Monetary Equilibria and Knightian Uncertainty

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Keywords: Money, Maximin expected utility, Conditional Pareto optimality, Indeterminacy, Overlapping generations model.

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1. Introduction

As has been traditional since Savage’s (1954) pioneering work, most studies on economies under uncertainty have assumed that a decision-maker behaves as if she assigns a single probability measure to uncertainty and maximizes the expected utility. Ellsberg’s example (1961), however, pointed out the importance of ambiguity, situations that a decision-maker imprecisely assigns probability measures to uncertainty, rather than risk described by Savage. For the last one and a half decades, the implications of ambiguity in economics have been explored. Along the line of the literature, this article explores implications of ambiguity on money in an overlapping generations (OLG) framework.

This article studies a pure-endowment stationary stochastic OLG economy under uncertainty.\(^1\) In each single period, a shock is realized in finite state space, one new agent is born, and that agent lives for two periods and then dies. In the economy, there are available a single physical (consumption) good in each period and an infinitely-lived outside asset with no dividend payment: fiat money. In the first period of her life, an agent divides an initial endowment into a consumption in this period and money holdings. In her second period, she buys goods from the next generation (in its first period) and consumes both these and a second-period endowment that realizes depending only on the shock in her second period. There is no storage technology or production.

So far, the model is similar to stochastic OLG models in the literature. However, our model is different from those in that each agent evaluates her consumption streams over two periods by the maximin expected utility à la Gilboa and Schmeidler (1989). Each agent faces uncertainty about her second-period consumption, which is represented by not a single probability distribution but a set of probability distributions. Such uncertainty is sometimes referred to as Knightian uncertainty after the economist who distinguished such a multi-prior situation from risk where uncertainty is summarized by a single probability distribution (Knight, 1921). Each agent is then supposed to use the “worst” probability distribution to finally evaluate consumption streams. Such a behavior of decision-makers was characterized using behavioral axioms by Gilboa and Schmeidler (1989) for an infinite state space and by Casadesus-Masanell, Klibanoff

\(^1\)Except for preferences, our model is almost the same as that of Labadie (2004).
Under these schemes, this paper proves two main results. First, we show that an open set of the “economy” exists in which multiple stationary monetary equilibria exist. Here, “economy” is defined by a pair of initial endowments in the agent’s first and second periods. Since we assume that each generation is identical and that the state space is finite, the economy is defined by a point in a finite-dimensional Euclidean space (apart from the preference structure). It is shown that for some open set of the economy, a continuum of equilibrium prices exists. We also provide several numerical examples that indicate that an increase in uncertainty (in the sense that the set of probability measures of each agent expands) enlarges not only the degree of indeterminacy of equilibrium prices and allocations but also the range of the economy in which indeterminacy is observed. It is well known (Gottardi, 1996) that when the agent’s preference is differentiable, each equilibrium price is locally isolated (although there might be many). In contrast, the preference in our model, which is represented by the maximin expected utility, may not be differentiable and this is why the continuum of equilibria shows up in our model.

Second, we conduct welfare analysis and prove that each of these stationary monetary equilibria is conditionally Pareto optimal; i.e., no other stationary allocations strictly Pareto dominate the equilibrium allocations. In particular, this is somewhat surprising since any equilibrium in the continuum of equilibria cannot be dominated by another in it.

Related Literature

We finally mention the relationship of our model to the existing literature. This paper builds on a number of contributions. Most obviously, it brings together the literature on stochastic OLG models. We should first note that this article is the first that introduces the maximin expected utility preferences to the stochastic OLG model of money.2 While our model is canonical except for preferences, the density of the stationary monetary equilibrium drastically differs from those in previous studies. In representative agent models, Manuelli (1990), Magill and Quinzii (2003), and Ohtaki (2011) reported uniqueness results for a stationary monetary equilibrium. Furthermore, Gottardi (1996) showed that a stationary monetary equilibrium

2One may find Fukuda (2008) who introduced firms with convex Choquet expected utility preferences to Diamond’s (1965) OLG model with capital accumulation.
generically exists and is locally isolated in a more complicated stochastic OLG model with many agents and many one-period securities.\(^3\) With the maximin expected utility preferences, in contrast, we show indeterminacy of stationary monetary equilibria.

Optimality of equilibrium allocations has been characterized per Manuelli (1990), Aiyagari and Peled (1991), Chattopadhyay and Gottardi (1999), Demang and Laroque (1999), Ohtaki (2013), and others. Among these studies, smoothness of preferences is presumed. However, it is also known that the maximin expected utility preferences are nonsmooth; \textit{i.e.}, they have points at which the expected utility function is not differentiable. Therefore, we cannot apply the characterizations developed by the above studies directly to the current model. Instead of applying the established characterizations, we show the optimality of stationary monetary equilibria more directly by tailoring Sakai’s (1988) result for our model.

The effect of the maximin expected utility preferences on indeterminacy of equilibrium has been argued for several economic models. In a seminal paper, Dow and Werlang (1992) showed the bid-ask spread in optimal portfolio choice problems. A related line of this argument has been pursued per Epstein and Wang (1994, 1995). Epstein and Wang extended Lucas’ (1978) dynamic asset pricing model by assuming that a representative agent evaluates consumption streams by the maximin expected utility preference. At an equilibrium of Epstein and Wang’s model, however, the representative agent does not hold any security and consumes all of her initial endowment in each period. Then, the equilibrium (supporting) prices are those prices under which the optimal decision of the representative agent is to hold no asset. Epstein and Wang showed that there exists uncountably many such equilibrium prices. However, in their model, the indeterminacy of equilibrium \textit{allocation} cannot happen because of the representative-agent setting. In our exchange economy, in contrast, indeterminacy of allocation (real indeterminacy) as well as price indeterminacy takes place.

Indeterminacy due to the maximin expected utility can also be observed even in the general equilibrium model. Chateauneuf, Dana, and Tallon (2000) and Dana (2004) showed

\(^3\)As shown per Gottardi (1996), a “zero-th order” stationary monetary equilibrium, of which money prices depend only on the current states, is locally isolated. In contrast, it has been shown per Spear, Srivastava, and Woodford (1990) that the first- and second-order stationary monetary equilibria, in which money prices may depend on the past states, are indeterminate.
that the Arrow-Debreu economy exhibits not only price but also real indeterminacy when there is no aggregate risk using the convex Choquet expected utility. Here, no aggregate risk means that the total endowment does not change over the state space. As argued by Rigotti and Shannon (2012), however, such indeterminacy in the Arrow-Debreu economy is not robust; i.e., equilibrium is generically determinate. To generate indeterminacy, therefore, ‘some other ingredient has to be inserted’ as mentioned by Mukerji and Tallon (2004).\(^4\) In contrast to the Arrow-Debreu economy, our model is a dynamic one with overlapping generations and money, and shows the existence of a nonempty open set of economies exhibiting real indeterminacy when aggregate risks do exist; i.e., when the sum of initial endowments of the old generation and the young generation in the same period changes over the state space.

_Construction of the Paper_

The paper is organized as follows. Section 2 presents details of the model and defines stationary monetary equilibrium. Section 3 demonstrates that stationary monetary equilibrium can be characterized by a system of inclusions rather than a system of equations. Section 4 shows the existence and indeterminacy of stationary monetary equilibrium. Section 5 examines the optimality of stationary monetary equilibria. Proofs are provided in the final section.

## 2. The Economy

We consider a stationary, pure-endowment stochastic overlapping generations economy of money, wherein agents assign multiple priors to uncertainty. Except for preferences, the present model is essentially the same as that of Labadie (2004).

### 2.1. Primitives

Time is discrete and runs from \( t = 1 \) to infinity. Uncertainty is modeled by a stationary Markov process with its finite state space \( S \). For each \( t \geq 0 \), we denote by \( s_t \) the state realized in period \( t \), called the _period \( t \) state_, where the (implicit) period 0 state \( s_0 \in S \) is treated as

\(^4\)Mukerji and Tallon (2001) therefore introduced an idiosyncratic component of the asset returns to argue the equilibrium indeterminacy. Recently, Mandler (2012) found equilibrium indeterminacy in the near Arrow-Debreu model with a productive asset and sequential trades.
In each period, the state realizes at the beginning of the period. We denote by $\Delta(S)$ or, more simply, $\Delta_S$ the set of all probability measures on $S$. There is a single perishable good, called the consumption good. No storage technology or production technology is available.

In each period, one new agent enters the economy after the realization of the state and lives for two periods. In the rest of this article, we concentrate on the stationary situation. Therefore, agents are distinguished by the state in which they are born, and not the time or history of realized states. An agent born in state $s_t$ and period $t$ is endowed with $\omega_{s_t}^1$ units of the consumption good in the first period of her life and $\omega_{s_{t+1}}^2$ in state $s_{t+1}$ in the second period. We assume that $(\omega_{s_t}^1, \omega_{s_{t+1}}^2) \in \mathbb{R}_{++} \times \mathbb{R}_{++}^S$ for all $s \in S$. Note that the second-period endowment is assumed to be independent of the shock in the first period. Therefore, the economy is represented by a point $(\omega_{s_t}^1, \omega_{s_{t+1}}^2)_{s, s' \in S}$ in the positive orthant of the finite-dimensional Euclidean space, $\mathbb{R}^{S \times +}$, given the agents’ preferences described below.

We denote by $c_{st} = (c_{s_{1t}}, (c_{s_{1t}^s, s'}), s \in S)$ the contingent consumption stream of an agent born in state $s_t$ and period $t$. An agent born in state $s_t$ and period $t$ is assumed to rank the consumption streams $c_{st}$ according to her lifetime utility function $U^{s_t} : \mathbb{R}_{++} \times \mathbb{R}_{++}^S \to \mathbb{R}$. Throughout this paper, we assume that agents have maximin expected utility (MMEU) preferences, i.e., there exist an increasing, strictly concave, and continuously differentiable real-valued function $u$ on $\mathbb{R}^S_{++}$ and a family of compact and convex subsets of $\Delta_S$, $(\mathcal{P}_s)_{s \in S}$, such that

$$\left( \forall s \in S \right) \left( \forall c \in \mathbb{R}_{++} \times \mathbb{R}_{++}^S \right) \quad U^s(c) = \min_{\pi \in \mathcal{P}_s} \sum_{s' \in S} u(c^1, c^2_{s'}) \pi_{s'}. \quad (1)$$

We sometimes write $\mathcal{P} = (\mathcal{P}_s)_{s \in S}$ and refer to transition correspondence. Because the MMEU preference crucially depends on $\mathcal{P}_s$ and since $s$ affects $U$ only through $\mathcal{P}_s$, we often write $U^s(c)$ as $U(\mathcal{P}_s)(c)$. We denote by $U(\pi)(c)$ rather than by $U(\{\pi\})(c)$ the MMEU preference when $\mathcal{P}_s = \{\pi\}$ for some $\pi \in \Delta_S$. Furthermore, since $\sum_{s' \in S} u(c^1, c^2_{s'}) \pi_{s'}$ is clearly continuous in $\pi$ for

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5This study implicitly considers a standard date-event tree as seen in, for example, Chattopadhyay (2001). Therefore, the initial state $s_0$ can be interpreted as the root of the date-event tree.

6An axiomatization of the maximin expected utility preferences over lottery acts was given by Gilboa and Schmeidler (1989). That of the MMEU preferences over Savage acts was given by Casadesus-Masanell, Klibanoff and Ozdenoren (2000). Their axiomatization does not depend on whether the state space is finite or infinite, and hence, may be applied to our situation with a finite state space.

7Let $H$ be a nonempty finite set. A real-valued function $f$ on $X \subseteq \mathbb{R}^H$ is nondecreasing if $f(x) \geq f(y)$ for all $x, y \in X$ such that $x_h \geq y_h$ for all $h \in H$ and is increasing if $f(x) > f(y)$ for all $x, y \in X$ such that $x_h > y_h$ for all $h \in H$ and $x_k > y_k$ for some $k \in H$. 

5
each $c$ and since each $P_s$ is a compact subset of $\Delta_S$ by the assumption, the minimum in (1) is actually attained. Hence, we may define

$$(\forall s \in S)(\forall c \in \mathbb{R}^+ \times \mathbb{R}^{S^+}_+ ) \quad M(P_s)(c) := \arg \min_{\pi \in P_s} U(\pi)(c),$$

which is the nonempty set of priors minimizing the expected utility given the consumption stream $c \in \mathbb{R}^+ \times \mathbb{R}^{S^+}_+$. We can prove the strict concavity of $U(P_s)(\cdot)$ in $c$.

**Theorem 1** $U(P_s)(\cdot)$ is strictly concave for all $s \in S$.

Therefore, we can say that strict concavity of index function $u$ is taken over to the lifetime utility $U^s$. We will invoke Theorem 1 when we characterize the equilibria and when we show their conditional Pareto optimality.

### 2.2. Equilibria

To describe the intergenerational trade, we introduce an infinitely-lived outside asset, which yields no dividends. This asset is so-called fiat money. The stock of fiat money is constant over date-events and is denoted by $M > 0$.

A *stationary equilibrium* is a pair $(q^*, c^*)$ of a contingent real money balance $q^* \in \mathbb{R}^S_+$ and a contingent consumption stream $c^* = (c^1_s, (c^2_{ss'})_{s' \in S})_{s \in S}$ such that there exists an $m^* \in \mathbb{R}^S$ satisfying that: for all $s \in S$, (i) $(c^*_1, m^*_s)$ belongs to the set

$$\arg \max_{(c_1^s, (c_2^{s'})_{s' \in S}, m_s) \in \mathbb{R}^+_+ \times \mathbb{R}^{S^+}_+ \times \mathbb{R}} \left\{ U^s(c^s) \right\}$$

$$(\forall s' \in S) \quad c^1_s = \omega^1_s - q^*_s m^*_s / M,$$

and (ii) $m^*_s = M$. It is a *stationary monetary equilibrium* if $q^* \gg 0$. Condition (i) requires that the pair of the consumption stream $(c^1_s, (c^2_{ss'})_{s' \in S})$ and money holding $m^*_s$ must be the solution of the (lifetime) utility-maximizing problem of the agent born in state $s$, whereas condition (ii) is the market clearing condition of fiat money. From (i), $(c^*_1, m^*_s)$ satisfies that $c^1_s = \omega^1_s - q^*_s m^*_s / M$.

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8 One can also show the (quasi-/strict quasi-) concavity of $U(P_s)(\cdot)$ under the assumption of the (quasi-/strictly quasi-) concavity of $u$.

9 In (i), we assume that the budget constraints hold with equalities. We can do this for the first budget constraint without loss of generality by the increase of $u$. For the other budget constraints, we simply assume it. Also note that we exclude corner solutions by assuming that $(c^1, (c^2_{s'})_{s' \in S}) \in \mathbb{R}^+_+ \times \mathbb{R}^{S^+}_+$ and (ii).
and (∀s’) \( c^{s'}_{s_s} = \omega^2_{s'} + q^s_{s'} m_s / M \), which together with (ii) implies that \( c^1_s = \omega^1_s - q^s_s \) and (∀s’) \( c^{s'}_{ss'} = \omega^2_{ss'} + q^s_{s'} \). It follows that

\[
(∀s, i, j \in S) \quad c^*_{s_s} + c^*_{is} = \omega^1_s + \omega^2_s = c^*_{is} + c^*_{js}.
\]

That is, we obtain the market-clearing conditions of the contingent consumption good (the Warlas law). Furthermore, the last equation implies that \( c^*_{ss'} \) is independent of \( s \) (which we write as \( c^*_{s_s} \)).

Note that, once the equilibrium real money balance \( q^* \) is chosen, the equilibrium consumption stream \( c^* \) is automatically and uniquely determined from the budget constraints. Therefore, we can identify a stationary (monetary) equilibrium \((q^*, c^*)\) with an equilibrium (positive) real money balance \( q^* \). We say that a stationary equilibrium \( q^* \) with its corresponding consumption stream \( c^* = (c^{s'}_s, (c^{s'}_{s'})_{s' \in S})_{s \in S} \) is fully-insured (with respect to the second-period consumptions) if, for all \( s', s'' \in S, c^{s'}_{s'} = c^{s''}_{s''} \).

3. Characterization of Stationary Equilibria

This section characterizes stationary equilibria whose existence will be established in the next section. Under the current framework, the agents’ optimization problems degenerate into a simple form: for all \( s \in S \), the agent solves

\[
\max_{m \in \mathbb{R}} U(P_s)(\omega^1_s - q^s_s m / M, (\omega^2_{s'} + q^s_{s'} m / M)_{s' \in S}).
\]

We denote by \( V(P_s)(m) \) the objective function in (2) for notational convenience. When \( P_s \) is a singleton for all \( s \in S \), the objective function is differentiable with respect to \( m \), and hence, a stationary equilibrium can be characterized by a system of equations, which is derived from the first-order condition of the optimization problem and the money market clearing condition \( m_s = M \) for all \( s \in S \). However, if \( P_s \) has multiple elements, the objective function may not be differentiable, in which case we can no longer characterize a stationary equilibrium by a system of equations. To handle this case, we will apply the following theorem, which is adapted from Aubin (1979, p.118, Proposition 6).
Theorem 2 (Aubin, 1979) Let Π be a nonempty subset of a metric space and let \{f_\pi\}_{\pi \in \Pi} be a collection of functions from \(\mathbb{R}\) to \(\mathbb{R}\). For each \(x \in \mathbb{R}\), define

\[ g(x) := \inf_{\pi \in \Pi} f_\pi(x) \quad \text{and} \quad M(x) := \{\pi \in \Pi | g(x) = f_\pi(x)\}. \]

Let \(x \in \mathbb{R}\). If (a) Π is compact, (b) there exists a neighborhood \(X\) of \(x\) such that functions \(\pi \mapsto f_\pi(y)\) are continuous (in the metric topology) for all \(y \in X\), and (c) for all \(\pi \in \Pi\), \(f_\pi\) is concave and differentiable, and \(g\) is then differentiable at \(x\) from both the left and right and it holds that

\[ D_-g(x) = \max_{\pi \in M(x)} Df_\pi(x) \quad \text{and} \quad D_+g(x) = \min_{\pi \in M(x)} Df_\pi(x). \]

By Theorem 1, \(U(P_s)(\cdot)\) is strictly concave. Hence, a real number \(m \in \mathbb{R}\) is a solution of the degenerate optimization problem (2) if and only if

\[ (\forall s \in S) \quad D_+V(P_s)(m) \leq 0 \leq D_-V(P_s)(m), \]

where \(D_-V(P_s)(m)\) and \(D_+V(P_s)(m)\) are the left and right derivatives of \(V(P_s)(m)\) taken with respect to \(m\), whose existence is guaranteed by Theorem 2.10

Given any \(q^* \in \mathbb{R}^S_+\) and any \(s \in S\), let \(c^m_{ss'}(q^*) := (\omega^1_s - q^*_{ss'}m/M, \omega^2_s + q^*_{ss'}m/M)\) and \(e^m_s(q^*) := (\omega^1_s - q^*_{ss'}m/M, (\omega^2_s + q^*_{ss'}m/M)_{s' \in S})\). We are now ready to characterize a stationary equilibrium by a system of inclusions.

Theorem 3 A nonnegative-valued function \(q^*\) on \(S\) is a stationary equilibrium if and only if

\[ (\forall s \in S) \quad \left\{ - \sum_{s' \in S} q^*_s u_1(c^{M}_{ss'}(q^*)\pi_{s'}) + \sum_{s' \in S} q^*_s u_2(c^{M}_{ss'}(q^*)\pi_{s'}) \right\} \pi \in M(P_s)(c^{M}_s(q^*)), \quad (3) \]

where \(u_i\) is the derivative of \(u\) with respect to the \(i\)-th argument.

This is an extension of the characterization of a stationary equilibrium in stochastic OLG models as considered by Magill and Quinzii (2003) and Ohtaki (2011), where agents have a unique prior. In fact, if \(P_s\) is a singleton for all \(s \in S\), this system of inclusions degenerates into a system of equations, which is the same result as the stochastic OLG model with a unique prior.

10Because the domain of \(U(P_s)(\cdot)\) is open (i.e., \(\mathbb{R}_+ \times \mathbb{R}^S_+\)), the domain of \(V(P_s)(\cdot)\) is also open. Therefore, the inequalities in the text certainly constitute a necessary and sufficient condition for the given problem.
When the prior is unique, the preference turns out to be differentiable. Gottardi (1996) proved that if the preference is differentiable, a stationary monetary equilibrium exists for “almost all” economies. Since the inclusions (3) are weaker requirements than their equality counterparts, we may conclude that a stationary monetary equilibrium exists for “almost all” economies in our model with Knightian uncertainty.

4. Existence and Indeterminacy of Stationary Monetary Equilibria

In the previous section, we observed that a stationary equilibrium can be characterized by a system of not equations but inclusions. However, we have not yet shown the existence of stationary monetary equilibria explicitly. Neither have we examined the number of equilibria. This section presents a condition under which a continuum of stationary monetary equilibria exists.

Let \( \omega^2 \in \mathbb{R}^S_+ \) be arbitrarily given. Consider an equilibrium real money balance \( q^* \in \mathbb{R}^S_+ \) such that for any \( s', s'' \in S \), \( \omega^2_{s'} + q^*_s \neq \omega^2_{s''} + q^*_s \) whenever \( s' \neq s'' \). In this case, there exists a neighborhood of \( q^* \) in which \( \mathcal{M}(\mathcal{P}_s)(c^M_s(q)) \) is “constant and a singleton” for each \( s \).

We refer to this “unique” measure (which depends on \( s \) since \( \mathcal{P}_s \) depends on \( s \)) by \( \mu_s \). The system of inclusions, (3), is then a system of simultaneous equations:

\[
(\forall s \in S) \quad 0 = - \sum_{s' \in S} q^*_s u_1(\omega^1_s - q^*_s, \omega^2_s + q^*_s) \mu_{ss'} + \sum_{s' \in S} q^*_s u_2(\omega^1_s - q^*_s, \omega^2_s + q^*_s) \mu_{ss'}. \]

Any solution to this system is an equilibrium of the economy and its local nature, including whether there exists a continuum of solutions, is the same as in the standard stochastic OLG model. There is nothing new here and we do not pursue this line further.

\[\text{To be more precise, Gottardi showed that, under smooth preferences, a stationary monetary equilibrium generically exists.}\]

\[\text{The basic idea is as follows. Let } s_1 := \arg \min_{s' \in S} \omega^2_{s'} + q^*_s \text{ and let } \mathcal{M}_1 \text{ be the set of probability measures in } \mathcal{P}_s \text{ that assign the largest probability to } s_1. \text{ Then, let } s_2 := \arg \min_{s_1' \in \mathcal{M}_1} \omega^2_{s_1'} + q^*_s \text{ and let } \mathcal{M}_2 \text{ be the set of probability measures in } \mathcal{M}_1 \text{ that assign the largest probability to } s_2. \text{ Continuing this process may lead to a single probability measure in } \mathcal{P}_s \text{ that is the unique element of } \mathcal{M}(\mathcal{P}_s)(c^M_s(q)). \text{ In general, however, whether this procedure works depends on the nature of } \mathcal{P}_s. \text{ In this sense, the argument of this paragraph stands only heuristically. For example, if } \mathcal{P}_s \text{ is characterized by } \varepsilon\text{-contamination, the above procedure will determine a single probability measure. For the } \varepsilon\text{-contamination, see Nishimura and Ozaki (2006) and references therein.}\]

\[\text{As stated in the previous footnote, there are possibilities that the system of equations turns out to be that of inclusions. Since we are concerned with a sufficient condition for multiple equilibria, it is harmless to neglect such a situation for our purpose.}\]
Next, we turn to a partially-insured equilibrium. Let \( S' \) be a subset of \( S \) with \( |S'| \geq 2 \). Consider an equilibrium real money balance \( q^* \in \mathbb{R}^S_{++} \) such that for any \( s', s'' \in S', \omega^2_{s'} + q^*_s = \omega^2_{s''} + q^*_s \). Then, \( \mathcal{M}(\mathcal{P}_s)(c^M_s(q^*)) \) includes all probability measures in \( \mathcal{P}_s \) that assign the same probability to \( S' \). However, since \( q^* \) need not be constant over \( S' \) (although \( \omega^2 + q^* \) need be), the set in (3) is not necessarily a singleton. Therefore, a continuum of solutions to the system of inclusions, (3), may arise. To spell out the configuration of endowments that allows such indeterminacy will become very complicated. However, importantly, we can choose \( q^* \) jurisdictionally so as to make \( \omega^2 + q^* \) constant over some set. This “endogenized flatness” can be further exploited to show indeterminacy of fully-insured equilibria. We do this in a new subsection.

4.1. Fully-insured Equilibria

In the rest of this section, we consider a fully insured stationary equilibrium such that

\[
(\exists d > 0)(\forall s \in S) \quad \omega^2_s + q^*_s = d, \tag{4}
\]

where \( d > 0 \) is a constant that represents the agent’s consumption in the second period. Then, we first observe that

\[
(\forall s \in S) \quad \mathcal{M}(\mathcal{P}_s)(c^M_s(q^*)) = \mathcal{P}_s.
\]

Furthermore, we have the following theorem.

**Theorem 4** Suppose that \( q^* \) satisfies (4) for some \( d \). Then, \( q^* \) is a stationary equilibrium if and only if, for all \( s \in S \),

\[
d - \max_{\pi \in \mathcal{P}_s} \sum_{s' \in S} \omega^2_{s'} \pi_{s'} \leq \frac{u_1(\omega^1_s + \omega^2_s - d, d)}{u_2(\omega^1_s + \omega^2_s - d, d)} \leq d - \min_{\pi \in \mathcal{P}_s} \sum_{s' \in S} \omega^2_{s'} \pi_{s'}. \tag{5}
\]

Here, \( u_1(\omega^1_s + \omega^2_s - d, d)/u_2(\omega^1_s + \omega^2_s - d, d) = u_1(\omega^1_s - q^*_s, \omega^2_s + q^*_s)/u_2(\omega^1_s - q^*_s, \omega^2_s + q^*_s) \) is a “marginal rate of substitution” between the consumption in the first period and that in the second period when the full insurance is attained at stationary monetary equilibrium. Therefore, the theorem tells us that \( q^* \) is a fully insured equilibrium if and only if the “marginal rate of substitution” belongs to some interval.
We exploit Theorem 4 to look for economies in which a continuum of equilibria arises. To be more precise, given an economy $\omega = (\omega_1, \omega_2) \in \mathbb{R}^S_+ \times \mathbb{R}^S_+$, we say that the economy exhibits a continuum of fully insured equilibria if there exists an open set $D \subseteq \mathbb{R}^S_+$ such that, for any $d \in D$, $(q_s)_{s \in S} := (d - \omega^2_s)_{s \in S}$ is a stationary monetary equilibrium. For notational ease, we define a function $f : \mathbb{R}^2_+ \rightarrow \mathbb{R}^2_+$ by

\[
(\forall (x, y) \in \mathbb{R}^2_+) \quad f(x, y) := \frac{u_1(x, y)}{u_2(x, y)}.
\]

Note that $f$ is continuous because $u$ is continuously differentiable by the assumption. We can now demonstrate the indeterminacy of fully insured stationary monetary equilibria.

**Theorem 5** Suppose that there exists $\hat{\omega}^2 \in \mathbb{R}^S_+$ such that

\[
\max_{s \in S} \min_{\pi \in P_s} \sum_{s' \in S} \hat{\omega}^2_s \pi_{s'} < \min_{s \in S} \max_{\pi \in P_s} \sum_{s' \in S} \hat{\omega}^2_s \pi_{s'}
\]

and that the function $f(\cdot, y)$ is surjective for each $y > 0$. There then exists a nonempty open set of economies, $\Omega \subseteq \mathbb{R}^S_+ \times \mathbb{R}^S_+$, each element of which exhibits a continuum of fully insured equilibria.

When the function $u$ is time-separable, one of the assumptions of Theorem 5 may be further simplified.

**Corollary 1** Suppose that (6) holds. Furthermore, assume that there exist continuously differentiable functions $v, w : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

\[
(\forall c_1, c_2 \in \mathbb{R}_+) \quad u(c_1, c_2) = v(c_1) + w(c_2).
\]

Finally, assume that $v$ satisfies the Inada conditions; i.e.,

\[
\lim_{c \downarrow 0} v'(c) = +\infty \quad \text{and} \quad \lim_{c \uparrow +\infty} v'(c) = 0.
\]

Then, all the assumptions of the previous theorem are satisfied.

Precisely, Theorem 5 demonstrates indeterminacy of the second-period consumption $d$. For any $d$ found in Theorem 5, define $q^*(d) : S \rightarrow \mathbb{R}_+$ by $q^*_s(d) := d - \omega^2_s > 0$ for all $s \in S$. One
can easily find that \( q^*(d) \neq q^*(d') \) for \( d \neq d' \) and that \( q^*(d) \) is a fully insured stationary monetary equilibrium. These imply the indeterminacy of fully insured stationary monetary equilibria, and therefore, that real indeterminacy (indeterminacy of equilibrium allocation) also arises.

**Remark 1 (Case of a Unique Prior).** When agents have a unique prior (i.e., when \( \mathcal{P}_s \) is a singleton for all \( s \in S \)), the lifetime utility function is differentiable. Then, according to Gottardi (1996), a stationary monetary equilibrium generically exists and is generically regular. Therefore, it is locally isolated.\(^{14}\) In contrast, we have shown the existence of a continuum of stationary monetary equilibria when agents assign not a single prior but the set of priors. This indeterminacy depends crucially on the fact that MMEU preferences might not be differentiable at some point.

**Remark 2 (Equilibrium Indeterminacy under MMEU).** At an equilibrium of the Epstein and Wang’s (1994, 1995) asset pricing model, the representative agent does not hold any security and consumes all of her initial endowment in each period.\(^{15}\) The equilibrium (supporting) prices are then those prices under which the optimal decision of the representative agent is to hold no asset. Epstein and Wang showed that there exist uncountably many such equilibrium prices. However, in their model, indeterminacy of equilibrium allocation cannot happen because of the representative-agent setting. In our exchange economy, in contrast, indeterminacy of allocation (real indeterminacy) as well as price indeterminacy takes place.

**Remark 3 (Robustness of Indeterminacy).** Indeterminacy due to the maximin expected utility can be observed even in the Arrow-Debreu model. Chateauneuf, Dana and Tallon (2000) and Dana (2004) argued indeterminacy due to convex Choquet expected utility preferences in the Arrow-Debreu economy. Dana (2004) showed not only price but also real indeterminacy when there is no aggregate risk. Here, no aggregate risk means that the total endowment does not change over the state space. As argued by Rigotti and Shannon (2012), however, such indeterminacy in the Arrow-Debreu economy is not robust; i.e., the equilibrium is generically

\(^{14}\)Moreover, if preferences are separable as in Corollary 1 and if the relative risk aversion of \( w \) is less than or equal to unity, the number of stationary monetary equilibria is at most one. See Ohtaki (2011) for more details.

\(^{15}\)This is a natural extension of Dow and Werlang’s (1992) no-trade result to a dynamic equilibrium setting.
determinate. In contrast to the Arrow-Debreu economy, our model is dynamic with overlapping generations and money, and it shows the existence of a nonempty open set of economies exhibiting real indeterminacy where aggregate risks do exist; i.e. where the sum of initial endowments of the old generation and young generation in the same period changes over the state space. Note that our robustness result depends on the “endogenized flatness”; i.e., the fact that we can choose equilibrium real money balances so that the second-period consumptions should be constant over all states.

4.2. An Increase in Uncertainty

This subsection shows that an increase in uncertainty implies both an increase of the “degree” of indeterminacy and the dilation of the set of economies that exhibit a continuum of fully insured equilibria. Given an economy $\omega$ and a transition correspondence $P$, define by $\hat{D}_P(\omega)$ the set of second-period consumptions for some fully insured stationary monetary equilibrium. That is, let

$$\hat{D}_P(\omega) := \left\{ d \in \mathbb{R}^{++} \mid d \text{ satisfies Eq.(5) under } P \text{ and } \omega \right\}.$$ 

Note that $\hat{D}_P(\omega)$ may be empty for some $\omega$. Additionally, define $\hat{\Omega}_P$ by

$$\hat{\Omega}_P := \left\{ \omega \in \mathbb{R}^{S+}_+ \times \mathbb{R}^{S+}_+ \mid \hat{D}_P(\omega) \neq \emptyset \right\}.$$ 

Note that $\hat{\Omega}_P$ is the set of economies in which a fully insured equilibrium exists. Finally, we write $P \subseteq Q$ when $P_s \subseteq Q_s$ for each $s \in S$. In accordance with the notion of Ghirardato and Marinacci (2002), we may say that $Q$ represents more uncertainty (and more uncertainty aversion given the MMEU preference) than $P$ when $P \subseteq Q$ holds. We can then prove the following theorem.

**Theorem 6** Let $P$ and $Q$ be two transition correspondences such that $P \subseteq Q$. Also assume that $u$ and $P$ satisfy all the assumptions of Theorem 5 (and hence, $\hat{\Omega}_P$ is nonempty by Theorem 5). Then, (a) $\hat{D}_P(\omega) \subseteq \hat{D}_Q(\omega)$ for any $\omega \in \hat{\Omega}_P$; and (b) $\hat{\Omega}_P \subseteq \hat{\Omega}_Q$.

Recall that $\hat{\Omega}_P$ and $\hat{D}_P(\omega)$ are the set of economies exhibiting indeterminacy (of fully insured equilibrium) and the set of second-period consumptions for some fully insured equilibrium. As mentioned above, therefore, the theorem indicates that an increase in uncertainty
(a) increases the degree of indeterminacy of fully insured stationary monetary equilibrium and, at the same time, (b) expands the set of economies exhibiting a continuum of fully insured equilibria.

4.3. A Two-state Example

In this subsection, we consider a stochastic OLG model where there are only two states. Let \( S = \{ \alpha, \beta \} \) and let \( u \) be defined by \((\forall c^1, c^2 \in \mathbb{R}_++): u(c^1, c^2) := \ln c^1 + \ln c^2.16\) Note that \( u \) thus defined satisfies the assumption of Corollary 1. Additionally, define \( P \) by

\[
(\forall s \in S) \quad P_s = \{(p, 1-p) \mid \varepsilon \leq p \leq \delta \}
\]

where \( 0 \leq \varepsilon < \delta \leq 1 \). The assumptions of Theorem 5 are then satisfied since any \( \hat{\omega}^2 \) such that \( \hat{\omega}^2_\alpha \neq \hat{\omega}^2_\beta \) satisfies (6). Therefore, Theorem 5 concludes that there is an open set of economies that exhibit a continuum of fully insured equilibria under given \( u \) and \( P \).

We maintain \( u \) and \( P \) defined in the previous paragraph throughout the rest of this subsection. Consider any economy where \( \omega^2_\alpha < \omega^2_\beta \). Then, (5) can be rewritten as

\[
(\forall s \in S) \quad (d - \omega^2_s)A_\delta(d) \leq \omega^1_s \leq (d - \omega^2_s)B_\varepsilon(d),
\]

where

\[
A_\delta(d) := \frac{d + (\delta \omega^2_\alpha + (1 - \delta) \omega^2_\beta)}{d - (\delta \omega^2_\alpha + (1 - \delta) \omega^2_\beta)} \quad \text{and} \quad B_\varepsilon(d) := \frac{d + (\varepsilon \omega^2_\alpha + (1 - \varepsilon) \omega^2_\beta)}{d - (\varepsilon \omega^2_\alpha + (1 - \varepsilon) \omega^2_\beta)}
\]

It is seen that \( A_\delta(d) < B_\varepsilon(d) \) because \( \varepsilon < \delta \). Furthermore, a decrease in \( \varepsilon \) raises \( B_\varepsilon(d) \) and an increase in \( \delta \) lowers \( A_\delta(d) \). Therefore, an increase in uncertainty (a dilation of \( P \)) slackens the inequalities, and hence leads to an increase in the degree of indeterminacy and the dilation of the set of economies that exhibit a continuum of fully insured equilibria, which is as predicted by Theorem 6.

In the rest of this section, we fix an economy that may exhibit a continuum of equilibria and represent such equilibria diagrammatically. We specify the economy by \( \omega = (\omega^1_\alpha, \omega^1_\beta, \omega^2_\alpha, \omega^2_\beta) = \)

\footnote{The maximin expected utility preferences of Cobb-Douglas form as in this example have been recently axiomatized by Faro (2012).}
Figure 1: “Edgeworth Box”: (a) There is no uncertainty, (b) There is uncertainty.

Notably, in this case, the agents’ indifference curves exhibit kinks on the 45-degree line (see Figure 1). Using these equations, we can depict the “budget line” in the box diagram as the green line in Figure 1(a). In addition, we can depict the indifference curves as in Figure 1 by restricting the lifetime utility on the budget set; i.e., by considering

\[ U^*(c^2_s, c^2_\beta) := U^*(\omega^1_s + \omega^2_s - c^2_s, c^2_\alpha, c^2_\beta). \]
Figure 2: “Edgeworth box”: \((ω_1^1, ω_1^2, ω_2^1, ω_2^2) = (6, 3, 1, 2)\) and \((ε, δ) = (1/4, 3/4)\)

Figure 1(b)). In the Edgeworth box (Figure 2), the economy clearly exhibits a continuum of fully insured equilibria.

5. Efficiency of Stationary Monetary Equilibrium Allocations

We have shown the existence of a continuum of stationary monetary equilibria. A question that naturally arises is whether allocations corresponding to these stationary monetary equilibria are optimal. The answer is that they are in some sense. To see this, we begin with some definitions.
Let $S_0 := \{s_0\} \cup S$. A stationary feasible allocation is a pair of $c^1 \in \mathbb{R}^S_+$ and $c^2 \in \mathbb{R}^{S_0 \times S}_+$ that satisfies that

$$\forall (s_-, s) \in S_0 \times S \quad c^1_s + c^2_{s-s} = \omega^1_s + \omega^2_s,$$

where $c^2_{s-s}$ is the consumption of the initial old, who is a one-period lived agent in state $s$ and period 1. From this equation and the assumption that the endowment depends only on the current state (see Subsection 2.1), it follows that $c^2_{s-s}$ is independent of $s_-$. Thus, we write a stationary feasible allocation as $c = (c^1_s, c^2_s)_{s \in S}$. A stationary feasible allocation $b = (b^1_s, b^2_s)_{s \in S}$ is conditionally Pareto superior to a stationary feasible allocation $c = (c^1_s, c^2_s)_{s \in S}$ if

$$\forall s \in S \quad U_s(b^1_s, (b^2_s)_{s' \in S}) \geq U_s(c^1_s, (c^2_{s'})_{s' \in S}) \quad \text{and} \quad b^2_s \geq c^2_s$$

with at least one strict inequality. The latter set of inequalities means that the initial old will not be worse off when moving from $c$ to $b$. A stationary feasible allocation $c$ is conditionally Pareto optimal if there is no other stationary feasible allocation that is conditionally Pareto superior to $c$.

We are now ready to state the main result of this section.

**Theorem 7** Every stationary monetary equilibrium achieves conditional Pareto optimality.

Theorem 7 claims that even when there exists a continuum of stationary equilibrium allocations because of MMEU preferences, each element of the continuum is conditionally Pareto optimal.\(^{18}\)

### 6. Proofs

**Proof of Theorem 1.** Let $s \in S$. Note that $U(\cdot)(c)$ is continuous for any given $c \in \mathbb{R}_+ \times \mathbb{R}^S_+$. Also let $c, b \in \mathbb{R}_+ \times \mathbb{R}^S_+$ be such that $c \neq b$ and let $\alpha \in (0, 1)$. We denote by $c_\alpha b$ the convex combination of $c$ and $b$; i.e., $c_\alpha b = \alpha c + (1 - \alpha)b$. We first claim that

$$U(\mathcal{P}_s)(c_\alpha b) > \min_{\pi \in \mathcal{P}_s} [\alpha U(\pi)(c) + (1 - \alpha)U(\pi)(b)] .$$

\(^{18}\)Note that the proof of Theorem 7 does not require continuous differentiability of $u$. Furthermore, we can replace the strict concavity of $u$ with its strict quasi-concavity. See Footnote 8.
Suppose, to the contrary, that
\[ U(\mathcal{P}_s)(c, b) \leq \min_{\pi \in \mathcal{G}_s} [\alpha U(\pi)(c) + (1 - \alpha) U(\pi)(b)] . \] 
(8)

By the compactness of \( \mathcal{P}_s \) and the continuity of \( U(\cdot)(d) \) for all \( d \in \mathbb{R}_+ \times \mathbb{R}^{S_+}_+ \), \( \mathcal{M}(\mathcal{P}_s)(c, b) \) and the set

\[ N(\mathcal{P}_s) = \arg \min_{\pi \in \mathcal{P}_s} [\alpha U(\pi)(c) + (1 - \alpha) U(\pi)(b)] \]

are nonempty. Hence, it follows from Eq.(8) that there exist \( \mu \in \mathcal{M}(\mathcal{P}_s)(c, b) \) and \( \nu \in N(\mathcal{P}_s) \) such that

\[
U(\mu)(c, b) = U(\mathcal{P}_s)(c, b) \\
\leq \min_{\pi \in \mathcal{P}_s} [\alpha U(\pi)(c) + (1 - \alpha) U(\pi)(b)] \\
= \alpha U(\nu)(c) + (1 - \alpha) U(\nu)(b),
\]

which implies that

\[
\alpha U(\mu)(c) + (1 - \alpha) U(\mu)(b) < U(\mu)(c, b) \leq \alpha U(\nu)(c) + (1 - \alpha) U(\nu)(b)
\]

by the strict concavity of \( u \). This contradicts the fact that \( \nu \) minimizes \( \alpha U(\cdot)(c) + (1 - \alpha) U(\cdot)(b) \) on \( \mathcal{P}_s \), which proves the claim.

Recall that \( N(\mathcal{P}_s) \) is nonempty. Let \( \nu \in N(\mathcal{P}_s) \). We then obtain that

\[
\min_{\pi \in \mathcal{G}_s} [\alpha U(\pi)(c) + (1 - \alpha) U(\pi)(b)] = \alpha U(\nu)(c) + (1 - \alpha) U(\nu)(b) \\
\geq \alpha U(\mathcal{P}_s)(c) + (1 - \alpha) U(\mathcal{P}_s)(b)
\]

by the definition of \( U(\mathcal{P}_s)(\cdot) \). Combining this with the claim proven above, it follows that

\[
U(\mathcal{P}_s)(c, b) > \min_{\pi \in \mathcal{G}_s} [\alpha U(\pi)(c) + (1 - \alpha) U(\pi)(b)] \\
\geq \alpha U(\mathcal{P}_s)(c) + (1 - \alpha) U(\mathcal{P}_s)(b).
\]

Therefore, \( U(\mathcal{P}_s)(\cdot) \) is strictly concave as desired.
Proof of Theorem 3. By Theorems 1 and 2, we can characterize a stationary equilibrium by a system of inequalities:

\[(\forall s \in S) \ D_+ V(P_s)(M) \leq 0 \leq D_- V(P_s)(M).\]

Since

\[
D_+ V(P_s)(m) = \min_{\pi \in M(P_s)(c^m_2(q^*))} \left( - \sum_{s' \in S} q^*_s u_1(\omega^1_s - q^*_s, d)_s + \sum_{s' \in S} q^*_s u_2(\omega^1_s - q^*_s, d)_s \right),
\]

\[
D_- V(P_s)(m) = \max_{\pi \in M(P_s)(c^m_2(q^*))} \left( - \sum_{s' \in S} q^*_s u_1(\omega^1_s - q^*_s, d)_s + \sum_{s' \in S} q^*_s u_2(\omega^1_s - q^*_s, d)_s \right)
\]

for all \(m \in \mathbb{R}\) and \(M(P_s)(c^m_s(q^*))\) is convex for all \(s \in S\), we obtain (3).

Proof of Theorem 4. Let \(s \in S\). Under (4), (3) is successively rewritten as

\[
0 \in \left\{ - \sum_{s' \in S} q^*_s u_1(\omega^1_s - q^*_s, d)_s + \sum_{s' \in S} q^*_s u_2(\omega^1_s - q^*_s, d)_s \pi | \pi \in P_s \right\},
\]

which is equivalent to

\[
0 \in \left\{ - q^*_s u_1(\omega^1_s - q^*_s, d)_s + u_2(\omega^1_s - q^*_s, d)_s \sum_{s' \in S} q^*_{s'} \pi | \pi \in P_s \right\}.
\]

Hence, it follows that

\[
\frac{q^*_s u_1(\omega^1_s - q^*_s, d)_s}{u_2(\omega^1_s - q^*_s, d)_s} \in \left\{ \sum_{s' \in S} q^*_{s'} \pi | \pi \in P_s \right\}
\]

Because \(P_s\) is compact and convex for each \(s\), the last expression is equivalent to

\[
\min_{\pi \in P_s} \sum_{s' \in S} q^*_{s'} \pi_{s'} \leq \frac{q^*_s u_1(\omega^1_s - q^*_s, d)_s}{u_2(\omega^1_s - q^*_s, d)_s} \leq \max_{\pi \in P_s} \sum_{s' \in S} q^*_{s'} \pi_{s'}.
\]

By substituting \(q^*_s = d - \omega^2_s\) into this expression, we obtain (5).

Proof of Theorem 5. Let \(\hat{\omega}^2\) be an element of \(\mathbb{R}^S_{++}\) that satisfies (6). That is, assume that

\[
\max_{s \in S} \min_{\pi \in P_s} \sum_{s' \in S} \hat{\omega}^2_{s'} \pi_{s'} < \min_{s \in S} \max_{\pi \in P_s} \sum_{s' \in S} \hat{\omega}^2_{s'} \pi_{s'}.
\]

The existence of such an \(\hat{\omega}^2\) is guaranteed by the assumption. There then exist \(m, M \in \mathbb{R}_{++}\) such that

\[
(\forall s \in S) \ \min_{\pi \in P_s} \sum_{s' \in S} \hat{\omega}^2_{s'} \pi_{s'} < m < M < \max_{\pi \in P_s} \sum_{s' \in S} \hat{\omega}^2_{s'} \pi_{s'}.
\]
Because \( \max_{\pi \in \mathcal{P}} \sum_{s' \in S} \omega_{s'}^2 \pi_{s'} \) and \( \min_{\pi \in \mathcal{P}} \sum_{s' \in S} \omega_{s'}^2 \pi_{s'} \) are continuous in \( \omega^2 \) for each \( s \in S \) according to the maximum theorem (Berge, 1963), there exists an open neighborhood, \( \Omega^2_{s'} \), of \( \hat{\omega}^2 \) for each \( s \in S \) such that any element of \( \Omega^2_{s'} \) satisfies

\[
\min_{\pi \in \mathcal{P}} \sum_{s' \in S} \omega_{s'}^2 \pi_{s'} < m < M < \max_{\pi \in \mathcal{P}} \sum_{s' \in S} \omega_{s'}^2 \pi_{s'}.
\]

Define \( \Omega^2 \) by \( \Omega^2 := \cap_{s \in S} \Omega^2_{s'} \). Obviously, \( \Omega^2 \) is open and nonempty since \( \hat{\omega}^2 \in \Omega^2 \). There then exist \( \tilde{\omega}_1^2, \tilde{\omega}_2^2, \ldots, \tilde{\omega}_{|S|}^2 \) such that \( \hat{\omega}^2 \in (\tilde{\omega}_1^2, \tilde{\omega}_2^2) \times \cdots \times (\tilde{\omega}_{|S|}^2, \tilde{\omega}_{|S|}^2) \subseteq \Omega^2 \), where we write \( S = \{1, 2, \ldots, |S|\} \). Let \( \hat{d} \) be any real number such that \( \hat{d} > \max_{s \in S} \tilde{\omega}_s^2 \). Then, from the above inequalities, it holds that, for any \( \omega^2 \in (\tilde{\omega}_1^2, \tilde{\omega}_2^2) \times \cdots \times (\tilde{\omega}_{|S|}^2, \tilde{\omega}_{|S|}^2) \) and for any \( s \in S \),

\[
\hat{d} - \max_{\pi \in \mathcal{P}} \sum_{s' \in S} \omega_{s'}^2 \pi_{s'} < \hat{d} - M < \hat{d} - m < \hat{d} - \min_{\pi \in \mathcal{P}} \sum_{s' \in S} \omega_{s'}^2 \pi_{s'}.
\]

Let \( s \in S \). Define a function \( g : \mathbb{R}^2_{++} \rightarrow \mathbb{R} \) by

\[
(\forall (\omega_1^2, \omega_2^2)) \quad g(\omega_1^2, \omega_2^2) := (\hat{d} - \omega_2^2) \frac{u_1(\omega_1^2 + \omega_2^2 - \hat{d}, \hat{d})}{u_2(\omega_1^2 + \omega_2^2 - \hat{d}, \hat{d})}.
\]

By the assumption that \( f \) is surjective with respect to its first argument, there exists \( \hat{\omega}_s^2 \in \mathbb{R}_{++} \) such that \( g(\hat{\omega}_s^2, \tilde{\omega}_s^2) \in (\hat{d} - M, \hat{d} - m) \), where \( \tilde{\omega}_s^2 \) is found in the first paragraph of this proof. Since \( g \) is continuous according to the continuity of \( f \), \( g^{-1}((\hat{d} - M, \hat{d} - m)) \) is open. Therefore, there exist open sets \( \hat{\Omega}_s^2 \) and \( \hat{\Omega}_s^2 \) such that \( \hat{\omega}_s^2 \in \hat{\Omega}_s^2 \), \( \tilde{\omega}_s^2 \in \hat{\Omega}_s^2 \) and \( \hat{\Omega}_s^2 \times \hat{\Omega}_s^2 \subseteq g^{-1}((\hat{d} - M, \hat{d} - m)) \).

Let \( \hat{\Omega}_s^2 := (\omega_1^2, \omega_2^2) \cap \hat{\Omega}_s^2 \). Then, \( \hat{\Omega}_s^2 \) is open and nonempty since \( \tilde{\omega}_s^2 \) is included in both sets. Finally, define \( \Omega \) by \( \Omega := \bigcap_{s \in S} \hat{\Omega}_s^2 \times \cdots \times \hat{\Omega}_1^2 \times \cdots \times \hat{\Omega}_s^2 \).

Let \( (\omega^1, \omega^2) \in \Omega \). Then, \( \hat{d} \) (found in the first paragraph of this proof) satisfies (5) with strict inequalities for each \( s \in S \). Since \( f \) is continuous, there exists an open neighborhood, \( D_s \), of \( \hat{d} \) such that, for any \( d \in D_s \), \( d \) satisfies (5) with strict inequalities. Define \( D \) by \( D := \cap_{s \in S} D_s \).

This is the desired set and the proof is complete.

**Proof of Corollary 1.** Under the stated assumptions, the function \( f \) will become \( f(x, y) = v'(x)/w'(y) \) for each \( x \) and \( y \). According to the Inada condition on \( v \), \( f(\cdot, y) \) is clearly surjective for each \( y \), which completes the proof.
Proof of Theorem 6. (a) Let \( \omega \in \hat{\Omega}_P \) and \( d \in \hat{D}_P(\omega) \). We can do this since \( \hat{\Omega}_P \) is nonempty by the assumption and Theorem 5 and since \( \hat{D}_P(\omega) \) is nonempty according to the definition of \( \hat{\Omega}_P \). Then, for each \( s \in S \),

\[
d - \max_{\pi \in \hat{\Omega}_s} \sum_{s' \in S} \omega_{s'}^2 \pi_{s'} \leq d - \max_{\pi \in \hat{P}_s} \sum_{s' \in S} \omega_{s'}^2 \pi_{s'} \\
\leq (d - \omega_s^2) f(\omega_s^1 + \omega_s^2 - d, d) \\
\leq d - \min_{\pi \in \hat{\Omega}_s} \sum_{s' \in S} \omega_{s'}^2 \pi_{s'} \\
\leq d - \min_{\pi \in \hat{D}_Q(\omega)} \sum_{s' \in S} \omega_{s'}^2 \pi_{s'},
\]

where the first and the last inequalities hold since \( P_s \subseteq Q_s \) and the second and the third inequalities hold according to the assumption that \( d \in \hat{D}_P(\omega) \). Therefore, \( d \in \hat{D}_Q(\omega) \), and hence, \( \hat{D}_P(\omega) \subseteq \hat{D}_Q(\omega) \).

(b) Let \( \omega \in \hat{\Omega}_P \). Then, \( \hat{D}_P(\omega) \) is nonempty, and hence, \( \hat{D}_Q(\omega) \) is also nonempty since \( \hat{D}_P(\omega) \subseteq \hat{D}_Q(\omega) \) according to (a). Therefore, \( \omega \in \hat{\Omega}_Q \), which completes the proof.

Proof of Theorem 7. Sakai (1988) proved that if \( U_s : \mathbb{R}_{++} \times \mathbb{R}^{S_{++}}_+ \rightarrow \mathbb{R} \) is increasing and strictly quasi-concave for all \( s \in S \), then, for any stationary feasible allocation \( c \) corresponding to a stationary monetary equilibrium, there exists no other stationary feasible allocation \( b \) that is conditionally Pareto superior to \( c \). (Sakai (1988) assumed that \( U_s \) is represented by an expected utility function with a utility index that satisfies some regularity conditions. However, his proof only requires that \( U_s \) should be increasing and strictly quasi-concave.) Clearly, \( U_s \) is increasing. Furthermore, Theorem 1 implies that \( U^s(\cdot) \) is strictly concave, and hence, it is strictly quasi-concave.\(^{19}\) Therefore, the proof is complete.

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\(^{19}\) See, for example, Theorem 1.E.1(i) of Takayama (1974).
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