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Abstract

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Abstract

This note is an extended version of Arai [2], in which convex risk measures describing the upper and lower bounds of a good deal bound are studied for the case where the set of 0-attainable claims is convex as an extension of Arai and Fukasawa [3]. Here a good deal bound is defined as a subinterval of a no-arbitrage pricing bound. An outline of good deal bounds is given firstly for the readers who are not familiar with good deal bounds. In addition, many examples of convex markets are also introduced; and precise proofs for all mathematical results are provided.

0 An outline of good deal bounds

For a given contingent claim in an incomplete financial market, its price is not determined uniquely under the no-arbitrage framework. Only a pricing bound, called a no-arbitrage pricing bound, is provided. Now, we give a concrete explanation. Let L be a linear space of measurable functions defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that L represents the set of all possible future cash-flows. We describe our market with $M \subset L$ the set of 0-attainable claims, that is, future payoffs which investors can replicate completely with 0 initial cost. Roughly speaking, the no-arbitrage pricing bound for claim $x \in L$ is given as

$$\left[\inf_{Q \in \mathcal{Q}_0} \mathbb{E}_Q[x], \sup_{Q \in \mathcal{Q}_0} \mathbb{E}_Q[x] \right], \tag{0.1}$$

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where $\mathcal{Q}_0 := \{Q \ll \mathbb{P} | \mathbb{E}[m] \leq 0 \text{ for any } m \in M\}$, and $\mathbb{E}_Q$ means the expectation under Q. We can think of $\mathcal{Q}_0$ as the set of all martingale measures. Remark that we do not care about the integrability condition of $Q \in \mathcal{Q}_0$ in this section to simplify our argument. In general, the no-arbitrage pricing bound is too wide to be useful as the collection of candidate prices from a practical point of view. Thus, we focus on narrowing the interval of candidate prices. Now, we call a "too good" price either for a seller or a buyer a good deal price. We construct then a sharper pricing bound, called a good deal bound by excluding good deal prices from the no-arbitrage pricing bound. Whether a price is a good deal, depends on the investor's risk preference.

0.1 Good deal bounds induced by subsets of $\mathcal{Q}_0$

For a subset $\widehat{\mathcal{Q}} \subset \mathcal{Q}_0$, the interval

$$\left[\inf_{Q \in \widehat{\mathcal{Q}}} \mathbb{E}_Q[x], \sup_{Q \in \widehat{\mathcal{Q}}} \mathbb{E}_Q[x]\right] \tag{0.2}$$

forms a good deal bound. This is a typical way to construct a sharper pricing bound. In this subsection we illustrate such good deal bounds. Research in this direction has been undertaken by Cochrane and Saá-Requejo [14], which suggested good deal bounds based on the Sharpe ratio. Here, the Sharpe ratio $\mathrm{SR}(x; Q)$ (from buyer's view) for $x \in L$ and $Q \in \mathcal{Q}_0$ is defined as

$$\mathrm{SR}(x; Q) := \frac{\mathbb{E}[x] - \mathbb{E}_Q[x]}{\sqrt{\mathrm{Var}(x)}}.$$

They formulated the upper bound of a good deal bound of x for a given level $\delta > 0$ by excluding measures $Q \in \mathcal{Q}_0$ satisfying $\mathrm{SR}(x; Q) \geq \delta$. In other words, when $\mathrm{SR}(x; Q) \geq \delta$, the value $\mathbb{E}_Q[x]$ is regarded as a good deal price for a buyer. Thus, it should be excluded from the candidate prices of x. Noting that $|\mathrm{SR}(x, Q)| \leq \sqrt{\mathrm{Var}(\frac{dQ}{d\mathbb{P}})}$ holds, we can represent the Sharpe ratio based good deal bound as (0.2) for $\widehat{\mathcal{Q}} = \{Q \in \mathcal{Q}_0 | \mathrm{Var}(\frac{dQ}{d\mathbb{P}}) \leq \delta^2\}$. In addition, Björk and Slinko [10] extended this methodology to continuous time models whose asset price has jumps.

Example 0.1 Consider a one-period trinomial model being composed of one riskless asset with zero interest rate and one risky asset. For $t = 0, 1$, let S_t be the price of the risky asset at time t. Denoting $\Omega = \{\omega_1, \omega_2, \omega_3\}$, we set

$$S_0 = 96, \qquad \begin{cases} S_1(\omega_1) = 120, \\ S_1(\omega_1) = 108, \\ S_1(\omega_1) = 80, \end{cases} \qquad \begin{cases} \mathbb{P}(\{\omega_1\}) = 1/3, \\ \mathbb{P}(\{\omega_2\}) = 2/21 \\ \mathbb{P}(\{\omega_3\}) = 4/7. \end{cases}$$

Note that S is a martingale under $\mathbb{P}$. Now, we consider a call option with strike price 110. Its no-arbitrage pricing bound is $[0, 4]$. Next, we calculate the Sharpe ratio based good deal bounds. For an equivalent martingale measure $Q \sim \mathbb{P}$, we denote $q_i := Q(\{\omega_i\})$ for $i = 1, 2, 3$. Note that the price of the call option under Q is given as $10q_1$, and $\mathrm{Var}(\frac{d\mathbb{P}}{dQ}) = \frac{11}{4}(3q_1 - 1)^2$. For level $\delta \in (0, \frac{\sqrt{11}}{10})$, Q satisfies $\mathrm{Var}(\frac{d\mathbb{P}}{dQ}) \leq \delta^2$ if and only if

$$-\frac{2\delta}{3\sqrt{11}} + \frac{1}{3} \leq q_1 \leq \frac{2\delta}{3\sqrt{11}} + \frac{1}{3}.$$

For example, taking $\delta = 0.3$, the corresponding good deal bound is given as $[\frac{10}{3} - \frac{2}{\sqrt{11}}, \frac{10}{3} + \frac{2}{\sqrt{11}}]$, namely, $[2.730, 3.936]$ approximately.

Next, Bernardo and Ledoit [5] considered, instead of the Sharpe ratio, the gain-loss ratio for x defined as $\mathbb{E}[x^+]/\mathbb{E}[x^-]$. Roughly speaking, we have

$$\sup_{m \in M} \frac{\mathbb{E}[m^+]}{\mathbb{E}[m^-]} = \inf_{Q \in \mathcal{Q}_0} \frac{\operatorname{esssup} \frac{dQ}{d\mathbb{P}}}{\operatorname{essinf} \frac{dQ}{d\mathbb{P}}}.$$

Thus, denoting $\widehat{\mathcal{Q}} = \{Q \in \mathcal{Q}_0 | \operatorname{esssup} \frac{dQ}{d\mathbb{P}} / \operatorname{essinf} \frac{dQ}{d\mathbb{P}} \leq \delta\}$, we can construct a good deal bound through (0.2). Moreover, we introduce utility-based good deal bounds, which suggested by Černý [12] firstly. Let U be a utility function, that is, an increasing continuous concave function from $\mathbb{R}$ to $\mathbb{R} \cup \{-\infty\}$. We regard $Q \in \mathcal{Q}_0$ as a "too good " pricing measure for x when $\mathbb{E}_Q[U(x)]$ is sufficiently large in comparison to $U(\mathbb{E}_Q[x])$. For example, when U is an exponential utility function, that is, $U(x) = -e^{-ax}/a$ for some $a > 0$, Q is regarded as a good deal measure for a given level $\delta > 0$ if $\mathbb{E}_Q[U(x)] \geq \delta^{-1} U(\mathbb{E}_Q[x])$. In this case, we can represent the corresponding good deal bound by (0.2) for $\widehat{\mathcal{Q}} = \{Q \in \mathcal{Q}_0 | \mathbb{E}[V(\frac{dQ}{d\mathbb{P}})] \leq e^{\delta}\}$, where $V(x) = x \log x$ the conjugate function of U. Note that good deal bounds induced by the Sharpe ratio and the gain-loss ratio also can be interpreted as utility-based ones, e.g. the Sharpe ratio case corresponds to the bound induced by a quadratic utility function $U(x) = -(a - x)^2$ for $x < a$. Furthermore, $\widehat{\mathcal{Q}}$s appearing in this subsection should be interpreted as the set of

$Q \in \mathcal{Q}_0$ to which a variant of distance from $\mathbb{P}$ is less than a given level δ. As a further research, Klöppel and Schweizer [28] derived dynamic versions of utility-based good deal bounds in continuous time. In particular, they studied deeply dynamic bounds based on exponential utility functions for exponential Lévy models, and mentioned relationships with dynamic coherent risk measures.

Moreover, Becherer [4] considered dynamic good deal bounds obtained by restrictions on optimal expected growth rates, and showed that such a bound is corresponding to one induced by a logarithmic utility function. In [4], for the case where asset price is given by an Ito process, he obtained a backward stochastic differential equation whose solution describes the upper bound of the good deal bound based on optimal expected growth rates. In addition, defining a coherent risk measure ρ linked to the upper bound, he obtained the optimal strategy minimizing the risk of the hedging error quantified by ρ. Note that the value of this residual risk gives the upper bound of the corresponding good deal bound.

0.2 Good deal bounds induced by convex risk measures

As seen in the previous subsection, a good deal bound can be described through a risk measure. Moreover, good deal bounds are closely related to Fundamental Theorem of Asset Pricing (FTAP). Actually, Jaschke and Küchler [24] provided an essential equivalence between good deal bounds and coherent risk measures, and showed a variant of FTAP. Staum [40] treated a similar problem for the noncoherent case. On the other hand, Arai and Fukasawa [3] studied convex risk measures describing the upper and lower bounds of a good deal bound for the case where M the set of 0-attainable claims forms a convex cone. The upper (resp. lower) bound of a good deal bound may be determined by the seller's (resp. the buyer's) attitude toward the risk associated with the claim. Denoting by $a(x)$ such an upper bound for a claim x, we suppose that a has the following properties:

1. $a(0) = 0$,
2. $a(x) \leq a(y)$ if $x \leq y$,
3. $a(x + c) = a(x) + c$ for any $c \in \mathbb{R}$,
4. $a(\lambda x + (1 - \lambda)y) \leq \lambda a(x) + (1 - \lambda)a(y)$ for any $\lambda \in [0, 1]$

for any x, $y \in L$. The last property represents the risk-aversion of the seller taking into account the impact of diversification. In brief, we suppose that ρ_a defined as $\rho_a(x) := a(-x)$ is a normalized convex risk measure. By the same sort argument as above, a functional b which refers to a lower good deal bound is given by a normalized convex risk measure ρ_b as $b(x) = -\rho_b(x)$. Since a good deal bound is given as a subinterval of the no-arbitrage pricing bound, a convex risk measure does not necessarily yield a good deal bound. Thus, [3] characterized such a convex risk measure, called a good deal valuation (GDV), and defined GDV as a normalized convex risk measure ρ with the Fatou property such that for any claim x, $\rho(-x)$ takes a value in the no-arbitrage pricing bound of x. This definition of GDV is given from sellers' viewpoint; for a GDV ρ and a claim x, $a(x) := \rho(-x)$ serves as an ask price of x. Nevertheless, it is easy to see that if ρ is a GDV, then $b := -\rho$ gives bid prices.

Now, we enumerate main contribution of [3] as follows:

1. Equivalent conditions for the existence of a GDV are given. Among others, they showed that a GDV exists under a condition weaker than the no-arbitrage one, which means that there may be GDVs even if the underlying market admits an arbitrage opportunity. Further they gave equivalent conditions for a given convex risk measure to be a GDV. In particular, they proved that any GDV is given as a risk indifference price. Although there is much literature which observed that a risk indifference price provides a good deal bound, the above reverse implication seems to be new.

2. As mentioned before, GDV may exist even in markets with free lunch. [3] observed the equivalence between the no-free-lunch condition (NFL) and the existence of a relevant convex risk measure which is a GDV. This result could be considered as a version of FTAP.

1 Introduction to good deal bounds with convex constraints

We aim at studying GDVs for the case where M is convex along with the argument of [3]. In addition to [3], there is much literature on good deal bounds from the point of view of risk measures, say, Bion-Nadal [8], Bion-Nadal and Di Nunno [9], [24] and [40]. But, no one studied good deal bounds for markets with convex constraints, whereas such models appear frequently in mathematical finance, say, illiquid market models, models with borrowing constraints and so on. Indeed, there is much literature treating models with convex constraints: Cuoco [15], Cvitanić and Karatzas

[16] and [17], Karatzas and Kou [25], Larsen and Žitković [30], Pennanen [32] and [33], Pennanen and Penner [34], and so forth. See also examples introduced in Section 3.

Our main contribution is threefold as follows:

1. We begin with a study for the functional ρ^0 defined as

   $$\rho^0(x) := \inf\{r \in \mathbb{R} | \text{ there exists } m \in M \text{ such that } r + m + x \geq 0\}.$$

   Remark that the superhedging cost for claim x is given by $\rho^0(-x)$, and the upper and lower bounds of the no-arbitrage pricing bound is expressed as $[-\rho^0(x), \rho^0(-x)]$. As seen in [3], ρ^0 is given as a coherent risk measure when M is a convex cone. In this case, the set $\mathcal{Q}_0$ of all probability measures Q such that $\sup_{m\in M} \mathbb{E}_Q[m] = 0$, plays a central role to discuss not only ρ^0 but also GDVs. On the other hand, excluding the cone property from M, ρ^0 is no longer coherent in general. In this setting, we need to consider, instead of $\mathcal{Q}_0$, the set, denoted by $\mathcal{Q}$, of all probability measures Q such that $\sup_{m\in M} \mathbb{E}_Q[m]$ is finite. In particular, we investigate properties of the largest minorant of ρ^0 with the Fatou property, since it is the first candidate of GDVs.

2. We shall enumerate equivalent conditions for the existence of a GDV; and introduce a set of equivalent conditions for a given convex risk measure to be a GDV. In addition, we introduce an example of a GDV which is not a risk indifference price. This shows that the structure of good deal bounds is much different from that for the case where M is a convex cone. Moreover, we give conditions for a GDV to be a risk indifference price; and for a risk indifference price to be a GDV.

3. We deal with the Kreps-Yan-type FTAP. Kreps [29] proved that, if M is a convex cone, $\mathcal{Q}_0 \neq \emptyset$ is equivalent to the NFL, that is, the weak closure of M does not include any nonzero nonnegative claims. Moreover, [3] showed that the existence of a relevant GDV is equivalent to the NFL. Thus, we expect naturally that, when M is convex, the equivalence holds true among the NFL, the existence of a relevant GDV and a condition related to $\mathcal{Q}$. Indeed, we shall see the equivalence between the first two conditions, but illustrate counterexamples for the last one. Some variants of FTAP for constrained models have been introduced by Carassus, Pham and Touzi [11], Evstigneev, Schürger and Taksar [21], [32], Rokhlin [37], Roux [38] and so on. Thus, our contribution is to treat FTAP comprehensively for models with convex constraints.

An outline of this note is as follows. In Section 2, we describe our model; and prepare some terminologies and mathematical preliminaries. In particular, since we take an Orlicz space (or heart) as L, we introduce some terminologies on Orlicz space. Section 3 illustrates many examples of convex markets. We study superhedging cost in Section 4. Section 5 is devoted to study properties of GDVs. FTAP will be discussed in Section 6; and conclusions are given in Section 7.

2 Preliminaries

Throughout this note, we fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that we denote by $\mathbb{N}$ the set of all positive integers; and by L^0 the set of all $\mathbb{R}$-valued measurable functions on $(\Omega, \mathcal{F})$. Moreover, for a set of measurable functions X, we denote X_+ (resp. X_-):$= \{x \in X | x \geq 0$ a.s. (resp. $\leq$)$\}$.

We start with definitions of Young function, Orlicz heart and Orlicz space.

Definition 2.1 1. An even lower semi-continuous convex function $\Phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is called a Young function, if it satisfies the following:

   (a) $\Phi(0) = 0$,

   (b) $\Phi(\alpha) \uparrow \infty$ as $\alpha \uparrow \infty$,

   (c) $\Phi(\alpha) < \infty$ for α in a neighborhood of 0.

2. For a Young function Φ, a space M^Φ of measurable functions on $(\Omega, \mathcal{F})$ defined as

$$M^\Phi := \{x \in L^0 | \mathbb{E}[\Phi(cx)] < \infty \text{ for any } c > 0\}$$

is called Orlicz heart with Φ. In addition, a space L^Φ defined as

$$L^\Phi := \{x \in L^0 | \mathbb{E}[\Phi(cx)] < \infty \text{ for some } c > 0\}$$

is called Orlicz space with Φ.

3. The complimentary function of Φ is defined as

$$\Psi(\beta) := \sup_{\alpha \in \mathbb{R}} \{\alpha\beta - \Phi(\alpha)\}$$

for any $\beta \in \mathbb{R}$. Note that Ψ is also a Young function.

Any Young function is continuous on $[0, \infty)$ except for possibly a single point at which it jumps to $+\infty$. Both M^Φ and L^Φ are Banach lattices with norm $\|x\| := \inf\{c > 0 | \mathbb{E}[\Phi(x/c)] \leq 1\}$ and pointwise ordering in the almost sure sense. When Φ is finite, $L^\Phi = M^\Phi$ if and only if we can find $c > 0$ and $\alpha_0 > 0$ such that $\Phi(2\alpha) \leq c\Phi(\alpha)$ for any $\alpha \geq \alpha_0$. Thus, when $\Phi(\alpha) = |\alpha|^p$ with $p \geq 1$, we have $M^\Phi = L^\Phi = L^p$. On the other hand, if $\Phi(\alpha) = e^{|\alpha|} - 1$, M^Φ is a proper subset of L^Φ. Moreover, if Φ takes the value ∞, say, $\Phi(\alpha) = |\alpha|$ if $|\alpha| \leq 1$; $= \infty$ otherwise, then $L^\Phi = L^\infty$ and $M^\Phi = \{0\}$. In this paper, we fix a Young function Φ; and denote by Ψ its complimentary function. Note that L^Ψ is the dual space of M^Φ, that is, the set of all continuous linear functionals on M^Φ. For example, when $M^\Phi = L^p$ for $p > 1$, $L^\Psi = L^{\frac{p}{p-1}}$. Moreover, the dual space of L^Φ may include a singular part. For more details on Orlicz space, see Edgar and Sucheston [20] and Rao and Ren [36].

Let L be either M^Φ or L^Φ, which is regarded as the set of all future cash-flows. We denote by L^* its dual space. This setting would be natural, since it covers wide classes including all L^p spaces with $p \in [1, \infty]$; and fits to utility maximization problems (see Arai [1], [3], Biagini and Frittelli [6] and Cheridito and Li [13]). Moreover, let $M \subset L$ denote the set of all 0-attainable claims: future payoffs which investors can purchase without initial cost. Thus, M has the monotonicity, i.e., if $m_1 \in M$ and $m_2 \leq m_1$, then m_2 is also in M. In [3], M is assumed to be a convex cone. Here M is said to be a cone if $cm \in M$ for any $c \geq 0$ whenever $m \in M$, which implies that investors may trade any amount of $m \in M$. As a typical example for such a case, we illustrate frictionless markets introduced in Example 2.1 of [3]. Let S be the underlying asset price process being an $\mathbb{R}^d$-valued semimartingale defined on $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t \in [0,T]})$, where $\{\mathcal{F}_t\}_{t \in [0,T]}$ is a filtration with the usual conditions. Then M is typically given in the form

$$M = \left\{ \int_0^T H_t \mathrm{d}S_t \middle| H \in \mathcal{H} \right\} \cap L - L_+, \tag{2.1}$$

where $\mathcal{H}$ is the set of the admissible strategies. Here $L_1 - L_2 := \{x_1 - x_2 | x_1 \in L_1, x_2 \in L_2\}$ for two subsets L_1, $L_2 \subset L$. Note that the term "$-L_+$" is corresponding to the monotonicity of M. The following are examples for $\mathcal{H}$ which forms a convex cone.

1. (Section 5 of Delbaen and Schachermayer [19]) Let $\mathcal{H}^1$ be the set of processes H of the form $H_t = \sum_{i=1}^n h_i 1_{(\tau_{i-1}, \tau_i]}(t)$, where $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n \leq T$ are stopping times and for each i, h_i is an $\mathcal{F}_{\tau_{i-1}}$-measurable random variable such that the stopped process S^{τ_n} and

$h_1, \ldots, h_n$ are bounded. Since $\mathcal{H}^1$ is a convex cone, so is M defined by (2.1) with $\mathcal{H} = \mathcal{H}^1$.

2. Letting $\mathcal{H}^2$ be the set of S-integrable predictable processes such that $\int_0^t H_s dS_s$ is uniformly bounded from below, M defined by (2.1) with $\mathcal{H} = \mathcal{H}^2$ is a convex cone because so is $\mathcal{H}^2$. It seems that $\mathcal{H}^2$ is reasonable as the set of admissible strategies as explained in Section 8 of [19].

3. Note that $\mathcal{H}^2$ may be reduced to $\{0\}$ when S is not necessarily locally bounded. As a natural framework for such cases, we can consider W-admissible strategies as in Biagini et al. [7]. Fix $W \in L$ with $W \geq 1$, and denote by $\mathcal{H}^3$ the set of S-integrable predictable processes H such that there exists a constant $c > 0$ satisfying $\int_0^t H_s dS_s \geq -cW$ for any $t \in [0, T]$. Then, $\mathcal{H}^3$ is a convex cone and so, M defined by (2.1) with $\mathcal{H} = \mathcal{H}^3$ also forms a convex cone.

As seen in the above, the convex cone property of M appears in some natural settings. On the other hand, as said in Section 1, M does not necessarily have the cone property when we take account of e.g. constraints on admissible strategies or the illiquidity of the market. Some examples for such models are introduced in Section 3. In this note, we assume that M is convex; and aim at a generalization of the results of [3] to the convex case by excluding the cone property from M.

For later use, we prepare some notation.

Definition 2.2 1. $\mathcal{P} := \{Q \ll \mathbb{P} | dQ/d\mathbb{P} \in L^\Psi\}$,

2. $L_1^* := \{g \in L^* | g(1) = 1, g(x) \geq 0 \text{ for any } x \in L_+\}$,

3. $\overline{L}^* := \{g \in L_1^* | \sup_{m \in M} g(m) < \infty\}$,

4. $\mathcal{Q} := \{Q \in \mathcal{P} | \sup_{m \in M} \mathbb{E}_Q[m] < \infty\}$,

5. $\mathcal{Q}^e := \{Q \in \mathcal{Q} | Q \sim \mathbb{P}\}$,

6. $\mathcal{Q}_0 := \{Q \in \mathcal{Q} | \sup_{m \in M} \mathbb{E}_Q[m] = 0\}$.

Remark 2.3 When M is a convex cone, $\sup_{m \in M} \mathbb{E}_Q[m]$ becomes either 0 or ∞ for $Q \in \mathcal{P}$, that is, $\mathcal{Q}$ and $\mathcal{Q}_0$ coincide. On the other hand, $\sup_{m \in M} \mathbb{E}_Q[m]$ may take a positive number in our setting.

2.1 Convex risk measure

We define convex risk measures and some related terminologies. In addition, we introduce a representation result.

Definition 2.4 1. A $(-\infty, \infty]$-valued functional ρ defined on L is called a convex risk measure if ρ satisfies, for any $x, y \in L$,

properness: $\rho(0) < \infty$,

monotonicity: $\rho(x) \geq \rho(y)$ if $x \leq y$,

cash-invariance: $\rho(x + r) = \rho(x) - r$ for any $r \in \mathbb{R}$,

convexity: $\rho(\lambda x + (1 - \lambda)y) \leq \lambda\rho(x) + (1 - \lambda)\rho(y)$ for any $\lambda \in [0, 1]$.

2. In addition, a convex risk measure ρ is a coherent risk measure if it satisfies

positive homogeneity: $\rho(\lambda x) = \lambda\rho(x)$ for any $x \in L$ and any $\lambda \geq 0$.

Definition 2.5 1. Let f be a $[-\infty, \infty]$-valued functional on L.

(a) If $f(0) = 0$, then f is said to be normalized.

(b) f is said to have the Fatou property if $\lim_{n\to\infty} f(-x_n) = f(-x)$ for any increasing sequence $\{x_n\} \subset L$ with $x_n \uparrow x$.

(c) f is said to be relevant if $f(-z) > 0$ for any $z \in L_+ \setminus \{0\}$.

(d) We define the penalty function for f as

$$f^*(g) := \sup_{x \in L}\{g(-x) - f(x)\} \tag{2.2}$$

for $g \in L_1^*$. In particular, we denote, for $Q \in \mathcal{P}$,

$$f^*(Q) := \sup_{x \in L}\{\mathbb{E}_Q[-x] - f(x)\}. \tag{2.3}$$

2. We denote by $\mathcal{R}$ the set of all normalized convex risk measures on L with the Fatou property.

Theorem 2.6 (Proposition 1 of [6]) *Any $\rho \in \mathcal{R}$ is represented as*

$$\rho(x) = \sup_{Q \in \mathcal{P}}\{\mathbb{E}_Q[-x] - \rho^*(Q)\}.$$

2.2 A separating result

We prepare a proposition, which will appear over and over again in the sequel. Now, we denote by $\overline{M}$ (resp. M^s) the closure of M in $\sigma(L, L^\Psi)$ (resp. in $\|\cdot\|$).

Proposition 2.7 *Let $B \subset L_+$ be a convex set including at least one positive constant.*

1. *If B is $\|\cdot\|$-compact and $M^s \cap B = \emptyset$, then there exists a $g \in \overline{L}^*$ such that*

$$\sup_{m \in M^s} g(m) < \inf_{x \in B} g(x). \tag{2.4}$$

2. *If B is $\sigma(L, L^\Psi)$-compact and $\overline{M} \cap B = \emptyset$, then there exists a $Q \in \mathcal{Q}$ such that*

$$\sup_{m \in \overline{M}} \mathbb{E}_Q[m] < \inf_{x \in B} \mathbb{E}_Q[x].$$

Proof. It suffices to show only the first assertion. By the conditions, the Hahn-Banach separating theorem implies the existence of $g \in L^*$ satisfying (2.4). Remark that $\sup_{m \in M^s} g(m) \geq 0$ because $0 \in M$. Thus, we have $g(1) > 0$, since B includes at least one positive constant. Without loss of generality, we may assume $g(1) = 1$. Moreover, since $L_- \subset M$, $g \in L_1^*$ holds true. In addition, Definition 2.2 implies that the LHS of (2.4) takes the value ∞ unless $g \in \overline{L}^*$. Thus, g belongs to $\overline{L}^*$. □

3 Examples for convex markets

In spite of that market models with convex constraints are discussed frequently, there is no study on good deal bounds for convex markets as mentioned in Section 1. Before stating main results, we introduce some examples such that M is convex, but not a cone.

Example 3.1 (A simple illiquid market model) As one of important financial risks, we focus on liquidity risk, which is caused by the effect of a large trader, a price impact and so forth. Some of illiquid market models are such that the corresponding M forms a convex set, but not a cone.

Now, we illustrate a simple example of such illiquid market models. We consider a one-period model in which one riskless asset with zero interest rate and one risky asset are tradable. For $t = 0, 1$, let S_t be the price of the risky asset at time t. We assume that $S_0 \in \mathbb{R}_+$ and $S := S_1 - S_0$ belongs to L. We take into account nonlinear illiquidity effects denoted by a function $f : \mathbb{R} \to \mathbb{R}$. More precisely, we assume that, for any $\alpha \in \mathbb{R}$, it costs $\alpha S_0 + f(\alpha)$ to get α units of the risky asset. It seems that $f(\alpha)$ describes the extra cost for purchasing α units of the risky asset. Now, suppose that f is a continuous convex function with $f(0) = 0$, non-increasing on $(-\infty, 0]$ and non-decreasing on $[0, \infty)$. For example, $f(\alpha) = e^{|\alpha|} - 1$ or $f(\alpha) = \alpha^2$. As a result, the set of all 0-attainable claims is expressed as

$$M = \{\alpha S - f(\alpha) | \alpha \in \mathbb{R}\} - L_+,$$

which forms a convex subset of L including L_-, but not necessarily a cone.

Example 3.2 (Constraints on number of shares) Consider a continuous trading model with maturity $T \in (0, \infty)$. Suppose that one riskless asset with zero interest rate and d risky assets are tradable; and the price of the risky assets is described by an $\mathbb{R}^d$-valued locally bounded RCLL special semimartingale S (see p.129 of Protter [35] for the definition of special semimartingales) defined on a complete probability space $(\Omega, \mathcal{F}, P; \mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]})$, where $\mathbb{F}$ is a filtration satisfying the so-called usual condition, that is, $\mathbb{F}$ is right-continuous, $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_0$ contains all null sets of $\mathcal{F}$. Let $L(S)$ be the set of all $\mathbb{R}^d$-valued S-integrable predictable processes; and $G_t(\vartheta) := \int_0^t \vartheta_s dS_s$ for any $t \in [0, T]$ and any $\vartheta \in L(S)$. Note that $\vartheta \in L(S)$ denotes the number of shares the investor holds; and the process $G(\vartheta)$ represents the gain process induced by a self-financing strategy ϑ. Now, we impose convex constraints on the set of all admissible strategies. That is, we consider the case where the set of 0-attainable claims is given as

$$M = \{G_T(\vartheta) | \vartheta \in L(S), \vartheta_t \in K \text{ for any } t \in [0, T]\} \cap L - L_+,$$

where K is a convex subset of $\mathbb{R}^d$ including 0. We introduce some concrete examples of K as follows:

1. (Rectangular constraints) $K = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$ for some fixed numbers $-\infty \leq a_i \leq 0 \leq b_i \leq \infty$, $i = 1, 2, \ldots, d$.

2. (Constraints on total number of shares) $K = \{(h_1, \ldots, h_d) \in \mathbb{R}^d | h_i \geq 0 \text{ for each } 1 \leq i \leq d, \sum_{i=1}^d h_i \leq c\}$ for some positive constant c.

3. (Short-sale constraints) $K = \{(h_1, \ldots, h_d) \in \mathbb{R}^d | h_i \geq -c \text{ for each } 1 \leq i \leq d\}$ for some positive constant c.

For more details, see [15], [16] and [17].

Example 3.3 (Constraints on amount invested) We consider the same model as the previous example; and assume that $S > 0$ and M is given as

$$M = \{G_T(\vartheta) | \vartheta \in L(S), \vartheta_t S_{t-} \in K \text{ for any } t \in [0,T]\} \cap L - L_+,$$

where K is a convex subset of $\mathbb{R}^d$ including 0, and $\vartheta S_- = (\vartheta^1 S^1_-, \dots, \vartheta^d S^d_-)$ represents the amount invested in each asset. The three examples for K introduced in Example 3.2 are also typical examples for the present setting.

Example 3.4 (a-admissible) We consider the same mathematical framework as Example 3.2. Let a be a positive real number. $\vartheta \in L(S)$ is said to be a-admissible if $G_t(\vartheta) \geq -a$ for any $t \in [0,T]$. When M is given as

$$M = \{G_T(\vartheta) | \vartheta \in L(S) \text{ is } a\text{-admissible } \} \cap L - L_+ \tag{3.1}$$

for fixed $a > 0$, it forms a convex set. On the other hand, when M is denoted by $M = \{G_T(\vartheta) | \vartheta \in L(S) \text{ is } a\text{-admissible for some } a > 0\} \cap L - L_+$, it is a convex cone. For more details, see Section 9 in [19].

Example 3.5 (W-admissible) In the previous example, when S is not necessarily locally bounded, M defined in (3.1) may become $\{0\}$. As a natural way to avoid it, we introduce W-admissibility. Let W be a random variable in L with $W \geq 1$. $\vartheta \in L(S)$ is said to be W-admissible if $G_t(\vartheta) \geq -W$ for any $t \in [0,T]$. Then,

$$M = \{G_T(\vartheta) | \vartheta \in L(S) \text{ is } W\text{-admissible}\} \cap L - L_+$$

formulates a convex market.

Example 3.6 (Predictably convexity) We introduce the predictably convexity, which brings us an important class of models with convex constraints. It has been undertaken by Föllmer and Kramkov [22]; and discussed in Chapter 9 of Föllmer and Schied [23] for discrete time models. See also [1], Klöppel and Schweizer [27]. Now, we define it as follows: A family of semimartingales $\mathcal{S}$ is said to be predictably convex if, for any $S^{(1)}, S^{(2)} \in \mathcal{S}$ and any $[0,1]$-valued predictable process h, $\int_0^\cdot h dS^{(1)} + \int_0^\cdot (1-h) dS^{(2)}$ belongs to $\mathcal{S}$. For the three examples of portfolio constraints in Example 3.2, their Ms are predictably convex. Here we consider the same continuous trading model as Example 3.2, provided that $S = (S^1, \dots, S^d)$ is possibly nonlocally bounded. Now, we fix an $\mathcal{F}_T$-measurable random variable $W \in L$

with $W \geq 1$ satisfying, for each $i = 1, \ldots, d$, there exists an $\mathbb{R}$-valued S^i-integrable predictable process ϑ^i such that

$$\mathbb{P}(\{\omega| \text{ there exists } t \in [0,T] \text{ such that } \vartheta_t^i(\omega) = 0\}) = 0$$

and $|\int_0^t \vartheta_s^i \mathrm{d}S_s^i| \leq W$ for any $t \in [0,T]$. In addition, we denote

$$\begin{aligned}\Theta^W \quad := \quad & \{\vartheta \in L(S)|\text{there exists } c > 0 \text{ such that } G_t(\vartheta) \geq -cW \\ & \text{for any } t \in [0,T]\},\end{aligned}$$

and $G(\Theta^W) := \{G(\vartheta)|\vartheta \in \Theta^W\}$. Let $\mathcal{S}$ be a predictably convex subset of $G(\Theta^W)$, and $\Theta^{\mathcal{S}}$ the corresponding subset of Θ^W to $\mathcal{S}$. That is, we can describe $\mathcal{S} = \{G(\vartheta)|\vartheta \in \Theta^{\mathcal{S}}\}$. Now, we denote

$$M = \left\{G_T(\vartheta)|\vartheta \in \Theta^{\mathcal{S}}\right\} - L_+, \tag{3.2}$$

which is convex.

4 Superhedging cost

Superhedging cost for a claim is defined as the lowest price of the claim which enables investors to construct an arbitrage opportunity by selling the claim and selecting a suitable strategy from M. More precisely, defining a functional ρ^0 on L as

$$\rho^0(x) := \inf\{r \in \mathbb{R}| \text{ there exists } m \in M \text{ such that } r + m + x \geq 0\}, \tag{4.1}$$

the superhedging cost for claim x is given by $\rho^0(-x)$; and the no-arbitrage pricing bound for x is given by $[-\rho^0(x), \rho^0(-x)]$. Note that GDVs will be defined by using ρ^0 in Section 5. Thus, we investigate properties of ρ^0 which we will need for studying GDVs.

Lemma 4.1 $(\rho^0)^*(g) = \sup_{m\in M} g(m)$ *for any* $g \in L_1^*$*, where* $(\rho^0)^*$ *is the penalty function for* ρ^0 *defined in (2.2).*

Proof. Since $\rho^0(-m) \leq 0$ for any $m \in M$, (2.2) implies that $(\rho^0)^*(g) \geq \sup_{m\in M}\left\{g(m) - \rho^0(-m)\right\} \geq \sup_{m\in M} g(m)$ for any $g \in L_1^*$. On the other hand, for any $x \in L$ with $\rho^0(x) < \infty$, we take an $r > \rho^0(x)$ arbitrarily. There is then an $m^{x,r} \in M$ satisfying $r + m^{x,r} + x \geq 0$. Since $g(m^{x,r}) \leq \sup_{m\in M} g(m)$ for any $g \in L_1^*$, we have $\sup_{m\in M} g(m) \geq g(-x) - r$, that is, $\sup_{m\in M} g(m) \geq g(-x) - \rho^0(x)$. In addition, this inequality also holds for any $x \in L$ with $\rho^0(x) = \infty$. Therefore, we have $\sup_{x \in L}\{g(-x) - \rho^0(x)\} \leq \sup_{m \in M} g(m)$ for any $g \in L_1^*$. Consequently, $(\rho^0)^*(g) = \sup_{m \in M} g(m)$ for any $g \in L_1^*$. □

Proposition 4.2 *$\overline{L}^* \neq \emptyset$ if and only if ρ^0 is a convex risk measure on L.*

Proof. "only if" part: Firstly, the monotonicity and cash-invariance are obvious. Next, we see $\rho^0 > -\infty$. Assuming that there exists an $x \in L$ with $\rho^0(x) = -\infty$, (4.1) implies that for any $c > 0$, we can find an $m^c \in M$ such that $-c + m^c + x \geq 0$. Thus, for any $g \in \overline{L}^*$, we have $g(x) \geq c - (\rho^0)^*(g)$ for any $c > 0$ by Lemma 4.1, that is, $g(x) = \infty$. This is a contradiction, so ρ^0 is $(-\infty, \infty]$-valued; and has the properness because $\rho^0(0) \leq 0$. Lastly, we see the convexity of ρ^0. Fix x_1, $x_2 \in L$ and $\lambda \in [0, 1]$ arbitrarily. Now, we assume that both $\rho^0(x_1)$ and $\rho^0(x_2)$ are finite. Otherwise, the convexity holds clearly. Taking $r_i > \rho^0(x_i)$ for $i = 1, 2$ arbitrarily, the convexity of M implies $\lambda r_1 + (1 - \lambda) r_2 \geq \rho^0(\lambda x_1 + (1 - \lambda) x_2)$, from which the convexity of ρ^0 follows.

"if" part: Suppose that $\overline{L}^* = \emptyset$. Assuming that there exists a $c > 0$ with $c \notin M^s$, Proposition 2.7 implies the existence of $g \in \overline{L}^*$ satisfying $c > \sup_{m \in M^s} g(m)$, which is a contradiction. As a result, any $c > 0$ is included in M^s. Now, for any $k \in \mathbb{N}$, we take an $m_k \in M$ satisfying $\|2^k - m_k\| \leq 1$. We define $x_n := \sum_{k=1}^n |2^k - m_k| 2^{-k}$ for any $n \in \mathbb{N} \cup \{\infty\}$. Then, x_n converges to x_∞ a.s.; and $\{x_n\}$ is a Cauchy sequence in $\|\cdot\|$. Hence, Lemma 4.3 provides $x_\infty \in L$. Noting that $x_\infty \geq x_n \geq \sum_{k=1}^n (2^k - m_k) 2^{-k} = n - \sum_{k=1}^n m_k 2^{-k}$ for any $n \in \mathbb{N}$, and $\rho^0(-m) \leq 0$ for any $m \in M$, we have

$$\rho^0(x_\infty) \leq -n + \sum_{k=1}^n 2^{-k} \rho^0(-m_k) + \left(1 - \sum_{k=1}^n 2^{-k}\right) \rho^0(0) \leq -n$$

for any $n \in \mathbb{N}$. Consequently, $\rho^0(x_\infty) = -\infty$, which is a contradiction. □

Lemma 4.3 *Let $\{x_n\}_{n \geq 1}$ be a Cauchy sequence on $(L, \|\cdot\|)$ which converges to x_∞ a.s. Then, $\{x_n\}$ converges to x_∞ in $\|\cdot\|$, that is, $x_\infty \in L$.*

Proof. Since $\{x_n\}$ is a Cauchy sequence, there exists an $x'_\infty \in L$ such that $\|x_n - x'_\infty\| \to 0$ by the completeness of $(L, \|\cdot\|)$. In addition, Proposition 2.1.10 (6) of [20] implies that x_n tends to x'_∞ in probability. Hence, $x_\infty = x'_\infty \in L$. □

Remark 4.4 In the proof of Proposition 4.2, we see $x_\infty \in L$. At first glance, it seems to be shown easier as follows: $\|x_\infty\| = \|\sum_{k=1}^{\infty} |2^k - m_k| 2^{-k}\| \leq \sum_{k=1}^{\infty} 2^{-k}\|2^k - m_k\| < \infty$, which implies $x_\infty \in L$. However, this is not accurate. Firstly, the former inequality is not trivial. Besides, even if $\|x_\infty\| < \infty$, x_∞ does not necessarily belong to L, since L may be M^Φ a proper subset of L^Φ.

Remark 4.5 When M is a convex cone, $(\rho^0)^*$ takes the values 0 and ∞ only. Thus, ρ^0 is a coherent risk measure if and only if $\overline{L}^* \neq \emptyset$. For more details, see [3].

Example 4.6 For the case where $M = [0,1] - L_+ = \{x \in L | x \leq 1\}$, ρ^0 becomes a convex risk measure. Indeed, $g(x) := \mathbb{E}[x]$ belongs to $\overline{L}^*$. On the other hand, setting $M = [0,\infty) - L_+ = \{x \in L | x \vee 0 \in L^\infty\}$, we have $\rho^0(0) = -\infty$, that is, ρ^0 is not a convex risk measure. In this case, $\overline{L}^*$ is empty evidently.

Remark 4.7 We consider the concept of no arbitrage of the first kind, which is weaker than the NFL and the no-free-lunch with vanishing risk. We call $z \in L_+ \backslash \{0\}$ an arbitrage of the first kind if, for any $\varepsilon > 0$, we can find an $m \in M$ such that $\varepsilon + m - z \geq 0$. For more details on arbitrage of the first kind, see Kardaras [26]. We can see immediately that, for $z \in L_+ \backslash \{0\}$, it is an arbitrage of the first kind if and only if $\rho^0(-z) = 0$. In other words, there is no arbitrage of the first kind if and only if ρ^0 is relevant.

Now, we define a functional $\widehat{\rho^0}$, which is closely related to superhedging cost, as follows:

$$\widehat{\rho^0}(x) := \begin{cases} \sup_{Q \in \mathcal{Q}} \{\mathbb{E}_Q[-x] - (\rho^0)^*(Q)\} & \text{if } \mathcal{Q} \neq \emptyset, \\ -\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{Q}$ and $(\rho^0)^*(Q)$ are defined in Definition 2.2 and (2.3), respectively. We introduce a proposition, some lemmas and examples related to $\widehat{\rho^0}$.

Proposition 4.8 *The following are equivalent:*

1. $\mathcal{Q} \neq \emptyset$.
2. $\widehat{\rho^0}$ *is the largest convex risk measure with the Fatou property less than* ρ^0.
3. *There exists a* $c \geq 0$ *such that* $\mathbb{P}(\overline{m} > c) < 1$ *for any* $\overline{m} \in \overline{M}$.

4. *There exists a* $c > 0$ *such that* $c \notin \overline{M}$.

Proof. 1⇔2: This equivalence is the very definition of $\widehat{\rho^0}$.
1⇒3: Supposing $\mathcal{Q} \neq \emptyset$ and taking a $Q \in \mathcal{Q}$ arbitrarily, we have $(\rho^0)^*(Q) \in [0, \infty)$ and $\mathbb{E}_Q[\overline{m}] - (\rho^0)^*(Q) \leq 0$ for any $\overline{m} \in \overline{M}$, that is, $Q(\overline{m} - (\rho^0)^*(Q) \leq 0) > 0$ for any $\overline{m} \in \overline{M}$. Hence, $\mathbb{P}(\overline{m} > (\rho^0)^*(Q)) < 1$ for any $\overline{m} \in \overline{M}$.
3⇒4: If $c \in \overline{M}$ for any $c > 0$, condition 3 is false, since $\mathbb{P}(c + 1 > c) = 1$ for any $c \geq 0$.
4⇒1: Taking a $c > 0$ which is not included in $\overline{M}$, Proposition 2.7 ensures that $\mathcal{Q}$ is nonempty, since $\{c\}$ is compact. □

Example 4.9 We illustrate an example in which $\rho^0 \neq \widehat{\rho^0}$ holds. Let $\Omega = \{\omega_k; k \in \mathbb{N}\}$, $\mathbb{P}(\{\omega_k\}) > 0$ for $k \in \mathbb{N}$, and

$$\begin{aligned} M &= \left\{ \sum_{k=1}^{\infty} \vartheta_k 1_{\{\omega_k\}} | 0 \leq \vartheta_k \leq 1 \text{ for any } k \in \mathbb{N}, \right. \\ &\qquad \left. \vartheta_k = 0 \text{ except for finitely many } k\text{s} \right\} - L_+. \end{aligned}$$

Since $\mathbb{P} \in \mathcal{Q}$, $\widehat{\rho^0}$ has the Fatou property by Proposition 4.8. On the other hand, letting $x_n := \sum_{k=1}^n 1_{\{\omega_k\}}$ for $n \in \mathbb{N}$, we have $\rho^0(-x_n) = 0$ for any $n \in \mathbb{N}$, although $\rho^0(-1) = 1$ and x_n tends to 1. Thus, ρ^0 does not possess the Fatou property. Another example in which $\rho^0 \neq \widehat{\rho^0}$ holds has been introduced in [3].

Lemma 4.10 *The following are equivalent:*

1. $\mathcal{Q} \neq \emptyset$ *and* $\inf_{Q \in \mathcal{Q}} (\rho^0)^*(Q) = 0$.
2. $-\rho^0(x) \leq -\widehat{\rho^0}(x) \leq \widehat{\rho^0}(-x) \leq \rho^0(-x)$ *for any* $x \in L$.
3. $\widehat{\rho^0}(0) = \rho^0(0) = 0$.
4. $\widehat{\rho^0}(0) = 0$.

Proof. 1⇒2: Proposition 4.8 yields that $\widehat{\rho^0} \leq \rho^0$. The convexity of $\widehat{\rho^0}$ implies that $\widehat{\rho^0}(x) + \widehat{\rho^0}(-x) \geq 2\widehat{\rho^0}(0) = -2\inf_{Q \in \mathcal{Q}} (\rho^0)^*(Q) = 0$, from which the implication follows.

2⇒3: Substituting 0 for x, we have $\rho^0(0) \geq \widehat{\rho^0}(0) \geq -\widehat{\rho^0}(0)$. Thus, $\rho^0(0) \geq \widehat{\rho^0}(0) \geq 0$ holds. In addition, (4.1) implies that $\rho^0(0) \leq 0$.
3⇒4: Obvious.
4⇒1: By the definition of $\widehat{\rho^0}$, $\mathcal{Q} \neq \emptyset$ is ensured. Moreover, $0 = \widehat{\rho^0}(0) = -\inf_{Q\in\mathcal{Q}}(\rho^0)^*(Q)$ holds true. □

Example 4.11 We consider a one-period illiquid market model as in Example 3.1. Now we treat a binomial model with $f(\alpha) = \alpha^2$. Let $\Omega = \{\omega_1, \omega_2\}$, $\mathbb{P}(\{\omega_i\}) > 0$ for $i = 1, 2$, and

$$S(\omega) = \begin{cases} 1 & \text{if } \omega = \omega_1, \\ -1 & \text{if } \omega = \omega_2. \end{cases}$$

The set of 0-attainable claims is given as

$$M = \{\alpha S - \alpha^2 | \alpha \in \mathbb{R}\} - L_+ = \{\alpha S - \alpha^2 | \alpha \in [-1/2, 1/2]\} - L_+.$$

For a probability measure Q, we denote $q := Q(\{\omega_1\})$, and identify Q with q. Thus, we regard $[0, 1]$ as the set of all probability measures. We have

$$(\rho^0)^*(q) = \sup_{\alpha\in[-\frac{1}{2},\frac{1}{2}]} \left\{\mathbb{E}_q[\alpha S] - \alpha^2\right\} = \sup_{\alpha\in[-\frac{1}{2},\frac{1}{2}]} \alpha(2q - 1 - \alpha) = \frac{(2q-1)^2}{4}.$$

Obviously $\mathcal{Q} \neq \emptyset$ and $\inf_{Q\in\mathcal{Q}}(\rho^0)^*(Q) = 0$ hold true. Besides, ρ^0 is consistent with $\widehat{\rho^0}$. For $x = \frac{1}{2}1_{\{\omega_1\}}$, we have $\rho^0(-x) = \frac{5}{16}$, and $\rho^0(x) = -\frac{3}{16}$, that is, its no-arbitrage pricing bound is given by $[\frac{3}{16}, \frac{5}{16}]$. □

Example 4.12 The condition "$\mathcal{Q} \neq \emptyset$" does not ensure "$\inf_{Q\in\mathcal{Q}}(\rho^0)^*(Q) = 0$". We consider the following simple model: Set $\Omega = \{\omega_1, \omega_2\}$, and $S = 1_{\{\omega_1\}} + \frac{1}{2}1_{\{\omega_2\}}$. Note that we do not need to specify Φ. Let us consider the case where M is given by $\{\vartheta S | \vartheta \in [0, 1]\} - L_+$. In this case, we have $\mathcal{P} = \mathcal{Q} \neq \emptyset$ and $\inf_{Q\in\mathcal{Q}}(\rho^0)^*(Q) = \inf_{Q\in\mathcal{Q}} \mathbb{E}_Q[S] = \frac{1}{2}$. Hence, $\widehat{\rho^0}(0) = -\frac{1}{2}$, that is, $\widehat{\rho^0}$ is not normalized.

Remark 4.13 Condition 1 in Lemma 4.10 is equivalent to $\mathcal{Q}_0 \neq \emptyset$ when M is a convex cone. Actually, it will play a similar role to "$\mathcal{Q}_0 \neq \emptyset$" in the convex cone case as in [3].

Example 4.14 By Lemma 4.10, $\rho^0(0) = 0$ whenever $\widehat{\rho^0}(0) = 0$, while its reverse implication does not hold. We reconsider Example 4.9. Now, we assume $\rho^0(0) < 0$. Letting $r = \frac{-\rho^0(0)}{2} (> 0)$, there exists an $m \in M$ such that $m \geq r$, which means $m(\omega_k) \geq r$ for any $k \in \mathbb{N}$. This is a contradiction. As a result, $\rho^0(0) = 0$. Next, we calculate $\widehat{\rho^0}(0)$. Note that $\mathcal{Q} \neq \emptyset$. For any $Q \in \mathcal{Q}$ and any $\varepsilon > 0$, we can find a finite set $A \subset \Omega$ satisfying $Q(A) > 1 - \varepsilon$. Thus, for any $\varepsilon > 0$, there exists an $m \in M$ such that $\mathbb{E}_Q[m] > 1 - \varepsilon$. So that, $(\rho^0)^*(Q) = 1$ for any $Q \in \mathcal{Q}$. Consequently, we have $\widehat{\rho^0}(0) = -1$.

Lemma 4.15 *If there exists a $Q \in \mathcal{Q}^e$ with $(\rho^0)^*(Q) = 0$, then $\widehat{\rho^0}$ is relevant.*

Proof. Letting Q be an element of $\mathcal{Q}^e$ with $(\rho^0)^*(Q) = 0$, we have, for any $z \in L_+ \backslash \{0\}$, $\widehat{\rho^0}(-z) \geq \mathbb{E}_Q[z] - (\rho^0)^*(Q) = \mathbb{E}_Q[z] > 0$. □

Example 4.16 Even if $\inf_{Q \in \mathcal{Q}} (\rho^0)^*(Q) = 0$, we may have $(\rho^0)^*(Q) > 0$ for any $Q \in \mathcal{Q}$. Now, we construct such an example. Set $\Omega = \{\omega_0, \omega_1, \dots\}$, $\mathbb{P}(\{\omega_k\}) = \frac{1}{2^{k+1}}$ for $k = 0, 1, 2, \dots$. Letting M be given as $\{\vartheta S | \vartheta \in [0,1]\} - L_+$, where $S(\omega_k) = \frac{1}{2^k}$ for any $k \in \mathbb{N}$, we have $(\rho^0)^*(Q) = \mathbb{E}_Q[S] > 0$ for any $Q \in \mathcal{Q}$. Defining a probability measure Q_k for each $k \in \mathbb{N}$ as $Q_k(\{\omega_l\}) = 1_{\{l=k\}}$ for $l \in \mathbb{N} \cup \{0\}$, we have $Q_k \in \mathcal{Q}$ for each $k \in \mathbb{N}$. Then $\inf_{Q \in \mathcal{Q}} (\rho^0)^*(Q) \leq \inf_{k \in \mathbb{N}} (\rho^0)^*(Q_k) = \inf_{k \in \mathbb{N}} \frac{1}{2^k} = 0$. As a result, the condition in Lemma 4.15 is stronger than Condition 1 in Lemma 4.10 in general.

Example 4.17 We consider the predictably convexity introduced in Example 3.6; and illustrate representations of $\mathcal{Q}$, $\mathcal{Q}_0$ and $(\rho^0)^*$ for predictably convex models. The following argument is based on Section 6 of [1]. Now, we assume that M defined in (3.2) is included in L; and define

$$\begin{aligned} \mathcal{P}(\mathcal{S}) \quad := \quad & \{Q \in \mathcal{P} | \text{ there exists increasing predictable process } A \text{ such that} \\ & G(\vartheta) - A \text{ is a } Q\text{-supermartingale for any } \vartheta \in \Theta^{\mathcal{S}}\}. \end{aligned}$$

When $Q \in \mathcal{P}(\mathcal{S})$, $G(\vartheta)$ is a special semimartingale under Q for any $\vartheta \in \Theta^{\mathcal{S}}$ (Lemma 6.2 of [1]). Fixing $Q \in \mathcal{P}(\mathcal{S})$, we denote by $M^\vartheta + A^\vartheta$ the canonical decomposition of $G(\vartheta)$ under Q. Note that this decomposition depends on Q. Now, we define $\mathcal{A} := \{A^\vartheta | \vartheta \in \Theta^{\mathcal{S}}\}$. In addition, for two stochastic processes X and Y, we define an order $\preceq$ as follows:

$$X \preceq Y \iff Y - X \text{ is an increasing process.}$$

Remark that the ordered set $(\mathcal{A}, \preceq)$ is directed upward (Lemma 6.4 of [1]). An increasing predictable process $A^{\mathcal{S}}$ is called an upper variation process of the ordered set $(\mathcal{A}, \preceq)$ if $A^{\mathcal{S}}$ satisfies the following two conditions:

1. $A \preceq A^{\mathcal{S}}$ for any $A \in \mathcal{A}$,
2. if an increasing predictable process $\widehat{A}$ satisfies $A \preceq \widehat{A}$ for any $A \in \mathcal{A}$, then $A^{\mathcal{S}} \preceq \widehat{A}$ holds.

The following assertions are from Theorem 6.9 and Theorem 5.2 in [1].

1. We have $\mathcal{Q} = \mathcal{P}(\mathcal{S})$ and

$$(\rho^0)^*(Q) = \begin{cases} \mathbb{E}_Q[\overline{A}^Q_T] < \infty, & \text{if } Q \in \mathcal{P}(\mathcal{S}), \\ \infty, & \text{otherwise,} \end{cases}$$

   where $\overline{A}^Q$ is an upper variation process for $Q \in \mathcal{P}(\mathcal{S})$.
2. $\mathcal{Q}_0 = \{Q \in \mathcal{P}(\mathcal{S}) |\, G(\vartheta) \text{ is a } Q\text{-supermartingale for any } \vartheta \in \Theta^{\mathcal{S}}\}$.

5 Good deal valuations

In this section, we investigate thoroughly properties of GDVs. Since any good deal bound is given as a subinterval of the no-arbitrage pricing bound, when we represent the upper and lower bounds of a good deal bound as functionals a and b respectively, we have $[b(x), a(x)] \subset [-\rho^0(x), \rho^0(-x)]$ for any $x \in L$. Now, we define a functional ρ as $\rho(-x) := a(x)$. It is then natural that ρ is a normalized convex risk measure as discussed in [3]. We call such a risk measure a GDV. Its precise definition is given as follows:

Definition 5.1 A convex risk measure $\rho \in \mathcal{R}$ is said to be a good deal valuation(GDV) if

$$\rho(-x) \in [-\rho^0(x), \rho^0(-x)] \text{ for any } x \in L, \tag{5.1}$$

where $\mathcal{R}$ is the set of all normalized convex risk measures on L with the Fatou property.

Note that we consider only convex risk measures having the Fatou property as GDVs in this paper. Although the definition (5.1) is given from the seller's view point, we can rewrite (5.1) as

$$-\rho(x) \in [-\rho^0(x), \rho^0(-x)] \text{ for any } x \in L, \tag{5.2}$$

which means that any GDV describes the lower bound of a good deal bound. Indeed, denoting $-\rho^b(x) := b(x)$, ρ^b satisfies (5.2). Furthermore, note that any GDV ρ satisfies $-\rho(x) \le \rho(-x)$ for any $x \in L$ since $\rho(x) + \rho(-x) \ge 2\rho(0) = 0$ by the convexity. Then, for any GDV ρ, the interval $[-\rho(x), \rho(-x)]$ provides a good deal bound. Note that the upper and lower bounds of a good deal bound are mostly described by different GDVs.

Now, we show equivalent conditions for the existence of a GDV.

Theorem 5.2 *The following are equivalent:*

1. $\mathcal{Q} \neq \emptyset$ *and* $\inf_{Q \in \mathcal{Q}} (\rho^0)^*(Q) = 0$.
2. $\widehat{\rho^0}$ *is a GDV.*
3. *There exists a GDV.*
4. $\mathbb{P}(\overline{m} > \varepsilon) < 1$ *for any* $\varepsilon > 0$ *and any* $\overline{m} \in \overline{M}$.
5. $c \notin \overline{M}$ *for any* $c > 0$.

Proof. 1⇒2: By Proposition 4.8 and Lemma 4.10.
2⇒3: Obvious.
3⇒1: Let ρ be a GDV. Since $\rho(-m) \le \rho^0(-m) \le 0$ for any $m \in M$, we have

$$\begin{aligned} \rho^*(Q) &= \sup_{x \in L} \{\mathbb{E}_Q[-x] - \rho(x)\} \ge \sup_{m \in M} \{\mathbb{E}_Q[m] - \rho(-m)\} \\ &\ge \sup_{m \in M} \mathbb{E}_Q[m] = (\rho^0)^*(Q). \end{aligned} \tag{5.3}$$

Thus, $\rho^*(Q) = \infty$ for any $Q \in \mathcal{P} \backslash \mathcal{Q}$. Supposing $\mathcal{Q} = \emptyset$, ρ equals to $-\infty$ identically by Theorem 2.6. This is a contradiction. In addition, we have $0 \le \inf_{Q \in \mathcal{Q}} (\rho^0)^*(Q) \le \inf_{Q \in \mathcal{Q}} \rho^*(Q) = 0$ since $\rho(0) = 0$.
1⇒4: Supposing that there exist an $\varepsilon > 0$ and an $\overline{m} \in \overline{M}$ such that $\mathbb{P}(\overline{m} > \varepsilon) = 1$, we have $\mathbb{E}_Q[\overline{m}] > \varepsilon$ for any $Q \in \mathcal{P}$. That is, $\sup_{m \in M} \mathbb{E}_Q[m] = \sup_{\overline{m} \in \overline{M}} \mathbb{E}_Q[\overline{m}] > \varepsilon$ for any $Q \in \mathcal{P}$. From the view of Lemma 4.1, either $\mathcal{Q} = \emptyset$ or $\inf_{Q \in \mathcal{Q}} (\rho^0)^*(Q) > 0$ holds true.
4⇒5: We can see this by contraposition.
5⇒1: We fix $c > 0$ arbitrarily. Since $c \notin \overline{M}$, Proposition 2.7 implies that there exists a $Q_c \in \mathcal{Q}$ such that $(\rho^0)^*(Q_c) = \sup_{\overline{m} \in \overline{M}} \mathbb{E}_{Q_c}[\overline{m}] < c$. By the arbitrariness of $c > 0$, we have $\inf_{Q \in \mathcal{Q}} (\rho^0)^*(Q) = 0$. □

Remark 5.3

1. As seen in Example 4.16, even if there exists a GDV, we may find an $m \in M$ such that $\mathbb{P}(m > 0) = 1$, that is, an arbitrage opportunity in a strong sense. In other words, all conditions in Theorem 5.2 are not sufficient for the no-arbitrage condition.

2. The condition $\mathcal{Q} \neq \emptyset$ is not sufficient for $\widehat{\rho^0}$ to be a GDV, since it is not necessarily normalized. See Example 4.12.

3. The first condition in Theorem 5.2 is stronger than $\overline{L}^* \neq \emptyset$. That is, ρ^0 is not necessarily a GDV even if $\overline{L}^* \neq \emptyset$. See Example 4.9.

Remark 5.4 Theorem 3.2 of [3] provided equivalent conditions for the existence of a GDV when M is a convex cone. Now, we shall compare Theorem 5.2 with it.

1. The third condition of Theorem 3.2 in [3]: "$\mathbb{P}(\overline{m} > 0) < 1$ for any $\overline{m} \in \overline{M}$" is sufficient, but not necessary for the existence of a GDV in our setting as seen in Example 4.16.

2. The fourth condition in [3]: "$1 \notin \overline{M}$" is equivalent to condition 5 in Theorem 5.2 when M is a convex cone, whereas condition 5 is stronger than "$1 \notin \overline{M}$" unless M is a cone.

Next, we enumerate equivalent conditions for a given $\rho \in \mathcal{R}$ to be a GDV.

Proposition 5.5 *Let $\rho \in \mathcal{R}$. The following are equivalent:*

1. *ρ is a GDV.*

2. *$\rho(-m) \leq 0$ for any $m \in M$.*

3. *$\rho^*(Q) \geq (\rho^0)^*(Q)$ for any $Q \in \mathcal{P}$, that is, ρ is represented as*

$$\rho(x) = \sup_{Q \in \mathcal{Q}} \{\mathbb{E}_Q[-x] - \rho^*(Q)\}.$$

4. *$\rho(-x) \in [-\widehat{\rho^0}(x), \widehat{\rho^0}(-x)]$ for any $x \in L$.*

5. *$\{\rho^0 \leq 0\} \subset \{\rho \leq 0\}$.*

Proof. 1⇒2: For any $m \in M$, we have $\rho(-m) \le \rho^0(-m) \le 0$ by (4.1).
2⇒3: This is from (5.3).
3⇒4: For any $x \in L$, we have

$$\rho(x) = \sup_{Q \in \mathcal{Q}} \{\mathbb{E}_Q[-x] - \rho^*(Q)\} \le \sup_{Q \in \mathcal{Q}} \{\mathbb{E}_Q[-x] - (\rho^0)^*(Q)\} = \widehat{\rho^0}(x).$$

Moreover, the convexity of ρ yields that $-\rho(-x) \le \rho(x) \le \widehat{\rho^0}(x)$.
4⇒1: Note that $\mathcal{Q} \ne \emptyset$ holds under condition 4. Thus, $\widehat{\rho^0} \le \rho^0$ by Proposition 4.8. In addition, $\widehat{\rho^0}(0) \ge 0$ since $\rho(0) = 0$. So that, $\rho^0(0) = \widehat{\rho^0}(0) = 0$ because $\rho^0(0) \le 0$. As a result, Lemma 4.10 ensures that ρ is a GDV.
3⇒5: Recall that $\mathcal{Q} \ne \emptyset$ is ensured under condition 3. We have

$$\begin{aligned}
\rho^0(x) \le 0 \quad &\Rightarrow \quad \text{for any } \varepsilon > 0, \text{ there exists } m \in M \text{ such that } \varepsilon + m + x \ge 0 \\
&\Rightarrow \quad \text{for any } \varepsilon > 0, \varepsilon + \mathbb{E}_Q[x] + (\rho^0)^*(Q) \ge 0 \text{ for any } Q \in \mathcal{Q} \\
&\Rightarrow \quad \text{for any } \varepsilon > 0, \varepsilon + \mathbb{E}_Q[x] + \rho^*(Q) \ge 0 \text{ for any } Q \in \mathcal{Q} \\
&\Rightarrow \quad \rho(x) \le \varepsilon \text{ for any } \varepsilon > 0 \\
&\Rightarrow \quad \rho(x) \le 0.
\end{aligned}$$

5⇒2: Remark that we have $\rho^0(-m) \le 0$ for any $m \in M$. Thus, $-m \in \{\rho \le 0\}$ for any $m \in M$. □

5.1 Relationship with risk indifference price

When M is a convex cone, $\rho \in \mathcal{R}$ is a GDV if and only if it is a risk indifference price, as shown in Theorem 3.4 of [3]. However, we cannot generalize this result to our setting. In this subsection, we investigate relationship between GDVs and risk indifference prices. We start with the definition of risk indifference prices.

Definition 5.6 For a given $[-\infty, \infty]$-valued functional f on L, we define a functional $I(f)$ on L as

$$I(f)(x) := \inf \left\{ r \in \mathbb{R} \middle| \inf_{m \in M} f(r + m + x) \le \inf_{m \in M} f(m) \right\}.$$

In particular, when ρ is a convex risk measure, $I(\rho)$ is said to be the risk indifference price induced by ρ; and is represented as

$$I(\rho)(x) = \inf \left\{ r \in \mathbb{R} \middle| \inf_{m \in M} \rho(m + x) - r \le \inf_{m \in M} \rho(m) \right\}.$$

$I(\rho)(-x)$ describes the risk indifference seller's price for x induced by ρ as introduced in Xu [41]. Selling x for a price greater than $I(\rho)(-x)$, the investor can find a suitable strategy from M so that the risk measured by ρ does not increase. For more details on $I(\rho)$, see [3], [27] and [41]. Now, we prepare a lemma as follows:

Lemma 5.7 *Let ρ be a convex risk measure on L. If $I(\rho)$ is $(-\infty,\infty]$-valued, then we have $\inf_{m\in M}\rho(m)\in\mathbb{R}$ and that $I(\rho)$ is a convex risk measure with*

$$I(\rho)^*(g)=\begin{cases}(\rho^0)^*(g)+\rho^*(g)+\inf_{m\in M}\rho(m), & \textit{if } g\in\overline{L}^*,\\ \infty, & \textit{otherwise.}\end{cases}$$

If $I(\rho)\in\mathcal{R}$ in addition, then $\mathcal{Q}\neq\emptyset$ and

$$I(\rho)(x)=\sup_{Q\in\mathcal{Q}}\left\{\mathbb{E}_Q[-x]-(\rho^0)^*(Q)-\rho^*(Q)-\inf_{m\in M}\rho(m)\right\}.$$

Proof. We can see the lemma by the same way as the proof of Lemma 2.10 in [3] together with the above Lemma 4.1. □

We illustrate an example of a GDV which is not a risk indifference price; and two examples of GDVs which are risk indifference prices.

Example 5.8 We consider the same illiquid market model as Example 4.11. Defining

$$\rho(x)=\sup_{q\in[0,1]}\left\{\mathbb{E}_q[-x]-\frac{|2q-1|}{4}\right\}, \tag{5.4}$$

we have $\rho(x)\leq 0$ whenever $\widehat{\rho^0}(x)\leq 0$. Thus, Proposition 5.5 implies that ρ is a GDV. Moreover, $\rho(-x)=\frac{1}{4}$ for $x=\frac{1}{2}1_{\{\omega_1\}}$ while $\widehat{\rho^0}(-x)=\frac{5}{16}$, namely, $\rho\neq\widehat{\rho^0}$. The good deal bound induced by ρ in (5.4) is degenerated to a singleton $\{\frac{1}{4}\}$, although its no-arbitrage pricing bound is $[\frac{3}{16},\frac{5}{16}]$ as seen in Example 4.11.

Next, we show that ρ is not a risk indifference price. Suppose that ρ is represented as $\rho=I(\eta)$ for some convex risk measure η. Since $(\rho^0)^*(q)\leq\rho^*(q)\leq\frac{|2q-1|}{4}$ by Proposition 5.5 and (2.3), we have $\rho^*(q)=(\rho^0)^*(q)$ for $q\in\{0,\frac{1}{2},1\}$. In addition, Lemma 5.7 implies that $\rho^*(q)=I(\eta)^*(q)=(\rho^0)^*(q)+\eta^*(q)+\inf_{m\in M}\eta(m)$. Thus, $\eta^*(q)+\inf_{m\in M}\eta(m)=0$ for $q\in\{0,\frac{1}{2},1\}$. Hence, the convexity of η^* implies that $\eta^*(q)+\inf_{m\in M}\eta(m)=0$ for any $q\in[0,1]$, that is, $\rho^*=(\rho^0)^*$, which is a contradiction. □

Example 5.9 (Exponential utility indifference price) For $\gamma > 0$, we set $\Phi(\alpha) = e^{\gamma|\alpha|} - 1$ and $L = M^\Phi$. For an agent having an initial capital $c \in \mathbb{R}$ and an exponential utility function with risk-aversion γ, the utility indifference seller's price $p(-x)$ for $x \in L$ is defined implicitly as

$$\sup_{m\in M} \mathbb{E}[-\exp\{-\gamma(c+m)\}] = \sup_{m\in M} \mathbb{E}[-\exp\{-\gamma(c+p(-x)+m-x)\}].$$

For more details, see [7]. Denoting $\rho_\gamma(x) := \frac{1}{\gamma}\log\mathbb{E}[\exp\{-\gamma x\}]$, we have $p(x) = I(\rho_\gamma)(x)$. Note that ρ_γ is called an entropic risk measure. Assuming $\mathbb{E}[m] \leq 0$ for any $m \in M$ additionally, we have $\inf_{m\in M}\rho_\gamma(m) = 0$, that is, $p(-m) \leq 0$ for any $m \in M$. Hence, p is a GDV. □

Example 5.10 (Shortfall risk measure) We consider an agent selling a claim x with price $r \in \mathbb{R}$; and selecting $m \in M$ as her strategy. Her shortfall risk is then defined as a weighted expectation of the shortfall of her final cash-flow $r + m - x$ with a loss function l. Note that l represents her attitude towards risk. Now, we assume that l is given as $l(\alpha) = \Phi(0 \wedge \alpha)$; and $L = M^\Phi$. For simplicity, we assume the continuity of l. To suppress the shortfall risk less than a certain level $\delta > 0$ which she can endure, the least price she can accept is given as

$$\rho_l(-x) := \inf\{r \in \mathbb{R} \mid \text{there exists } m \in M \text{ such that } \mathbb{E}[l(r+m-x)] \leq \delta\}.$$

As seen in [1], ρ_l is a convex risk measure with the Fatou property under mild conditions. We define $\widehat{\rho}_l$ as $\widehat{\rho}_l(x) := \rho_l(x) - \rho_l(0)$. Denoting $\rho_l^1(x) := \inf\{r \in \mathbb{R} | \mathbb{E}[l(r+x)] \leq \delta\}$, we have $\widehat{\rho}_l = I(\rho_l^1)$. As seen in the previous example, supposing $\mathbb{E}[m] \leq 0$ for any $m \in M$, we have $\inf_{m\in M}\rho_l^1(m) = \rho_l^1(0)$, from which $\widehat{\rho}_l$ is a GDV. □

As seen in Example 4.11, a GDV is not necessarily a risk indifference price. Accordingly, the following theorem gives sufficient conditions for a GDV to be a risk indifference price; and for a risk indifference price to be a GDV.

Theorem 5.11 *Let $\rho \in \mathcal{R}$. We consider the following conditions:*

1. *There exists an $\eta \in \mathcal{R}$ with $\inf_{m\in M}\eta(m) = 0$ such that $\rho = I(\eta)$.*

1′. *There exists a convex risk measure η with $\eta(0) = \inf_{m\in M}\eta(m)$ such that $\rho = I(\eta)$.*

2. *There exists a convex set $A \subset L$ including 0 with $A + L_+ \subset A$ such that for any $x \in L$*

$$\rho(x) = \inf\{r \in \mathbb{R} | \text{there exists } m \in M \text{ such that } r + m + x \in A\}. \quad (5.5)$$

3. *ρ is a GDV.*

Then 1⇒1′ ⇔2⇒3 holds. Moreover, when $\rho^* - (\rho^0)^*$ is convex and M is given as $M = M_0 - L_+$ for some $\sigma(L, L^\Psi)$-compact convex set M_0 including 0, all the above conditions are equivalent.

Proof. 1⇒1′ is obvious.
1′ ⇒2: Denoting $\eta' := \eta - \inf_{m\in M} \eta(m)$, we have $\eta'(0) = \inf_{m\in M} \eta'(m) = 0$. Let $A := \{x \in L | \eta'(x) \leq 0\}$ and $A_\rho := \{x \in L | \rho(x) \leq 0\}$. Note that A is a convex set including 0 with $A + L_+ \subset A$. We have

$$\{x \in L| \text{ there exists } m' \in M \text{ such that } m' + x \in A\} \subset A_\rho,$$

since $\rho(x) = I(\eta)(x) = \inf_{m\in M} \eta'(m + x) \leq \eta'(m' + x) \leq 0$ if x belongs to the LHS. Thus, we have

$$\begin{aligned} \rho(x) &= \inf\{r \in \mathbb{R} | x + r \in A_\rho\} \\ &\leq \inf\{r \in \mathbb{R} | \text{there exists } m \in M \text{ such that } m + x + r \in A\}. \end{aligned}$$

As for the reverse inequality, we have, for any $\varepsilon > 0$,

$$\begin{aligned} \rho(x) &= \inf\{r \in \mathbb{R} | \inf_{m\in M} \eta'(m + x) \leq r\} \\ &\geq \inf\{r \in \mathbb{R} | \text{there exists } m \in M \text{ such that } \eta'(m + x) \leq r + \varepsilon\} \\ &= \inf\{r \in \mathbb{R} | \text{there exists } m \in M \text{ such that } m + x + r \in A\} - \varepsilon. \end{aligned}$$

By the arbitrariness of ε, we obtain (5.5).
2⇒1′: Denote $\eta(x) := \inf\{r \in \mathbb{R} | r + x \in A\}$. Noting that $\eta > -\infty$ by $\eta \geq \rho$; and $\eta(0) = 0$ by $0 \in A$, we obtain that η is a normalized convex risk measure by the conditions on A. Hence, it suffices to see

$$\inf_{m\in M} \eta(m + x) = \rho(x), \quad (5.6)$$

since $\inf_{m\in M} \eta(m) = 0$ holds if (5.6) holds. Remark that $\inf_{m\in M} \eta(m + x) = \infty \Leftrightarrow r + m + x \notin A$ for any $r \in \mathbb{R}$ and any $m \in M \Leftrightarrow \rho(x) = \infty$. Then, we suppose that both $\inf_{m\in M} \eta(m + x)$ and $\rho(x)$ are less than ∞. For any $r > \inf_{m\in M} \eta(m + x)$, there exists an $m \in M$ such that $r + m + x \in A$.

Thus, $\rho(x) \leq r$. On the other hand, for any $r > \rho(x)$, there exists an $m \in M$ such that $r + m + x \in A$, that is, $\eta(m + x) \leq r$, which implies that $\inf_{m\in M} \eta(m + x) \leq r$. As a result, we have (5.6).
2⇒3: As seen in the above, $\rho = I(\eta)$ holds under condition 2. Then, Lemma 5.7 provides that $\rho^* = I(\eta)^* = \eta^* + (\rho^0)^*$. Since $\eta^* \geq 0$ by $\eta(0) = 0$, we have $\rho^* \geq (\rho^0)^*$. Proposition 5.5 implies that ρ is a GDV.

As for the second assertion, it suffices to see the implication 3⇒1. Define $\widetilde{\rho}(x) := \sup_{Q\in\mathcal{Q}}\{\mathbb{E}_Q[-x] - \rho^*(Q) + (\rho^0)^*(Q)\}$. Since $\widetilde{\rho} \geq \rho > -\infty$ and $\widetilde{\rho}(0) \leq 0$ by Proposition 5.5, $\widetilde{\rho}$ is a convex risk measure with the Fatou property. Remark that $\widetilde{\rho}(m) \geq \sup_{Q\in\mathcal{Q}}\{-\rho^*(Q)\} = \rho(0) = 0$ for any $m \in M$, that is, $\widetilde{\rho}(0) = 0$ and $\inf_{m\in M} \widetilde{\rho}(m) = 0$. Thus, we have

$$\begin{aligned}
I(\widetilde{\rho})(x) &= \inf_{m\in M} \widetilde{\rho}(m + x) - \inf_{m\in M} \widetilde{\rho}(m) = \inf_{m\in M_0} \widetilde{\rho}(m + x) \\
&= \inf_{m\in M_0} \sup_{Q\in\mathcal{Q}} \{\mathbb{E}_Q[-m - x] - \rho^*(Q) + (\rho^0)^*(Q)\} \\
&= \sup_{Q\in\mathcal{Q}} \inf_{m\in M_0} \{\mathbb{E}_Q[-m - x] - \rho^*(Q) + (\rho^0)^*(Q)\} \\
&= \sup_{Q\in\mathcal{Q}} \{\mathbb{E}_Q[-x] - \rho^*(Q)\} = \rho(x),
\end{aligned}$$

since the minimax theorem (Theorem 3.1 of Simons [39]) is applicable by the compactness of M_0 and the convexity of $\rho^* - (\rho^0)^*$. □

Remark 5.12 1. Theorem 3.4 in [3] asserts that, when M is a convex cone, the following are equivalent for $\rho \in \mathcal{R}$: (a) ρ is a GDV; (b) there exists $\eta \in \mathcal{R}$ such that $\rho = I(\eta)$; and (c) condition 2 in Theorem 5.11. Now, recall that $\inf_{m\in M} \eta(m) = 0$ automatically holds in the convex cone markets. That's because condition 1 in Theorem 5.11 is stronger than the above condition (b).

2. When M is a convex cone, $\widetilde{\rho}$ coincides with ρ; and $\inf_{m\in M} \rho(m) = 0$ holds. Thus, we do not need the minimax theorem to see the implication 3⇒1 in the convex cone case.

3. Madan and Cherny [31] developed a theory for bid and ask prices. They gave a framework of bid and ask prices which are expressed in a similar way with (5.5), employing the concept of acceptability indices and acceptability levels. More precisely, [31] formulated an ask price $a(x)$, corresponding to $\rho(-x)$ in (5.5), as follows:

$$a(x) = \inf\{r \in \mathbb{R} | \text{there exists } m \in M \text{ such that } \alpha(r + m + x) \geq \gamma\}$$

for an acceptability level $\gamma > 0$; and an acceptability index α, which is defined as a function from L to $[0, \infty]$ satisfying (a) $\alpha(x)$, $\alpha(y) \geq \gamma \implies \alpha(x+y) \geq \gamma$, (b) $\alpha(x) \geq \gamma$, $y \geq x \implies \alpha(y) \geq \gamma$, and (c) $\alpha(x) \geq \gamma \implies \alpha(cx) \geq \gamma$ for any constant $c > 0$. Note that M is assumed to be a convex cone in [31].

5.2 Extension to conical market

Here we consider a conical market generated by the convex constrained market M. We define a convex cone set generated by M as

$$M' := \{cm | c \geq 0, m \in M\};$$

and regard it as the set of all 0-attainable claims in the extended market. Now, for a given $\rho \in \mathcal{R}$, we denote

$$\rho'(x) := \sup_{Q \in \mathcal{Q}_0} \{\mathbb{E}_Q[-x] - \rho^*(Q)\}.$$

Note that ρ' is a convex risk measure on L with the Fatou property whenever $\mathcal{Q}_0 \neq \emptyset$, and vice versa. In addition, $\rho' \in \mathcal{R}$ if and only if $\inf_{Q \in \mathcal{Q}_0} \rho^*(Q) = 0$. We show the following proposition:

Proposition 5.13 *For any GDV ρ (for the market M), if $\rho' \in \mathcal{R}$, then ρ' is the largest GDV for the extended conical market M' smaller than ρ.*

Proof. Since $\mathbb{E}_Q[m'] \leq 0$ for any $m' \in M'$ and $Q \in \mathcal{Q}_0$, we have

$$\rho'(-m') = \sup_{Q \in \mathcal{Q}_0} \{\mathbb{E}_Q[m'] - \rho^*(Q)\} \leq \sup_{Q \in \mathcal{Q}_0} \{-\rho^*(Q)\} = 0$$

for any $m' \in M'$, which means that ρ' is a GDV for M' by Proposition 5.5.

Now, ρ' is smaller than ρ, that is, $\rho'(x) \leq \rho(x)$ for any $x \in L$. Taking ρ_1 a GDV for M' smaller than ρ arbitrarily, we show $\rho' \geq \rho_1$. Denoting by ρ_1^* the penalty function of ρ_1, we have $\rho_1^*(Q) = \sup_{x \in L} \{\mathbb{E}_Q[x] - \rho_1(-x)\} \geq \sup_{x \in L} \{\mathbb{E}_Q[x] - \rho(-x)\} = \rho^*(Q)$ for any $Q \in \mathcal{Q}$. Note that, for any $Q \notin \mathcal{Q}_0$, there exists an $m_1' \in M'$ such that $\mathbb{E}_Q[m_1'] > 0$, that is, $\sup_{m' \in M'} \mathbb{E}_Q[m'] = \infty$ by the cone property of M'. Hence, for any $Q \in \mathcal{Q} \setminus \mathcal{Q}_0$, we have

$$\rho_1^*(Q) \geq \sup_{m' \in M'} \{\mathbb{E}_Q[m'] - \rho_1(-m')\} \geq \sup_{m' \in M'} \mathbb{E}_Q[m'] = \infty.$$

Consequently, we obtain

$$\rho'(x) = \sup_{Q \in \mathcal{Q}_0} \{\mathbb{E}_Q[-x] - \rho^*(Q)\} \geq \sup_{Q \in \mathcal{Q}_0} \{\mathbb{E}_Q[-x] - \rho_1^*(Q)\} = \rho_1(x)$$

for any $x \in L$. $\square$

5.3 Coherent good deal valuations

When M is a convex cone, $\widehat{\rho^0}$ is coherent, that is, there is a coherent GDV whenever a GDV exists. On the other hand, in our setting, since $\widehat{\rho^0}$ is not necessarily coherent, there might be no coherent GDV even if a GDV exists. Now, we illustrate an equivalent condition for the existence of a coherent GDV.

Proposition 5.14 *$\mathcal{Q}_0 \neq \emptyset$ if and only if there exists a coherent GDV.*

Proof. Suppose $\mathcal{Q}_0 \neq \emptyset$. Taking a $Q \in \mathcal{Q}_0$, we define $\rho_Q(x) := \mathbb{E}_Q[-x]$ for any $x \in L$. Note that ρ_Q is in $\mathcal{R}$ and coherent. We have then $\rho_Q(-m) = \mathbb{E}_Q[m] \leq \sup_{m \in M} \mathbb{E}_Q[m] = 0$ for any $m \in M$, from which ρ_Q is a GDV.

To see the reverse implication, let ρ be a coherent GDV. Since ρ is coherent, ρ^* takes the values 0 and ∞ only. Defining $\widetilde{\mathcal{Q}} := \{Q \in \mathcal{Q} | \rho^*(Q) = 0\}$, we have that $\widetilde{\mathcal{Q}}$ is nonempty and $\rho(x) = \sup_{Q \in \widetilde{\mathcal{Q}}} \mathbb{E}_Q[-x]$. Proposition 5.5 implies that, for any $m \in M$ and any $\widetilde{Q} \in \widetilde{\mathcal{Q}}$, $0 \geq \rho(-m) = \sup_{Q \in \widetilde{\mathcal{Q}}} \mathbb{E}_Q[m] \geq \mathbb{E}_{\widetilde{Q}}[m]$. Thus, $\sup_{m \in M} \mathbb{E}_{\widetilde{Q}}[m] = 0$ for any $\widetilde{Q} \in \widetilde{\mathcal{Q}}$, that is, $\widetilde{\mathcal{Q}} \subset \mathcal{Q}_0$. $\square$

6 Fundamental Theorem of Asset Pricing

In this section, we prove a Kreps-Yan type FTAP with convex constraints. Basically, the Kreps-Yan theorem ([29] or Section 5 in [19]) asserts, very roughly speaking, the equivalence between the existence of an equivalent martingale measure and the NFL: $\overline{M} \cap L_+ = \{0\}$. [3] proved, for the case where M is a convex cone, the equivalence among the NFL, $\mathcal{Q}_0 \cap \mathcal{Q}^e \neq \emptyset$ and the existence of a relevant GDV. Noting that $\mathcal{Q}$ and $\mathcal{Q}_0$ coincide when M is a convex cone, and taking Theorem 5.2 and Lemma 4.15 into account, we naturally expect the equivalence between the NFL and either condition 1 or 1′ of the following theorem, whereas neither of them actually holds. On the other hand, the equivalence between the NFL and the existence of a relevant GDV still holds. The following is an FTAP for markets with convex constraints:

Theorem 6.1 *As for the following conditions, we have 1′ ⇒4⇔3⇔2⇒1.*

1. $\mathcal{Q}^e \neq \emptyset$ *and* $\inf_{Q\in\mathcal{Q}^e}(\rho^0)^*(Q) = 0$.

1′. *There exists a* $Q \in \mathcal{Q}^e$ *with* $(\rho^0)^*(Q) = 0$.

2. $\overline{M} \cap L_+ = \{0\}$.

3. *There exists a relevant GDV.*

4. $\widehat{\rho^0}$ *is a relevant GDV.*

Proof. 2⇒1: For each $\delta \in (0,1]$, we define a set B_δ as

$$B_\delta := \{x \in L | 0 \leq x \leq 1, \mathbb{E}[x] \geq \delta\}. \tag{6.1}$$

Note that B_δ is compact in $\sigma(L, L^\Psi)$ and $\overline{M} \cap B_\delta = \emptyset$. Thus, Proposition 2.7 ensures the existence of $Q_\delta \in \mathcal{Q}$ satisfying

$$\sup_{\overline{m}\in\overline{M}} \mathbb{E}_{Q_\delta}[\overline{m}] < \inf_{x\in B_\delta} \mathbb{E}_{Q_\delta}[x]. \tag{6.2}$$

Now, we denote $Q^{(k)} := Q_{2^{-k}} \in \mathcal{Q}$ ($Q_{2^{-k}}$ is defined in (6.2) for $\delta = 2^{-k}$) for any $k \in \mathbb{N}$; $\alpha_k := \left\|\frac{dQ^{(k)}}{d\mathbb{P}}\right\|_\Psi \vee 1$ ($\|y\|_\Psi := \inf\{c > 0 | \mathbb{E}[\Psi(y/c)] \leq 1\}$); and $C_n := \sum_{k=n}^\infty \frac{2^{-k}}{\alpha_k} < \infty$ for any $n \in \mathbb{N}$. Moreover, we define $\beta_k^n := \frac{2^{-k}}{C_n\alpha_k}$ for any $k \geq n$; and $\widetilde{Q}^{(n)} := \sum_{k=n}^\infty \beta_k^n Q^{(k)}$ for any $n \in \mathbb{N}$. Note that $\widetilde{Q}^{(n)}$ is a probability measure equivalent to $\mathbb{P}$, since $\sum_{k=n}^\infty \beta_k^n = 1$ and $Q^{(k)}(A) > 0$ for any $A \in \mathcal{F}$ with $\mathbb{P}(A) > 2^{-k}$ by (6.2). Now, we denote $\gamma_i := \sum_{k=n}^{n+i} \beta_k^n \frac{dQ^{(k)}}{d\mathbb{P}}$ for $i = 1, 2, \ldots$. Then, $\{\gamma_i\}$ is a Cauchy sequence in $\|\cdot\|_\Psi$; and Lemma 4.3 yields $\frac{d\widetilde{Q}^{(n)}}{d\mathbb{P}} \in L^\Psi$. Moreover, noting that $2^{-n} \in B_{2^{-k}}$ for any $k \geq n$, we have, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\sup_{\overline{m}\in\overline{M}} E_{\widetilde{Q}^{(n)}}[\overline{m}] &= \sup_{\overline{m}\in\overline{M}} \sum_{k=n}^\infty \beta_k^n E_{Q^{(k)}}[\overline{m}] \leq \sum_{k=n}^\infty \beta_k^n \sup_{\overline{m}\in\overline{M}} E_{Q^{(k)}}[\overline{m}] \\
&< \sum_{k=n}^\infty \beta_k^n \inf_{x\in B_{2^{-k}}} E_{Q^{(k)}}[x] \leq \sum_{k=n}^\infty \beta_k^n 2^{-n} = 2^{-n},
\end{aligned}$$

which implies $\widetilde{Q}^{(n)} \in \mathcal{Q}^e$ with $(\rho^0)^*(\widetilde{Q}^{(n)}) < 2^{-n}$. As a result, we obtain $\inf_{Q\in\mathcal{Q}^e}(\rho^0)^*(Q) = 0$.
4⇒3: Obvious.
3⇒2: Let ρ be a relevant GDV. Since $\rho(-z) > 0$ for all $z \in L_+\backslash\{0\}$ by the relevance, it suffices to see that $\rho(-\overline{m}) \leq 0$ for any $\overline{m} \in \overline{M}$. If there

exists an $\overline{m} \in \overline{M}$ with $\rho(-\overline{m}) > 0$, then we can find a $Q \in \mathcal{Q}$ such that $\mathbb{E}_Q[\overline{m}] > \rho^*(Q) \geq (\rho^0)^*(Q) = \sup_{m \in M} \mathbb{E}_Q[m]$ by Proposition 5.5. This is a contradiction.

2⇒4: Since condition 2 implies condition 1, $\widehat{\rho^0}$ is a GDV by Theorem 5.2. Next, we show the relevance of $\widehat{\rho^0}$. For any $z \in L_+ \backslash \{0\}$, $z \wedge 1$ belongs to B_δ for some $\delta \in (0, 1]$, where B_δ is defined in (6.1). Since $B_\delta \cap \overline{M} = \emptyset$, Proposition 2.7 implies the existence of $Q \in \mathcal{Q}$ satisfying $\sup_{\overline{m} \in \overline{M}} \mathbb{E}_Q[\overline{m}] < \inf_{x \in B_\delta} \mathbb{E}_Q[x]$. Then, we have $0 \leq \sup_{\overline{m} \in \overline{M}} \mathbb{E}_Q[\overline{m}] < \mathbb{E}_Q[z \wedge 1] \leq \mathbb{E}_Q[z]$. Consequently, $\widehat{\rho^0}$ is relevant.

$1' \Rightarrow$4: Theorem 5.2 and Lemma 4.15 imply that $\widehat{\rho^0}$ is a relevant GDV. □

Remark 6.2 We can regard Theorem 6.1 as a generalization of Corollary 9.32 in [23].

Example 6.3 For the model introduced in Examples 4.11 and 5.8, $\widehat{\rho^0}$ is relevant, and condition $1'$ in Theorem 6.1 holds. □

In order to complete Theorem 6.1, we illustrate counterexamples for the implications which are not shown.

Example 6.4 (Counterexample for 1⇒2) Setting $\Omega = \{\omega_k | k \in \mathbb{N}\}$ and $\mathbb{P}(\{\omega_k\}) = 2^{-k}$ for each $k \in \mathbb{N}$, we define random variables S_k, $k \in \mathbb{N}$ as

$$S_k(\omega) = \begin{cases} 1, & \text{if } \omega = \omega_k, \\ 0, & \text{if } \omega \neq \omega_k, \end{cases}$$

and $M = \mathrm{co}\{S_1, S_2, \dots\} - L_+$. Remark that any element $m \in \mathrm{co}\{S_1, S_2, \dots\}$ is expressed as $m = \sum_{k=1}^{\infty} \lambda_k S_k$, where the sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ satisfies $\lambda_k \geq 0$ for any $k \geq 1$, $\sum_{k=1}^{\infty} \lambda_k = 1$ and $\lambda_k = 0$ except for finitely many ks. This model then does not satisfy condition 2.

Next, we make sure of condition 1. To this end, we define for each $n \in \mathbb{N}$,

$$Q_n(\{\omega_k\}) := \begin{cases} \frac{1}{n}, & \text{if } k \leq n-1, \\ \frac{2^{n-k-1}}{n}, & \text{otherwise}. \end{cases}$$

We can see that each Q_n is a probability measure equivalent to $\mathbb{P}$; and $\frac{\mathrm{d}Q_n}{d\mathbb{P}} \leq \frac{2^{n-1}}{n}$. For $m \in \mathrm{co}\{S_1, S_2, \dots\}$ with $m = \sum_{k=1}^{\infty} \lambda_k S_k$, we have

$$\mathbb{E}_{Q_n}[m] = \sum_{k=1}^{\infty} \lambda_k Q_n(\{\omega_k\}) \leq \sum_{k=1}^{\infty} \frac{\lambda_k}{n} = \frac{1}{n}.$$

Thus, we obtain, for any $n \in \mathbb{N}$,

$$(\rho^0)^*(Q_n) = \sup_{m \in M} \mathbb{E}_{Q_n}[m] = \sup_{m \in \mathrm{co}\{S_1, S_2, \dots\}} \mathbb{E}_{Q_n}[m] \le \frac{1}{n},$$

from which condition 1 follows.

Example 6.5 (Counterexample for 4⇒1′) We take $\Omega = \{\omega_k | k \in \mathbb{Z}\}$ and a probability measure $\mathbb{P}$ with $\mathbb{P}(\{\omega_k\}) > 0$ for each $k \in \mathbb{Z}$. Further, we define random variables S_k, $k \in \mathbb{Z}$ as

$$S_k(\omega) = \begin{cases} 1, & \text{if } \omega = \omega_k, \\ -1, & \text{if } \omega = \omega_{k-1}, \\ 0, & \text{otherwise,} \end{cases}$$

and $M = \mathrm{co}\{S_k | k \in \mathbb{Z}\} - L_+$.

Now, we see that this model satisfies condition 4. We define, for each $i \in \mathbb{Z}$ and $j \in \mathbb{N}$,

$$Q^i_j(\{\omega_k\}) := \left(\frac{1}{j} - \frac{|k-i|}{j^2}\right) \vee 0.$$

We can see that each Q^i_j is a probability measure with bounded density $\frac{\mathrm{d}Q^i_j}{\mathrm{d}\mathbb{P}}$. For any $m \in M$ with representation $m = \sum_{k=1}^{\infty} \lambda_k S_k$, we have

$$\mathbb{E}_{Q^i_j}[m] = \sum_{k=1}^{\infty} \lambda_k \mathbb{E}_{Q^i_j}[S_k] = \sum_{k=1}^{\infty} \lambda_k \left\{ Q^i_j(\{\omega_k\}) - Q^i_j(\{\omega_{k-1}\}) \right\} \le \frac{1}{j^2}.$$

We have then $(\rho^0)^*(Q^i_j) = \frac{1}{j^2}$, that is, $Q^i_j \in \mathcal{Q}$. Thus, $\mathcal{Q} \neq \emptyset$ and $\inf_{Q \in \mathcal{Q}} (\rho^0)^*(Q) = 0$, which ensure that $\widehat{\rho^0}$ is a GDV. On the other hand, taking a $z \in L_+ \setminus \{0\}$ arbitrarily, we can find an $\varepsilon > 0$ and an $i \in \mathbb{Z}$ satisfying $z \ge \varepsilon 1_{\{\omega_i\}}$. Hence, we have $\mathbb{E}_{Q^i_j}[z] \ge \varepsilon Q^i_j(\{\omega_i\}) = \frac{\varepsilon}{j}$ for any $j \in \mathbb{N}$. For a sufficient large j, we have $\mathbb{E}_{Q^i_j}[z] > (\rho^0)^*(Q^i_j)$, that is, $\widehat{\rho^0}(-z) > 0$.

Next, we see that condition $1'$ does not hold. Since condition 4 holds, so does condition 1 by Theorem 6.1, that is, $\mathcal{Q}^e$ is nonempty. For any $Q \in \mathcal{Q}^e$, there exists a $k_Q \in \mathbb{Z}$ such that $Q(\{\omega_{k_Q}\}) - Q(\{\omega_{k_Q - 1}\}) > 0$. Indeed, since $K_0 := \{k \in \mathbb{Z} | Q(\{\omega_k\}) \ge Q(\{\omega_0\})\}$ is finite, we can take $k_Q = \min K_0$. Hence, we have, for any $Q \in \mathcal{Q}^e$,

$$(\rho^0)^*(Q) = \sup_{m \in M} \mathbb{E}_Q[m] \ge \mathbb{E}_Q[S_{k_Q}] = Q(\{\omega_{k_Q}\}) - Q(\{\omega_{k_Q - 1}\}) > 0,$$

which denies condition $1'$.

Here we give an equivalent condition to condition $1'$ in Theorem 6.1.

Proposition 6.6 *There exists a relevant coherent GDV if and only if $\mathcal{Q}^e \cap \mathcal{Q}_0$ is nonempty.*

Proof. "if" part: This is shown by a similar way with Proposition 5.14. Taking a $Q \in \mathcal{Q}^e \cap \mathcal{Q}_0$, we define $\rho_Q(x) := \mathbb{E}_Q[-x]$ for any $x \in L$. Then, ρ_Q is a coherent GDV which is relevant.

"only if" part: This follows from the Halmos-Savage theorem (see e.g. Theorem 25 in [18]). Now, we give just a sketch of proof.

Let ρ be a relevant coherent GDV. Denoting $A := \{x \in L | \rho(-x) \leq 0\}$, we have $\sup_{x\in A} \mathbb{E}_Q[x] = \rho^*(Q)$ for any $Q \in \mathcal{Q}$. Now, we consider B_δ defined in (6.1). Since $A \cap B_\delta = \emptyset$ for any $\delta \in (0,1]$, the same separating argument as Proposition 2.7 implies that, for any $\delta \in (0,1]$, there exists a $Q \in \mathcal{Q}$ such that

$$\rho^*(Q) = \sup_{x\in A} \mathbb{E}_Q[x] < \inf_{z\in B_\delta} \mathbb{E}_Q[z],$$

since A is $\sigma(L, L^\Psi)$-closed. Now, for each $\delta \in (0,1]$, we denote

$$\mathcal{Q}_\delta := \{Q \in \mathcal{Q} | \inf_{z\in B_\delta} \mathbb{E}_Q[z] > \rho^*(Q)\}.$$

Remark that $\mathcal{Q}_\delta$ is stable for countable unions. Then, we can find a $Q_\delta \in \mathcal{Q}_\delta$ satisfying $\mathbb{P}(\{\frac{dQ_\delta}{d\mathbb{P}} > 0\}) = \max_{Q\in\mathcal{Q}_\delta} \mathbb{P}(\{\frac{dQ}{d\mathbb{P}} > 0\})$. In addition, we can see that $\mathbb{P}(\{\frac{dQ_\delta}{d\mathbb{P}} > 0\}) = 1$ by contradiction. Since $\delta \in B_\delta$, we have $\rho^*(Q_\delta) < \delta$, from which $\rho^*(Q_\delta) = 0$ holds, since ρ^* takes the values 0 and ∞ only. Hence, $(\rho^0)^*(Q_\delta) = 0$ by Proposition 5.5, that is, $Q_\delta \in \mathcal{Q}^e \cap \mathcal{Q}_0$. □

Remark 6.7 As mentioned at the beginning of this section, [3] showed in their Theorem 4.2 the equivalence among $\mathcal{Q}^e \cap \mathcal{Q}_0 \neq \emptyset$, the NFL and the existence of a relevant GDV for the case where M is a convex cone. Theorem 6.1 together with Examples 6.4 and 6.5 implies that, under convex constraints, there is no equivalent condition corresponding to $\mathcal{Q}^e \cap \mathcal{Q}_0 \neq \emptyset$, which is shown to be equivalent to the existence of a relevant coherent GDV in Proposition 6.6.

6.1 An extension theorem

We assume that any $x \in L$ is priced at $\rho(-x)$, where ρ is a GDV. Then $x - \rho(-x)$ is a 0-attainable claim. Now, we extend our market by adding all these claims to M. More precisely, the set of 0-attainable claims for the extended market is represented as

$$\begin{aligned} M^\rho &:= \{x - \rho(-x) | x \in L, \rho(-x) < \infty\} - L_+ \\ &= \{x \in L | \rho(-x) = 0\} - L_+ = \{x \in L | \rho(-x) \leq 0\}. \end{aligned}$$

Remark that M^ρ is a convex set including M. Since M^ρ is closed in $\sigma(L, L^\Psi)$ by Theorem 2.6, the NFL for the extended market is equivalent to $M^\rho \cap L_+ = \{0\}$, which is the no-arbitrage condition. We have the following theorem:

Theorem 6.8 *Let ρ be a GDV. The following are equivalent:*

1. *ρ is relevant.*
2. *$-\widehat{\rho^0}(x - z) < \rho(-x)$ for any $x \in L$ and $z \in L_+ \backslash \{0\}$.*
3. *$-\rho^0(x - z) < \rho(-x)$ for any $x \in L$ and $z \in L_+ \backslash \{0\}$.*
4. *$M^\rho \cap L_+ = \{0\}$.*

Proof. The implications 2⇒3⇒1⇔4 are shown by the same way as Theorem 4.3 in [3]. Then we have only to see the implication 1⇒2. For any $z \in L_+ \backslash \{0\}$, there exists $Q_z \in \mathcal{Q}$ such that $\mathbb{E}_{Q_z}[z/2] > \rho^*(Q_z)$ by the relevance of ρ. Thus, Proposition 5.5 implies that $\mathbb{E}_{Q_z}[z] > 2\rho^*(Q_z) \geq \rho^*(Q_z) + (\rho^0)^*(Q_z)$. Therefore, for any $x \in L$, we have

$$\begin{aligned} -\widehat{\rho^0}(x - z) &= \inf_{Q \in \mathcal{Q}} \left\{\mathbb{E}_Q[x - z] + (\rho^0)^*(Q)\right\} \leq \mathbb{E}_{Q_z}[x - z] + (\rho^0)^*(Q_z) \\ &< \mathbb{E}_{Q_z}[x] - \rho^*(Q_z) \leq \sup_{Q \in \mathcal{Q}} \{\mathbb{E}_Q[x] - \rho^*(Q)\} = \rho(-x). \end{aligned}$$

□

7 Conclusions

We study properties of good deal bounds for incomplete markets with convex constraints. In Section 4, we study properties of superhedging cost ρ^0 and its largest minorant with the Fatou property $\widehat{\rho^0}$. Next, we see that the existence of a GDV is equivalent to "$\mathcal{Q} \neq \emptyset$ and $\inf_{Q \in \mathcal{Q}} (\rho^0)^*(Q) = 0$" in Theorem 5.2; and enumerate equivalent conditions for a given $\rho \in \mathcal{R}$

to be a GDV in Proposition 5.5. Moreover, we introduce an example of a GDV which is not a risk indifference price; and look into relationship between GDVs and risk indifference prices. Furthermore, we prove an FTAP under convex constraints in Theorem 6.1. Among others, the equivalence between the NFL and the existence of a relevant GDV is proved. Moreover, we illustrate counterexamples to see that neither $\mathcal{Q}^e \cap \mathcal{Q}_0 \neq \emptyset$ nor $\inf_{Q \in \mathcal{Q}^e} (\rho^0)^*(Q) = 0$ is equivalent to the NFL.

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