi_*(v) \leq d(v)$. From the discussion above for detection of negative cycles, if $\pi_*(v) > -\infty$ then $d(v) > -\infty$. It follows that in the remainder case, where $d(v) = -\infty$, $\pi_*(v) = -\infty$. □

From the discussion of subsection 2.2, we can derive the following algorithm to find the solution of Problem 3, which turns out to be the Moore-Bellman-Ford algorithm to find the shortest path.

Algorithm 2

1. $k := 0$; $\boldsymbol{\pi}(v^*) := 0$; $I(v^*) := \{v^*\}$;
   $\boldsymbol{\pi}(v) := \infty$ and $I(v) := \emptyset$ for all $v \in V \setminus \{v^*\}$;
2. $k := k + 1$; $\boldsymbol{S} := \left\{ v \in V | L(\boldsymbol{\pi}^V, v) < 0 \text{ and } \boldsymbol{\pi}(v) > -\infty \right\}$;
3. If $\boldsymbol{S} = \emptyset$, then stop;
4. for all $v \in \boldsymbol{S}$,
   $$I(v) := \arg\min \{c(e) + \boldsymbol{\pi}(\psi_-(e)) | \psi_+(e) = v\},$$
   $$\boldsymbol{\pi}(v) := \max \{x \in X^v | x \leq \min \{c(e) + \boldsymbol{\pi}(\psi_-(e)) | \psi_+(e) = v\}\};$$
5. Jump to 2.

Theorem 3 *Let π_*^V be the solution of Algorithm 2. Then, π_*^V is the maximum solution of Problem 3.*

To rewrite Algorithm 2 into the Moore-Bellman-Ford Algorithm, we present the following lemma.

Lemma 1 *In step 2 of round $k \leq m$, for all $v \in V$, the following two claims hold.*

1. $\boldsymbol{\pi}(v) \geq \min\{c(e) + \boldsymbol{\pi}(\psi_-(e)) | \psi_+(e) = v\}$.

2. the following three statements are equivalent.

   (a) $\min\{c(e) + \boldsymbol{\pi}(\psi_-(e)) | \psi_+(e) = v\}$ *is the length of the shortest walk P from v to v^* such that $|P| \leq k$,*

   (b) there exists a walk P from v to v^* such that $|P| \leq k$,

   (c) $\min\{c(e) + \boldsymbol{\pi}(\psi_-(e)) | \psi_+(e) = v\} \neq \infty$.

Proof. When $k = 1$, both the claims are obviously true. Assume that in the cases of $1, 2, \ldots,$ and k, both the claims hold for all $v \in V$. By induction hypothesis, in step 2 of round $k+1$, for any edge e such that $\psi_+(e) = v$, the following three statements are equivalent: (i) $\boldsymbol{\pi}(\psi_-(e))$ is the length of the shortest walk P from $\psi_-(e)$ to v^* such that $|P| \leq k$, (ii) such a walk exists, (iii) $\boldsymbol{\pi}(\psi_-(e)) \neq \infty$. Thus, both the claims hold for $k+1$. □

From Lemma 1, we obtain the Moore-Bellman-Ford Algorithm.

Proposition 3 *In step 4 of Algorithm 2, $\pi(v)$ is updated as follows:*

4'. for all $v \in S$,

$$\boldsymbol{\pi}(v) := \begin{cases} \min\{c(e) + \boldsymbol{\pi}(\psi_-(e)) | \psi_+(e) = v\}, & \text{if } k \leq m \\ -\infty & \text{if } k > m. \end{cases}$$

Theorem 4 (Moore-Bellman-Ford) *The algorithm obtained by substituting step 4' for step 4 of Algorithm 2 terminates with the solution of the shortest-path problem within $O(mn)$.*

4 Concluding Remarks

4.1 Fixed point approach

Misra (2001) showed that the solution of the shortest-path problem without a negative circuit is the maximum solution of the following system of equations:

$$\text{for all } v \in V, \quad \pi(v) = \min \{c(e) + \pi(\psi_-(e)) | \psi_+(e) = v\}.$$

From this fact, Misra derived Dijkstra's Algorithm. From this observation, Theorems 2 and 3 presented above can be directly obtained.

4.2 Discrete convex analysis

Murota and Shioura (2012) examined Dijkstra's Algorithm from the viewpoint of discrete convex analysis. They describe the algorithm as a steepest ascent algorithm for L-concave minimization. Discrete convex analysis is a predominant theory covering many areas of combinatorial optimization, which is based on matroid theory and super-modular theory. To illustrate the relation of two theories, consider the analysis of two-sided markets. Metzler function plays a central role in that area since "the substitute property" is equivalent to the definition of Metzler function (Masuzawa 2012). On the other hand, discrete convex analysis provides more detailed information especially on the relation between utility function and the price system (see Fujishige and Yang 2003; Farooq and Shioura 2005). In this sense, I think, discrete convex analysis is a detailed theory while Metzler function provides a general theory.

References

H. Adachi (2000), On a Characterization of Stable Matchings, Economics Letters 68, 43-9

R. Farooq and A. Shioura (2005), A Note on the Equivalence between Substitutability and $M^\natural$-convexity, *Pacific Journal of Optimization 1*, 243-252.

S. Fujishige and Z. Yang (2003), A Note on Kelso and Crawford's Gross Substitutes Condition, *Mathematics of Operations Research 28*, 463-469.

B. Korte and J. Vygen (2008), *Combinatorial Optimization: Theory and Algorithms, 4-th Edition*, Springer-Verlag: Berlin Heidelber.

T. Masuzawa (2004), Punishment Strategies Make the α-Coalitional Games Ordinally Convex and Balanced, *International Journal of Game Theory 32*, 479-483.

T. Masuzawa (2008), Computing the Cores of Strategic Games with Punishment-Dominance Relations, *International Journal of Game Theory 38*, 185-201.

T. Masuzawa (2012a), STV Elections by Feedback Counting: An Approach from Cooperative Game Theory, Keio/Kyoto Global COE Discussion Paper Series, DP2012-015

T. Masuzawa (2012b), Punishment-Dominance Condition on Stable Two-sided Matching Algorithms, Keio/Kyoto Global COE Discussion Paper Series DP2012-018

T. Masuzawa (2012c), A Simple and Constructive Proof of the Existence of a Competitive Price Under the Gross Substitute Property, Keio/Kyoto Global COE Discussion Paper Series DP2012-022

L. Metzler (1973), *Collected Papers*, Cambridge, Massachusetts: Harvard University Press.

J. Misra (2001), A Walk over the Shortest Path: Dijkstra's Algorithm Viewed as Fixed-Point Computation, *Information Processing Letters 77*, 197–200.

K. Murota and A. Shioura (2012), Dijkstra's Algorithm and L-Concave Function Maximization, METR 2012-05, Department of Mathematical Informatics, University of Tokyo.