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Chaowen Yu*

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Keywords: Maschler-Owen value, NTU Shapley value, axiomatization, prize game, hyperplane game

JEL Classification Numbers: C7, D7

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1. Introduction

A non-transferable utility (NTU) game in coalitional form assigns for each coalition of players a set of payoffs that is feasible for players in the coalition. Given a class of NTU games, a *value function* (or simply *value*) is a function which associates to each NTU game in the class, an element of the set of feasible payoffs for the grand coalition (i.e., the set of all players in the game).

The purpose of this paper is to compare the axiomatic properties of two well-known values, the *Maschler-Owen value* (Maschler and Owen [1989]) and the *NTU Shapley value* (Shapley [1953]) on the same domain of NTU games.

Despite its importance, few studies have examined the difference of axiomatic properties of these two values on the same domain. The main reason for this may be that it is difficult to find the well-known class of NTU games on which the nonemptiness of both values are ensured. To avoid this difficulty, there may be two paths of research: One is to compare two values on a quite restricted domain of games on which both values are nonempty.[1] The other is to extend the definition of these values consistently so as to ensure the nonemptiness of two values on some relevant class of games. Our research follows this second path. For this purpose, we define the new value, the *generalized Shapley value* and compare the axiomatic properties of this value to that of the Maschler-Owen value on the domain of games consisting of all *monotonic hyperplane games* and all *monotonic prize games* (Hart [1994]).

We provide seven axioms and show that on our domain, the Maschler-Owen value satisfies all axioms except *Zero-Additivity* whereas the generalized Shapley value satisfies all axioms except *Marginality*. In Hart [1994], it was shown that on the domain of all monotonic hyperplane games, Macshler-Owen value is a unique value satisfying *Efficiency*, *Symmetry*, and *Marginality*. We show that this result is also valid on our larger domain. We also show that on our domain the generalized Shapley value can be characterized by the six axioms *Efficiency, Symmetry, TU-Marginality, Zero-Additivity, Scale Invariance*, and *Dependence only on Non-negative Payoffs*. The independence of these axioms are also given.

2. Preliminaries

Let $\mathbb{N} = \{1, 2, \cdots\}$ be the set of positive integers and let $\mathbb{R}$ be the set of real numbers. We interpret $\mathbb{N}$ as the set of potential players, and nonempty finite subset of $\mathbb{N}$ is called a *coalition*. Let N be a coalition. We denote by $\mathbb{R}^N$ the set of all real-valued functions on N and write 1_N an element of $\mathbb{R}^N$ for which $(1_N)_i = 1$ for every $i \in N$. For every two vectors $x, y \in \mathbb{R}^N$, $x \leqslant y$ means $x_i \leqslant y_i$ for every $i \in N$, $x < y$ means $x_i \leqslant y_i$ for every $i \in N$ and $x_i < y_i$ for some $i \in N$ and $x \ll y$ means $x_i < y_i$ for any $i \in N$. $\mathbb{R}^N_+$ and $\mathbb{R}^N_{++}$ represent the set of non-negative and positive elements of $\mathbb{R}^N$, respectively, i.e., $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : 0 \leqslant x\}$ and $\mathbb{R}^N_{++} = \{x \in \mathbb{R}^N : 0 \ll x\}$. If x and λ are elements in $\mathbb{R}^N$, define $\lambda x \in \mathbb{R}^N$ by $(\lambda x)_i = \lambda_i x_i$ and denote the inner product $\sum_{i \in N} \lambda_i x_i$ by $\lambda \cdot x$. For

[1] For example, see Hart [2004].

every subsets A of $\mathbb{R}^N$, the boundary of A is denoted by ∂A and for two sets A and B of $\mathbb{R}^N$ and a vector $\lambda \in \mathbb{R}^N$, we write $A+B = \{a+b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$.

A *non-transferable utility (NTU) game in coalitional form* (or simply a *game*) is a pair (N, V), where N is a coalition (the set of *players*), and V (the *coaltional function*) is a mapping which associates to each coalition $S \subseteq N$, a nonempty proper subset $V(S) \subsetneq \mathbb{R}^S$ of *feasible payoff vectors for* S such that each $V(S)$ is closed and comprehensive (i.e., $x \in V(S)$, $y \in \mathbb{R}^S$ and $y \leqslant x$ imply $y \in V(S)$). We also assume that $0 \in V(S)$ for every coalition $S \subseteq N$ just for a convenient normalization.

In this paper we consider the following special classes of NTU games:

- *Hyperplane games*: each $V(S)$ can be written by $V(S) = \{x \in \mathbb{R}^S : \lambda^S \cdot x \leqslant c^S\}$ for some $\lambda^S \in \mathbb{R}^S_{++}$ and $c^S \in \mathbb{R}$.
- *Prize games*: there exists a hyperplane game (N, W) such that for each coalition $S \subseteq N$, $V(S)$ can be written by $V(S) = (W(S) \cap \mathbb{R}^S_+) - \mathbb{R}^S_+$.

For the interpretation of prize games, see Hart [1994].

A game (N, V) is a *transferable utility (TU) game* (or simply a *TU game*) if there is a mapping $v : 2^N \setminus \{\emptyset\} \to \mathbb{R}$ such that for each coalition $S \subseteq N$, $V(S) = \{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leqslant v(S)\}$. Clearly, every TU game is a hyperplane game. For TU games, we will write (N, V) or (N, v) interchangeably.

A game (N, V) is said to be *monotonic* if $V(S) \times \{0^{T \setminus S}\} \subseteq V(T)$ for any pair of coalitions S, T with $S \subseteq T \subseteq N$. Thus, a game is monotonic if feasible payoffs which members of a coalition S can get when they cooperate are still feasible when members of a larger coalition T cooperate. The set of all monotonic hyperplane games and the set of all prize games are written by $\mathscr{H}_{mo}$ and $\mathscr{P}_{mo}$, respectively. We also write $\mathscr{D}_{mo} = \mathscr{H}_{mo} \cup \mathscr{P}_{mo}$.

We close this section by stating the definition of the *value function*.[2] A *value function* φ on some class of games is a mapping which associates to each game (N, V) in the domain, an element $\varphi(N, V)$ of $V(N)$.

3. Results

In this section we will give the definitions of generalized Shapley value (Shapley [1953]) and Maschler-Owen value (Maschler-Owen [1989]), and compare the set of axioms these values satisfy.

3.1. The TU Shapley value

For a TU game (N, v), the *TU Shapley value* $Sh(N, v) \in \mathbb{R}^N$ of (N, v) is defined by

$$Sh_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi} [v(P_i^\pi \cup \{i\}) - v(P_i^\pi)],$$

[2]Unlike many papers on this field analyzing axiomatic properties of value *correspondences*, we restrict our analysis to value *functions*. The reason why we do so is that we have difficulties in defining *Marginality* (defined below) on value *correspondences*. In contrast, if each element of a value correspondence is not a payoff (i.e., an element of $\mathbb{R}^N$) but a payoff configuration (i.e., an element of $\prod_{S \subseteq N, S \neq \emptyset} \mathbb{R}^S$, see Hart [1985]), there is a relevant definition of *Marginality* on value correspondences. See Hart [2005].

where Π denotes the set of all orders on N (i.e., bijections of N onto $\{1, 2, \cdots, |N|\}$) and $P_i^\pi = \{j \in N : \pi(j) < \pi(i)\}$ for each $\pi \in \Pi$ and each $i \in N$.

3.2. The NTU Shapley value

Let (N, V) be a game. For each $\lambda \in \mathbb{R}^N_{++}$, and each coalition $S \subseteq N$, define

$$v_\lambda(S) = \sup\{\lambda_S \cdot x : x \in V(S)\}.$$

If $v_\lambda(S) < \infty$ for every coalition $S \subseteq N$, (N, v_λ) is a TU game. In this case, we say the vector λ *defines* a TU game v_λ on (N, V). A *NTU Shapley value* of a game (N, V) is a set of elements x of $V(N)$ satisfying the following properties:

(i) there exists a $\lambda \in \mathbb{R}^N_{++}$ which defines a TU game v_λ on (N, V);

(ii) $\lambda x = Sh(v_\lambda)$.

We denote the NTU Shapley value of (N, V) by $\Psi(N, V)$.

It can be easily seen that the NTU Shapley value of a monotonic hyperplane game (N, V) is empty unless it has the common normal vector (i.e., for each coalition $S \subseteq N$, there is a scalar $\alpha \in \mathbb{R}_{++}$ such that $\lambda^S = \alpha(\lambda_i^N)_{i \in S}$ where $V(T) = \{x \in \mathbb{R}^T : \lambda^T \cdot x \leqslant c^T\}$ for some $c^T \in \mathbb{R}$ for any coalition $T \subseteq N$).

3.3. The generalized Shapley value

Now we define the *generalized Shapley value* $\tilde{\Psi}$ on $\mathscr{D}_{mo}$ as follows:

- For each monotonic prize game $(N, V) \in \mathscr{P}_{mo}$ with $V(N) = \{x \in \mathbb{R}^N : \lambda^N \cdot x \leqslant c^N\} \cap \mathbb{R}^N_+ - \mathbb{R}^N_+$, it is clear that the vector λ^N defines a TU game v_{λ^N} on (V, N). Then, we define $\tilde{\Psi}(N, V) = \{(\lambda^N)^{-1} Sh(N, v_{\lambda^N})\}$.

- For each monotonic hyperplane game $(N, V) \in \mathscr{H}_{mo}$, let (N, W) be a prize game with $W(S) = V(S) \cap \mathbb{R}^S_+ - \mathbb{R}^S_+$. Then, we define the generalized Shapley value of (N, V) by $\tilde{\Psi}(N, V) = \tilde{\Psi}(W, V)$.

That is, we extend the NTU Shapley value so that it would be nonempty-valued on $\mathscr{H}_{mo}$. Under this definition, the generalized Shapley value is a value function on $\mathscr{D}_{mo}$. Note that for a monotonic prize game (N, V), we may have $\tilde{\Psi}(N, V) \subsetneq \Psi(N, V)$ because there may be some vector $\lambda \in \mathbb{R}^N_{++}$ other than λ^N defining a TU game (N, v_λ) on (N, V). We restrict the Shapley value on monotonic prize games in such a way so as to avoid the case where $\Psi(N, V) \neq \tilde{\Psi}(N, V)$ for some monotonic hyperplane game (especially, some monotonic TU game) (N, V) with nonempty NTU Shapley value (i.e., $\Psi(N, V) \neq \emptyset$).

3.4. The Maschler-Owen value

The *Maschler-Owen value* is defined as follows. Let (N, V) be a game and Π be the set of all bijections of $|N|$ onto N. For each $\pi \in \Pi$, define $d_{\pi(1)}$ by

$$d_\pi(1) = \max\{x : x \in V(\pi(1))\}.$$

If $d_\pi(1), d_\pi(2), \cdots, d_\pi(i)$ are defined for $i \geqslant 1$, let $d_\pi(i+1)$ be defined as

$$d_\pi(i+1) = \max\left\{x : (x, d_\pi(1), \cdots, d_\pi(i)) \in V\left(\bigcup_{1 \leqslant j \leqslant i+1} \{\pi(j)\}\right)\right\}.$$

Now we write $D_\pi = (d_\pi(\pi^{-1}(i)))_{i \in N}$. Then the Maschler-Owen value of the game (N, V) is

$$\Phi(N, V) = \frac{1}{|\Pi|} \sum_{\pi \in \Pi} D_\pi.$$

3.5. Axioms

Now we shall introduce several axioms for value functions:

Axiom 1: *Efficiency*: $\varphi(N, V) \in \partial V(N)$.

Axiom 2: *Symmetry*: For each game (N, V) and each bijection π of N onto N, $\varphi_{\pi(i)}(N, \pi V) = \varphi_i(N, V)$ for each $i \in N$, where πV is a game defined by $(\pi V)(\pi S) = V(S)$ for each coalition $S \subseteq N$.

Axiom 3: *Marginality*: If $(N, V), (N, W)$ is a pair of games and $D_i(S, V, \varphi) = D_i(S, W, \varphi)$ for any coalition S with $i \in S \subseteq N$ where $D_i(S, V, \varphi)$ is defined by

$$D_i(S, V, \varphi) = \max\{x_i \in \mathbb{R} : (x_i, \varphi(S \setminus \{i\}, V)) \in V(S)\},$$

then $\varphi_i(N, V) = \varphi_i(N, W)$.

Axiom 4: *TU-Marginality*: If a pair $(N, v), (N, w)$ of TU games satisfies $v(S) - v(S \setminus \{i\}) = w(S) - w(S \setminus \{i\})$ for any coalition S with $i \in S \subseteq N$, then $\varphi_i(N, v) = \varphi_i(N, w)$.

Axiom 5: *Zero-Additivity*: Let (N, v_0) be a TU game with $v_0(S) = 0$ for any coalition $S \subseteq N$. If (N, V) is a game and $\varphi(N, V) \in \partial(V + V_0)(N)$, then $\varphi(N, V + V_0) = \varphi(N, V)$.

Axiom 6: *Scale Invariance*: If λ is a vector in $\mathbb{R}^N_{++}$ and (N, V) is a game, then $\varphi(N, \lambda V) = \lambda\varphi(N, V)$.

Axiom 7: *Dependence only on Non-negative Payoffs*: If (N, V), (N, W) is a pair of games

with $V(S) \cap R^S_+ = W(S) \cap \mathbb{R}^S_+$ for any coalition $S \subseteq N$, then $\varphi(N, V) = \varphi(N, W)$.

Zero-Additivity is a variant of Aumann's [1985] *conditional additivity* and *TU-Marginality* is just a restriction of marginality axiom to TU-games. We provide Axiom 7 just for the technical reason. All other axioms are well-known and can be found in Aumann [1985], Hart [1994], and Young [1985].

We can easily check the following claims.

Claim 1. *On $\mathscr{D}_{mo}$, the Maschler-Owen value satisfies all axioms in the above except Zero-Additivity.*

Proof: It only suffices to show that the Maschler-Owen value does not satisfy *Zero-Additivity*. Let $N = \{1, 2, 3\}$ and for each coalition $S \subseteq N$, $V(S) = \{x \in \mathbb{R}^S : \lambda^S \cdot x \leqslant c^S\}$ where $\lambda^N = (1, 1, 1)$, $\lambda^{\{1,2\}} = (1, 1/2)$, $\lambda^{\{1,3\}} = (1, 1/2)$, $\lambda^{\{2,3\}} = (1, 1/2)$, $\lambda^{\{1\}} = \lambda^{\{2\}} = \lambda^{\{3\}} = 1$ and $c^S = |S| - 1$. We also define a monotonic prize game (N, W) by $W(S) = (V(S) \cap \mathbb{R}^S_+) - \mathbb{R}^S_+$ for each coalition $S \subseteq N$. Then, by a simple computation, the Maschler-Owen value of the game (N, W) is given by

$$\Phi(N, W) = (1/2, 2/3, 5/6),$$

and so $\Phi(N, W) \in \partial(W + V_0)(N)$. However, the game $(N, W + V^0)$ is a TU game (N, w) where $w(S) = 2$ when $|S| \geqslant 2$ and $w(S) = 0$ for other coalitions $S \subseteq$ and so

$$\Phi(N, W + V^0) = Sh(N, w) = (2/3, 2/3, 2/3) \in \partial W(N).$$

Therefore, $\Phi(N, W + V^0) \neq \Phi(N, W)$ and so the Maschler-Owen value does not satisfy *Zero-Additivity* on $\mathscr{D}_{mo}$.

□

Claim 2. *On $\mathscr{D}_{mo}$, the generalized Shapley value satisfies all axioms in the above except Marginality.*

Proof: We shall only show that the generalized Shapley value does not satisfy *Marginality*. Let (N, v) be a TU game for which $N = \{1, 2, 3\}$ and

$$v(S) = \begin{cases} 2 & \text{if } S = N, \\ 1 & \text{if } |S| = 2, \\ 0 & \text{if } |S| = 1. \end{cases}$$

Let (N, V^p) be a monotonic prize game with $V^p(S) = \{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leqslant v(S)\} \cap \mathbb{R}^S_+ - \mathbb{R}^S_+$ for each coalition $S \subseteq N$. Now consider another monotonic prize game (N, W) defined by $W(S) = V^p(S)$ for each $S \subsetneq N$ and $W(N) = \{x \in \mathbb{R}^3 : x_1 + 2x_2 + 2x_3 \leqslant 3\} \cap \mathbb{R}^3_+ - \mathbb{R}^3_+$. Since $D_1(N, V^p, \tilde{\Psi}) = 1 = D_1(N, W, \tilde{\Psi})$ and $W(S) = V^p(S)$ for any $S \subsetneq N$, we can easily confirm that

$$D_1(S, V^p, \tilde{\Psi}) = D_1(S, W, \tilde{\Psi})$$

for each $S \subseteq N$ with $i \in S$. However, a simple computation verifies that

$$\tilde{\Psi}_1(N, V) = \frac{2}{3} \neq 1 = \tilde{\Psi}_1(N, W).$$

Therefore, the generalized Shapley value does not satisfy *Marginality* on $\mathscr{D}_{mo}$.

□

3.6. Axiomatization

In the following, we shall show that both the Maschler-Owen value and the generalized Shapley value can be characterized by a subfamily of Axioms 1-7.

For the proof of our main results, we need the following lemma which is a variant of Theorem 2 in Young [1985].

Lemma 1 (Young [1985]). The value function in the class of monotonic TU-games satisfies Efficiency, Symmetry and Marginality if and only if it is the TU Shapley value.

Although Theorem 1 can be verified in the similar way to Theorem 5.1 in Hart [1994], we provide the proof for the sake of completeness.

Theorem 1. *The value function on $\mathscr{D}_{mo}$ satisfies Axioms 1-3 if and only if it is the Maschler-Owen value.*

Proof: By Claim 1, the Maschler-Owen value satisfies Axioms 1-3. Let φ be a solution on $\mathscr{D}_{mo}$ satisfying Axioms 1-3. Since φ satisfies Axioms 1-3 on the class of monotonic TU games, it follows from Lemma 1 that φ coincides the TU Shapley value on this class. Now let (N, V) be a game in the domain $\mathscr{D}_{mo}$ and we assume that for any coalition $S \subsetneq N$, $\varphi(S, V) = \Phi(S, V)$. Fix $i \in N$. Then, for each $S \subseteq N$ with $i \in S$, $D_i(S, V, \varphi) = D_i(S, V, \Phi)$. Let w be a TU-game such that for each coalition S containing i, $w(S \setminus \{i\}) = \sum_{T \subsetneq S \setminus \{i\}} D_i(T \cup \{i\}, V, \varphi)$ and $w(S) = \sum_{T \subseteq S \setminus \{i\}} D_i(T \cup \{i\}, V, \varphi)$. Note that since (N, V) is monotonic, we have $D_i(S, V, \varphi) \geqslant 0$ for each coalition $S \subseteq N$ with $i \in S$ and so (N, w) is a monotonic TU game by the definition. Also, for each $S \subseteq N$ with $i \in S$, $D_i(S, w) = D_i(S, V)$. Then by *Marginality*, it follows that

$$\begin{aligned}
\varphi_i(N, V) &= \varphi_i(N, w) = Sh_i(N, w) \\
&= \frac{1}{|\{S \subseteq N : i \in S\}|} \sum_{S \subseteq N, i \in S} D_i(S, V, \varphi) \\
&= \frac{1}{|\{S \subseteq N : i \in S\}|} \sum_{S \subseteq N, i \in S} D_i(S, V, \Phi) \\
&= \Phi_i(N, V),
\end{aligned}$$

which completes the proof. To see why the last equality holds, we first note that by the

definition, for any coalition $T \subseteq N$ with $T \not\ni i$, it follows that

$$\Phi(T,V) = \frac{1}{|\pi \in \Pi : \pi(1,\cdots,|T|) = T|} \sum_{\pi\in\Pi:\pi(1,\cdots,|T|)=T} D_\pi|_T, \qquad (*)$$

where $D_\pi|_T$ denotes the restriction of D_π to the coalition T. Since $\partial V(S) \cap \mathbb{R}^S_+$ is convex for each $S \subseteq N$ with $i \in S$, we have

$$\frac{\sum_{\pi\in\Pi:\pi(1,\cdots,|S|-1)|=S\setminus\{i\},\pi(|S|)=i} D_\pi|_S}{|\pi \in \Pi : \pi(1,\cdots,|S|-1)| = S \setminus \{i\},\ \pi(|S|) = i|} \in \partial V(S) \qquad (**)$$

Then it follows from $(*)$ and $(**)$ that

$$\frac{\sum_{\pi\in\Pi:\pi(1,\cdots,|S|-1)|=S\setminus\{i\},\pi(|S|)=i} d_\pi(|S|)}{|\pi \in \Pi : \pi(1,\cdots,|S|-1)| = S \setminus \{i\},\ \pi(|S|) = i|} = D_i(S,V,\Phi).$$

Hence it holds that

$$\Phi_i(N,V) = \sum_{\pi\in|\Pi|} D_\pi|_{\{i\}} = \frac{1}{|\{S \subseteq N : i \in S\}|} \sum_{S\subseteq N, i\in S} D_i(S,V,\Phi).$$

□

Theorem 2. *The value function on $\mathscr{D}_{mo}$ satisfies Axioms 1-2 and Axioms 4-7 if and only if it is the generalized Shapley value.*

Proof: By Claim 2, the generalized Shapley value satisfies these six axioms. Conversely, let φ be a value on $\mathscr{D}_{mo}$ satisfying the six axioms. Since φ satisfies *Efficiency*, *Symmetry* and *TU-Marginality*, it follows from Lemma 1 that $\varphi(N,v) = \tilde{\Psi}(N,v)$ for any monotonic TU game (N,v). Now, let (N,V) be a monotonic prize game and let $\lambda^N \in \mathbb{R}^N_{++}$ be a normal vector to $V(N)$ (i.e., $V(N) = (\{x \in \mathbb{R}^N : \lambda^N \cdot x \leqslant c^N\} \cap \mathbb{R}^N_+) - \mathbb{R}^N_+$ for some scalar $c^N \in \mathbb{R}$)[3]. We also let W be the monotonic prize game defined by $W = \lambda^N V$. Then by *Scale Invariance*, we must have $\varphi(N,V) = (\lambda^N)^{-1}\varphi(N,W)$ where $(\lambda^N)^{-1} = (1/\lambda^N_1, 1/\lambda^N_2, \cdots, 1/\lambda^N_N)$. We can easily see that 1_N defines a TU game (N, w_{1_N}) on (N,W) and this game coincides with the game $(N, W + V^0)$. Since $\varphi(N,W) \in \partial(W + V_0)(N)$ and $\tilde{\Psi}(N,W) \in \partial(W + V_0)(N)$, it follows from *Zero-Additivity* that

$$\begin{aligned}\varphi(N,V) &= (\lambda^N)^{-1}\varphi(N,W) = (\lambda^N)^{-1}\varphi(N,W+V^0)\\ &= (\lambda^N)^{-1}Sh(N,w_{1_N}) = (\lambda^N)^{-1}\tilde{\Psi}(N,W+V^0)\\ &= (\lambda^N)^{-1}\tilde{\Psi}(N,W) = \tilde{\Psi}(N,V).\end{aligned}$$

Finally, for a monotonic hyperplane game (N,V), by *Dependence only on Non-negative Payoffs*, $\varphi(N,V) = \varphi(N,W) = \tilde{\Psi}(N,W) = \tilde{\Psi}(N,V)$ where (N,W) is a monotonic prize game defined by $W(S) = V(S) \cap \mathbb{R}^S_+ - \mathbb{R}^S_+$ for each coalition $S \subseteq N$, which completes the proof of Theorem 2.

□

[3] If $0 \in \partial V(N)$, we let $\lambda^N = 1_N$.

3.7. Independence of the axoims

To show the axioms appearing in Theorem 1 and 2 are independent we will prove that dropping one axiom in each theorem allows value functions φ and ψ on $\mathscr{D}_{mo}$ with $\varphi \neq \Phi$ and $\psi \neq \tilde{\Psi}$.

- The Maschler-Owen value (Theorem 1)
  - Without *Efficiency*: Let $\varphi(N, V) = 0$ for each game $(N, V) \in \mathscr{D}_{mo}$.
  - Without *Symmetry*: Let (N, V) be a game in the domain $\mathscr{D}_{mo}$ and let π be the increasing ordering of $\{1, 2, \cdots, |N|\}$ onto N. Define D_π as in the definition of the Maschler-Owen value and let $\varphi(N, V) = D_\pi$.
  - Without *Marginality*: Let φ be the generalized Shapley value.
- The generalized Shapley value (Theorem 2)
  - Without *Efficiency*: Let $\psi(N, V) = 0$ for each game $(N, V) \in \mathscr{D}_{mo}$.
  - Without *Symmetry*: For a game (N, V) with one player, let $\psi(N, V) \in \partial V(N, V)$. Now let (N, V) be a game with two player and let π be the increasing ordering of $\{1, 2\}$ onto N. Define D_π as in the definition of Maschler-Owen value and let $\psi(N, V) = D_\pi$. For a game (N, V) with $|N| \geqslant 3$, let $\psi(N, V) = \tilde{\Psi}(N, V)$. It can be easily verified that this value function actually satisfies *Zero-Additivity*.
  - Without *TU-Marginality*: For a game (N, V) in the domain $\mathscr{D}_{mo}$, there are a vector $\lambda^N \in \mathbb{R}^N_{++}$ and a real $c^N \in \mathbb{R}_+$ such that we have either $V(N) = \{x \in \mathbb{R}^N : \lambda^N \cdot x \leqslant c^N\}$ or $V(N) = \{x \in \mathbb{R}^N : \lambda^N \cdot x \leqslant c^N\} \cap \mathbb{R}^N_+ - \mathbb{R}^N_+$. In either case, let $\psi_i(N, V) = \frac{c^N}{|N|\lambda^N_i}$.
  - Without *Zero-Additivity*: Let ψ be the Maschler-Owen value.
  - Without *Scale Invariance*: As above $V(N)$ can be written by either $V(N) = \{x \in \mathbb{R}^N : \lambda^N \cdot x \leqslant c^N\}$ or $V(N) = \{x \in \mathbb{R}^N : \lambda^N \cdot x \leqslant c^N\} \cap \mathbb{R}^N_+ - \mathbb{R}^N_+$ for each game $(N, V) \in \mathscr{D}_{mo}$. Then for each $i \in N$, we define

$$\psi_i(N, V) = \begin{cases} \tilde{\Psi}_i(N, V) & \text{if there exists } \alpha \in \mathbb{R}_{++} \text{ with } \alpha\lambda^N = 1_N, \\ \frac{c^N}{|N|\lambda^N_i} & \text{otherwise.} \end{cases}$$

  - Without *Dependence only on Non-negative Payoffs*: For each game $(N, V) \in \mathscr{D}_{mo}$, let

$$\psi(N, V) = \begin{cases} \tilde{\Psi}(N, V) & \text{if } (N, V) \in \mathscr{P}_{mo}, \\ \Phi(N, V) & \text{if } (N, V) \in \mathscr{H}_{mo}. \end{cases}$$

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