KEIO/KYOTO JOINT
GLOBAL CENTER OF EXCELLENCE PROGRAM
Raising Market Quality-Integrated Design of “Market Infrastructure”

KEIO/KYOTO GLOBAL COE DISCUSSION PAPER SERIES

DP2012-022

A Simple and Constructive Proof of the Existence of a Competitive Price Under the Gross Substitute Property

Takuya Masuzawa*

Abstract

In this note, we demonstrate, in a simpler and more direct manner than done in existing literature, the existence of an equilibrium in a market whose excess demand function has the gross substitute property.

*Takuya Masuzawa
Faculty of Economics, Keio University

KEIO/KYOTO JOINT GLOBAL COE PROGRAM

Raising Market Quality-Integrated Design of “Market Infrastructure”

Graduate School of Economics and Graduate School of Business and Commerce,
Keio University
2-15-45 Mita, Minato-ku, Tokyo 108-8345, Japan

Institute of Economic Research,
Kyoto University
Yoshida-honmachi, Sakyo-ku, Kyoto 606-8501, Japan

A Simple and Constructive Proof of the Existence of a Competitive Price Under the Gross Substitute Property

Takuya Masuzawa

December 17, 2012

Abstract

In this note, we demonstrate, in a simpler and more direct manner than done in existing literature, the existence of an equilibrium in a market whose excess demand function has the gross substitute property.

1 Introduction

The gross substitute property of the excess demand function of a market was mainly discussed as a sufficient condition for both the global stability of the tâtonnement process and the uniqueness of the equilibrium as well as a regularity condition for the comparative statics (see Arrow and Hahn, 1971, chapters 9-11). Assuming the condition, Kuga (1965), Nikaido (1968, chapter 6), Greenberg (1977), and Hildenbrand and Kirman (1988, chapter 3) demonstrated simple proofs of the existence of a competitive price without Brouwer's fixed point theorem or Kakutani's. Some of these authors defined some evaluation functions from the excess demand and demonstrated that the optimizers of these functions are the competitive prices. However, the competitive price is not constructively obtained in any of their proofs and the relation between the global stability of the tâtonnement process and the existence of a competitive price is not clear.

In this note, we prove the existence of an equilibrium price under the gross substitute property by constructing the price. Recently, it has been shown that the algorithm for the construction of a competitive price is also commonly used in various areas: the algorithm to determine the α-core of games with punishment-dominance relations such as Cournot's oligopoly game and public provision game in Masuzawa (2008); the tax-cut algorithm of single transferable voting to achieve proportional representation (see Masuzawa 2012a); and the deferred acceptance algorithm to find a stable matching of two-sided markets by Gale and Shapley[1].

2 The proof

Let $\Re$ be the set of the real numbers. By $\Re_{++}$, we denote the set of the positive real numbers. Similarly, by $\Re_{+}$, we refer to the set of the non-negative real numbers. Let N be the finite set of commodities and $Z : \Re^N_{++} \to \Re^N$ be the excess demand function. For all $x^N, y^N \in \Re^N$, we write $x^N \geq y^N$ iff $x^i \geq y^i$ for all $i \in N$. We say that $p^N \in \Re^N_{++}$ is an *equilibrium price* if $Z^i(p^N) \leq 0$ for all $i \in N$.

We assume the following conditions.

Weak Walras' Law: $\sum_{i\in N} p^i \cdot Z^i(p^N) \leq 0$ for all $p^N \in \Re^N_{++}$.

Weak Gross Substitute Property: If $p^i \leq q^i$,

then $Z^j(p^{N\setminus\{i\}}, p^i) \leq Z^j(p^{N\setminus\{i\}}, q^i)$ for all $j \in N \setminus \{i\}$

Continuity: $Z : \Re^N_{++} \to \Re^N$ is continuous.

Boundary Condition: For all $p^N_* \in \Re^N_+ \setminus \{0\}$ such that

$P = \{i \in N : p^i_* = 0\} \neq \emptyset$, $\lim \sum_{i\in N} Z^i(p^N) \to \infty$ as $p^N \to p^N_*$.

Theorem 1 *Under the weak Walras' law, the continuity, the weak gross substitute property, and the boundary condition, $Z : \Re^N_{++} \to \Re^N$ has an equilibrium price.*

Proof: By the boundary condition and the gross substitute property, there exist β_k $(k = 1, 2, \ldots, n)$ such that (i) $1 = \beta_1 > \beta_2 > \cdots > \beta_n > 0$ and (ii) $\sum^n_{i\in N\setminus S} Z^i(p^N) > 0$ if $|S| \geq k$, $p^i \geq \beta_k$ for all $i \in S$ and $p^i \leq \beta_{k+1}$ for all

[1] See Masuzawa (2012b).

$i \in N \setminus S$. By the weak gross substitute property and the continuity, there exists the maximum of $p^N \in [\beta_n, 1]^N$ such that

$$\text{for all } i \in N, Z^i(p^N) \geq 0 \text{ or } p^i = \beta_n. \tag{A}$$

Indeed, the maximum p^N is obtained by

$$p^i := \max\left\{q^i : q^N \in [\beta_n, 1]^N \text{and, for all } i \in N, Z^i(q^N) \geq 0 \text{ or } q^i = \beta_n.\right\}. \tag{B}$$

Assume that $p^i < 1$ and $Z^i(p) > 0$ for some $i \in N$. By the weak gross substitute property and the continuity, for a sufficiently small $\epsilon > 0$, (i) $Z^j(p^{N\setminus\{i\}}, p^i + \epsilon) \geq 0$ or $p^j = \beta_n$ for all $j \in N \setminus \{i\}$, and (ii) $Z^i(p^{N\setminus\{i\}}, p^i + \epsilon) > 0$, which contradicts the definition of the maximum. Thus, we have

$$\text{for all } i \in N,\ \ Z^i(p) \leq 0 \text{ if } p^i < 1. \tag{C}$$

If $T := \{i : p_i = 1\} = \emptyset$, it is evident that p is a competitive price. Assume that $S_1 := \{i : p_i = 1\} \neq \emptyset$ and define $S_k = \{i \in N : \beta_k < p^i\}$ for all $k \geq 2$. Then, by the definition of β_k and (C), we can see by induction that $|S_k| \geq k$ for all $k \geq 2$. Thus, for all $i \in N$, $p^i > \beta_n$. By the weak Walras' law, for all $i \in N$, we have $p^i \cdot Z^i(p^N) = 0$, and thus $Z^i(p^N) = 0$. □

Note that in the discussion above, β_n is not constructively obtained while it characterizes the competitive price. If we assume the following condition, we can characterize the competitive price without $\{\beta_n\}$.

Strict Walras' Law: $\sum_{i \in N} p^i \cdot Z^i(p^N) = 0$ for all $p^N \in \Re^N_{++}$.

Proposition 1 *Under the strict Walras' law, the weak gross substitute property, the continuity, and the boundary condition, the maximum of (A) is also the maximum of $q^N \in [0, 1]^N$ such that*

$$\textit{for all } i \in N, Z^i(q^N) \geq 0. \tag{D}$$

Proof: Let $p^N \in [0, 1]^N$ be the maximum of (A). If $T := \{i : p_i = 1\} \neq \emptyset$, from the proof of Theorem 1, then $Z^i(p^N) = 0$ for all $i \in N$. Assume that

$T := \{i : p_i = 1\} = \emptyset$. Then, from the proof of Theorem 1, $Z^i(p^N) \leq 0$. By the strict Walras' law, we have $Z^i(p^N) \geq 0$ for all $i \in N$. Thus, p^N satisfies condition (D). Thus, the maximum of (D), q^N, exists and $q^N \geq p^N$. Further, q^N also satisfies (A), and thus $p^N \geq q^N$. $\square$

The algorithm described in Masuzawa (2008) computes the maximum that satisfies constraint (A) if the domain is restricted to a finite set. Its continuous version is as follows.

Procedure to construct a sequence that converges to an equilibrium:

1. $k := 0$;
2. for all $i \in N$, $p_0^i := 1$;
3. for all $i \in N$,
   $p_{k+1}^i := \max \left\{ p^i \in \Re_+ : Z^i(p_k^{N\setminus\{i\}}, p^i) \geq 0 \text{ and } p^i \leq p_k^i \right\} \cup \{0\}$;
4. $k := k + 1$; Jump to 3.

Note that the procedure is a kind of Walras' algorithm: if the excess supply of a good is positive, the price is decreasingly updated.

Theorem 2 *Under the strict Walras' law, the continuity, the weak gross substitute property, and the boundary condition, the sequence $\{p_k^N\}$ converges to an equilibrium price of $Z : \Re_{++}^N \to \Re^N$.*

Proof: First, note that $p_k^N \geq p_*^N$ for all k, where p_*^N is the maximum of (D). Second, since the sequence $\{p_k^N\}$ is monotonically decreasing, it has the limit p_{**}^N. Thus, $p_{**}^N \geq p_*^N$. Assume that $Z^i(p_{**}^N) < 0$ and $p_{**}^i > 0$. By the continuity, for a sufficiently large k, $Z^i(p_k^{N\setminus\{i\}}, p_{k+1}^i) < 0$ and $p_{k+1}^i > 0$, which contradicts the definition of p_{k+1}^i and the continuity. It follows that $p_{**}^N = p_*^N$.$\square$

Acknowledgements

I am grateful to Chaowen Yu for the discussion about the role of the boundary condition of the excess demand function, and to Mikio Nakayama for his helpful comments.

References

K. Arrow, F. H. Hahn (1971), *General Competitive Analysis*, San Francisco, Holden-Day.

J. Greenberg (1977), An Elementary Proof of the Existence of a Competitive Equilibrium with Weak Gross Substitutes, *The Quarterly Journal of Economics 91(3)*: 513-516.

W. Hildenbrand, A.P. Kirman (1988), *Equilibrium Analysis: Variation on Themes by Edgeworth and Walras.* North-Holland. Amsterdam.

K. Kuga (1965), Weak Gross Substitutability and the Existence of Competitive Equilibrium, *Econometrica 33*, 593-599.

T. Masuzawa (2008), Computing the Cores of Strategic Games with Punishment-Dominance Relations, *International Journal of Game Theory 38*, 185-201.

T. Masuzawa (2012a), STV Elections by Feedback Counting: An Approach from Cooperative Game Theory, Keio/Kyoto Global Coe Discussion Paper Series, DP2012-015

T. Masuzawa (2012b), Punishment-Dominance Condition on Stable Two-sided Matching Algorithms, Keio/Kyoto Global Coe Discussion Paper Series DP2012-018

H. Nikaido (1968), *Convex Structures and Economic Theory*, Academic Press: New York.