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Takuya Masuzawa*

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In this article, we consider a many-to-one two-sided matching market and define a canonical strategic form game, in which any worker applies to the top k firms and is assigned to the most preferred firm that does not reject him/her. Under the substitute property of firms' preferences, the game satisfies the punishment-dominance condition. The deferred-acceptance algorithm by Gale and Shapley (Amer. Math. Monthly 69: 1962), which finds the maximum and minimum of stable matchings, is described as an instance of the algorithm by Masuzawa (Int. Jour. Game Theory 38: 2008), which determines the α-cores of the strategic form games with the punishment-dominance condition.

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October 31, 2012

Abstract

In this article, we consider a many-to-one two-sided matching market and define a canonical strategic form game, in which any worker applies to the top k firms and is assigned to the most preferred firm that does not reject him/her. Under the substitute property of firms' preferences, the game satisfies the punishment-dominance condition. The deferred-acceptance algorithm by Gale and Shapley (Amer. Math. Monthly 69: 1962), which finds the maximum and minimum of stable matchings, is described as an instance of the algorithm by Masuzawa (Int. Jour. Game Theory 38: 2008), which determines the α-cores of the strategic form games with the punishment-dominance condition.

1 Introduction

The purpose of this article is to bridge two areas of cooperative game theory from the algorithmic point of view: the strategic analyses of NTU convex games and the stable matching of many-to-one two-sided markets. We show that the algorithm to determine the α-core of games with the punishment-dominance condition and the deferred acceptance algorithm by Gale and Shapley (1962) to find a stable matching of a two-sided market are based on the same principle.

The supermodularity of a set function $v : 2^N \to \Re$ has an important role in combinatorial optimization problems. Shapley (1971) introduced the concept into game theory and considered games whose characteristic function is supermodular, *TU convex games*. Vilkov (1977) extended the concept to the

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situation described by a vector-valued set correspondence, $V : 2^N \twoheadrightarrow \Re^N$, called NTU convexity, and some modified concepts have been developed (see Hendrickx et al, 2001; Masuzawa 2012a). The *punishment-dominance condition* was introduced by Masuzawa (2003) as a sufficient condition for a strategic form game to be NTU convex. A game satisfies the punishment-dominance condition if for any player i and any pair of his/her strategies x and y, the change of strategy from x to y makes the others' payoffs unanimously worse off or unanimously better off independently of their strategies[1]. The n-person prisoner's dilemma game, the Cournot oligopoly game, and the public good provision game all satisfy this condition. Furthermore, Masuzawa (2008) proposed efficient algorithms to determine and to find the α-core strategy.

On the other hand, Gale and Shapley (1962) considered many-to-one matching markets, in which any worker is employed by at most one firm, but a firm may employ many workers. An outcome of markets is stable if any firm or any worker does not have incentive to cancel some contracts and there exists no pair of one firm and one worker who can improve their payoff by making a new contract between them and canceling some existing contract. They proposed an algorithm to find a stable outcome, called *the deferred acceptance algorithm*. The key factor for a many-to-one matching market to have at least one stable outcome and for the deferred acceptance algorithm to work is *the substitute property* of choice correspondence of firms developed by Roth and Sotomayor (1990). A choice correspondence has the substitute property if any exclusion of a worker from the possibilities does not harm the other workers. One obvious relation between two-sided markets and the punishment-dominance condition is that the choice correspondence satisfies the substitute property iff its representation by characteristic vectors satisfies the punishment-dominance condition.

We explain the existence and the algorithm of the stable matchings in terms of the punishment-dominance condition. We define a canonical strategic form game in which the set of players is the set of workers. In the canonical game, any worker applies to the top k firms as a strategy, and any firm, who is not a player of the canonical game, declines some of the offers according to the preference. Finally, any worker is employed by his/her most preferred firm who does not reject him/her. We show that the game satis-

[1] It is also called *monotone-externality*.

fies the punishment-dominance condition and that the deferred acceptance algorithm is an instance of Masuzawa's algorithm.

The remainder of this article is organized as follows. In Section 2, we formulate the punishment-dominance condition and a generic model of many-to-one two-sided matching market. In Section 3, for a two-sided market, we define a canonical strategic form game and derive the deferred acceptance algorithm from the theory of games with the punishment-dominance condition. Further, we show that with slight modification, the game satisfies Tarski's monotonicity, and that the stable outcomes are characterized as the fixed points.

2 Preliminaries

In this section, we first introduce the tools of our analysis, a strategic form game and the punishment-dominance condition, and we briefly review the theory of the punishment-dominance condition. Next, following the formulation of Hatfield and Milgrom (2005), we introduce the target of the analysis, two-sided matching markets in a generic way.

2.1 The punishment-dominance condition

We denote the set of real numbers by $\Re$. Let N be a finite set of players. The nonempty subsets of players are called *coalitions* and typically denoted by $S, T \dots$. For any $S \subset N$, a typical element in $\Re^S$ is denoted by a^S or b^S, and the i-th coordinate is denoted by a^i or b^i respectively. Given any pair of elements, $a^S, b^S \in \Re^S$, we write $a^S \gg b^S$ iff $a^i > b^i$ for all $i \in S$. Similarly, we write $a^S \geq b^S$ iff $a^i \geq b^i$ for all $i \in S$. The set of strategies of $i \in N$ is denoted by X^i. We abbreviate $\prod_{i \in S} X^i$ by X^S, a typical element of which is denoted by x^S, y^S, or z^S, the projection of which onto X^T is denoted by x^T, y^T, or z^T for $T \subset S$. On the other hand, for $x^S \in X^S$ and $x^T \in X^T$ such that $S \cap T = \emptyset$, by (x^S, x^T) we refer to an element of $X^{S \cup T}$ such that x^S and x^T are the projections of it onto X^S and X^T respectively. By u^i, we refer to the payoff function of i from X^N to $\Re$ and we abbreviate $(u^i)_{i \in S}$ by u^S. A strategic form game G is a list $(N, (X^i)_{i \in N}, (u_i)_{i \in N})$.

A strategy profile $y^N \in X^N$ is an α*-core strategy* if there exists no pair of $S \subset N$ ($S \neq \emptyset$) and $x^S \in X^S$ such that $u^S(x^S, z^{N \setminus S}) \gg u^S(y^N)$ for all $z^{N \setminus S} \in x^{N \setminus S}$. The *punishment-dominance relations* are introduced by

Masuzawa (2003) to state a sufficient condition for the existence of an α-core strategy.

Definition 1 A binary relation *PD* on X^i is a punishment-dominance relation iff the following condition holds:
if $x^i PD y^i$, then $u^{N\setminus\{i\}}(x^i, z^{N\setminus\{i\}}) \geq u^{N\setminus\{i\}}(y^i, z^{N\setminus\{i\}})$ for all $z^{N\setminus\{i\}} \in X^{N\setminus\{i\}}$.

The following condition is sufficient for an α-core strategy to exist (Masuzawa, 2003; 2006).

Condition 1 (Punishment-dominance condition) For all $i \in N$, there exists a *complete* punishment-dominance relation on X^i.

For the economic situations that satisfy the punishment-dominance condition, see Masuzawa (2006). Assume that X^N is finite and let *PD* be a complete and anti-symmetric punishment-dominance relation. Then, we can define $\max_{PD} X^i$ and $\min_{PD} X^i$. Further, we define $up(x^i)$ and $down(x^i)$ as follows:

$$\begin{aligned} up(x^i) &:= \min_{PD}\{y^i \in X^i | y^i PD x^i\} \qquad \text{for all } x^i \in X^i \setminus \max_{PD} X^i, \\ down(x^i) &:= \max_{PD}\{y^i \in X^i | x^i PD y^i\} \qquad \text{for all } x^i \in X^i \setminus \min_{PD} X^i. \end{aligned}$$

Under the punishment-dominance condition, $y^N \in X^N$ is an α-core strategy iff there exists no pair of $S \subset N$ ($S \neq \emptyset$) and $x^S \in X^S$ such that $u^i(x^S, z^{N\setminus S}) > u^i(y^N)$ for all $i \in S$, where $z^j = \min_{PD} X^j$ for all $j \in N \setminus S$. Thus, the solution of the following problem characterizes the α-core strategy.

Problem 1

Input: $a^N \in R^N$, a finite strategic form game G with Condition 1, and a complete and anti-symmetric punishment-dominance relation *PD*.

Output: the maximum of $x^N \in X^N$ such that
for all $i \in N$, $u^i(x^N) > a^i$ or $x^i = \min_{PD} X^i$.

We say that $x^N \in X^N$ is *feasible* for Problem 1 if $u^i(x^N) > a^i$ or $x^i = \min_{\succ^i} X^i$ for all $i \in N$. Further it is *strongly feasible* for Problem 1 if $u^i(x^N) > a^i$ for all $i \in N$.

Claim 1 (Masuzawa 2008) If y^N satisfies the constraint of Problem 1 and $u^i(y^N) > a^i$ for some $i \in N$, then $u^i(x^N) > a^i$, where x^N is the maximum of Problem 1

From Claim 1, $y^N \in X^N$ is an α-core iff for the maximum x^N of Problem 1 for $a^N := u^N(y^N)$, $T := \{i | u^i(x^N) > a^i\} \neq \emptyset$. Thus, any algorithm to solve Problem 1 determines the α-core strategies of G with the punishment-dominance condition.

Algorithm 1 (Masuzawa 2008)

1. $x^i := \max_{PD}(X^i)$ for all $i \in N$;
2. $S := \{i | x^i \neq \min_{PD}(X^i) \text{ and } u^i(x^N) \leq a^i\}$;
3. if $S = \emptyset$ then stop;
4. $x^i := \mathit{down}(x^i)$ for all $i \in S$;
5. Jump to 2.

Proposition 1 (Masuzawa 2008) Algorithm 1 terminates with the solution x^{*N} of Problem 1.

Proof: Since x^i is monotonically decreasing, Algorithm 1 necessarily terminates. Let x^{*N} be feasible. If $x^i = x^{*i}$ and $x^N \geq x^{*N}$, then $u^i(x^N) \geq u^i(x^{*N})$, and thus $i \notin S$. It follows that $x^N \geq x^{*N}$ at any time during the computation.□

Weak inequality If we replace the constraint of Problem 1 by

$$\text{for all } i \in N,\ u^i(x^N) \geq a^i \quad \text{or} \quad x^i = \min_{PD} X^i$$

then, we can recover the exactness of Algorithm 1 by replacing Step 2 of Algorithm 1 by $S := \{i | x^i \neq \min_{PD}(X^i) \text{ and } u^i(x^N) < a^i\}$.

Acceleration Note that we can replace Step 4 by

4′. $x^i := \mathit{down}(x^i)$ for some $i \in S$;

since this change preserves the property that $x^N \geq x^{*N}$ at any time during the computation if x^{*N} is feasible. Further, by similar discussion, Step 4 can be replaced by the following:

$$4''. \text{ for all } i \in S \textbf{ while } u(x^i, x^{N\setminus\{i\}}) \leq a^i \text{ and } x^i \neq \min_{PD} X^i$$
$$\textbf{begin } x^i := \mathit{down}(x^i) \textbf{ end };$$

Sometimes, the replacement accelerates the computation.

2.2 Matching model

Following Hatfield and Milgrom (2005), we formulate a generic model of many-to-one matching market. Let (W, F, C, f, w) be a bipartite graph such that $(w, f) : C \to W \times F$, where $(w(c), f(c))$ is a pair of vertexes of edge c. An element $i \in W$ is a worker, and $j \in F$ is a firm. An element $c \in C$ denotes a possible contract between $w(c)$ and $f(c)$. For all $i \in W$, all $j \in F$, and all $D \subset C$, we define D^i and D_j by $D^i := \{c \in D | w(c) = i\}$ and $D_j := \{c \in D | f(c) = j\}$.

Any firm $j \in F$ has a choice correspondence, $Ch_j : 2^{C_j} \to 2^{C_j}$, such that $Ch_j(D) \subset D$ for all $D \subset C_j$. We assume the *substitute property*, which was introduced into matching theory by Roth (1984b) and further developed by Roth and Sotomayor (1990).

Assumption 1 (Substitute property)
If $c \in Ch_j(D)$ and $D' \subset D$, then $c \in Ch_j(D' \cup \{c\})$.

In other words, any alternative once chosen is also chosen if it remains available even after other alternatives become unavailable. Further, we assume the *independence of irrelevant alternatives*:

Assumption 2 (IIA: Independence of irrelevant alternatives)
If $Ch_j(D) \subset D' \subset D$, then $Ch_j(D) = Ch_j(D')$.

On the other hand, any worker has a linear order $\succsim^i$ on $C^i \cup \{\phi^i\}$, called preference, the strict part of which is denoted by $\succ^i$. The singular element ϕ^i denotes the state in which i does not have any contract. Note that we mean by ϕ^i the unemployment state of i in strict sense rather than the situation in which i has no contracts with any firm but engages in housework. To describe the state of housework, one has only to introduce an imaginary firm

j_i^* and an imputed contract c_i^* such that $Ch_{j_i^*}(C_{j_i^*}) := \{c_i^*\}$ and $w(c_i^*) = i$. Thus, without loss of generality, we assume the following:

Assumption 3 (Housework condition)
For all $i \in W$, there exists $j \in F$ such that $i \in Ch_j(D \cup \{i\})$ for all $D \subset C_j$.

Assumption 4 (Unemployment condition)
For all $i \in W$ and all $c \in C^i$, $c \succ^i \phi^i$.

Then, we can state the definition of stability.

Definition 2 (Feasibility and stability) A set of contracts $D \subset C$ is *feasible* iff for all $i \in W$, $|D^i| \leq 1$. $D \subset C$ is *individually stable* iff (i) for all $i \in W$, $|D^i| = 1$ and (ii) for all $j \in F$, $Ch_j(D_j) = D_j$. $D \subset C$ is *pairwise* stable iff there exists no $c \in C \setminus D$ such that $c \succ^{w(c)} c'$ for $c' \in D^{w(c)}$ and $c \in Ch_{f(c)}\left(D_{f(c)} \cup \{c\}\right)$. $D \subset C$ is *stable* iff it is individually and pairwise stable.

3 Results

3.1 Construction of the canonical strategic form game

We define a strategic form game to analyze stable sets of contracts, called *the canonical strategic form game*. The set of players of the canonical strategic form game G is $N := W$ rather than $F \cup W$. The set of strategies X^i of player i is $X^i := C^i$. An element $x^i \in C^i$ indicates *the admissible set of contracts over x^i for i*, which is defined by

$$A^i(x^i) := \{c \in C^i | c \succsim^i x^i\}.$$

In other words, a player $i \in N$ applies for the top k contracts for some integer k. For all $x^N \in \prod_{i \in N} X^i$, the *aggregate admissible set* is defined by

$$A(x^N) := \cup_{i \in N} A^i(x^i).$$

The *available set* $B_j(x^N)$ and the *demand set* $d_j(x^N)$ of $j \in F$ under $x^N \in X^N$ are defined by

$$B_j(x^N) := \{c \in A(x^N) | f(c) = j\},$$
$$d_j(x^N) := Ch_j(B_j(x^N)).$$

The *aggregate demand* is defined by

$$d(x^N) = \cup_{j \in F} d_j(x^N).$$

The employer of worker i is determined by

$$e^i(x^N) = \begin{cases} \max_{\succ^i}\{c \in d(x^N) | w(c) = i\} & \text{if } \{c \in d(x^N) | w(c) = i\} \neq \emptyset, \\ \phi^i & \text{otherwise.} \end{cases}$$

In other words, any worker $i \in W$ is assigned to c such that (i) c is demanded by some firm, and (ii) i prefers c to any other such $c' \in C$.

The payoff of $i \in N$ in G is defined by $v^i(e^i(x^N))$, where v^i is a utility function that represents $\succ^i$:

$$u^i(x^N) := v^i(e^i(x^N)).$$

Note that game G describes a realistic situation in which workers apply to firms and the firms decide whether to hire them.

3.2 Properties and algorithms

First, by the substitute property, we can easily see that $\succsim^i$ is a punishment-dominance relation.

Proposition 2 (Completeness of punishment-dominance relation) For all $i \in N$, all $x^i, y^i \in C^i$, all $j \in N \setminus \{i\}$, and all $x^{N\setminus\{i\}} \in \prod_{j' \in N \setminus \{i\}} X^{j'}$,

$$u^j(x^{N\setminus\{i\}}, x^i) \geq u^j(x^{N\setminus\{i\}}, y^i) \text{ if } x^i \succsim^i y^i.$$

To see this, note that any increase of $c^i \in C^i$ does not decrease the set $d(x^N) \setminus C^i$, and thus does not decrease the set $\{c \in d(x^N) | w(c) = k\}$ for any other worker k.

Second, the stable set of contracts is characterized as follows.

Proposition 3 The set of contract $D := \{x^i | i \in N\}$ is stable if and only if for all $i \in N$,

$$x^i \succsim^i e^i(x^N) \text{ and } e^i(x^N) \neq \phi^i.$$

Proof. The if-part: $d_j(x^N) = D_j$ since $x^i \succsim^i e^i(x^N) \neq \phi^i$ for all $i \in N$. By IIA, $Ch_j(D_j) = D_j$. Thus, individual stability holds. To see the pairwise stability, assume that $c \succ_{w(c)} x^i$ for some i. Then, $c \in B_j(x^N)$ for some $j \in F$. By IIA, $Ch_j(D_j \cup \{c\}) = D_j$, and thus, $c \notin Ch_j(D_j \cup \{c\})$.
The only-if-part: Assume that D is stable. Let i be any worker and define $c^* := e^i(x^N)$. First, consider the case such that $c^* \neq \phi^i$, and define $j^* := f(c^*)$. Since $D_{j^*} \subset B_{j^*}(x^N)$ and $c^* \in Ch_{j^*}(B_{j^*}(x^N))$, by the substitute property $c^* \in Ch_{j^*}(D_{j^*} \cup \{c^*\})$. By the pairwise stability, $x^i \succsim^i c^*$. Since i is arbitrarily chosen, for all $i' \in N$, $x^{i'} \succsim^{i'} e^{i'}(x^N)$. Thus, for all $j \in F$, $d_j(x^N) = Ch_j(D_j)$. Suppose to the contrary that $c^{i^*} = \phi^{i^*}$. Then, by definition, $x^{i^*} \in D$ but $x^{i^*} \notin d^j(x^N) = Ch_j(D_j)$ for all $j \in F$, which contradicts the definition of individual stability. □

On the other hand, the following problem characterizes the maximum of the stable set of contracts.

Problem 2

Input: the canonical strategic form game of a two-sided matching market.

Output: the maximum of x^N such that for all $i \in N$,

$$e^i(x^N) \succ^i \phi^i \text{ or } x^i = \min_{\succ^i} C^i.$$

Claim 2 Any stable set of contracts satisfies the constraint of Problem 2.

Under the housework condition, we obtain the following:

Claim 3 (Strong feasibility) There exists $y^N \in X^N$ such that $e^i(y^N) \neq \phi^i$ for all $i \in N$.

Claim 4 If $x^N \in X^N$ is the maximum of Problem 2, then $e^i(y^N) \neq \phi^i$ for all $i \in N$.

It then follows that:

Proposition 4 If $x^N \in X^N$ is the maximum of Problem 2, then $D := \{x^i | i \in N\}$ is stable.

Proof. Suppose to the contrary that $e^i(x^N) \succ^i x^i$ for some $j \in N$. By the substitute property,

$$e^i(x^{N\setminus\{i\}}, up(x^i)) \succsim^i e^i(x^{N\setminus\{i\}}, x^i) \succsim^i up(x^i) \succ^i \phi^i,$$

and by the punishment-dominance condition,

$$e^j(x^{N\setminus\{i\}}, up(x^i)) \succsim^j e^j(x^{N\setminus\{i\}}, x^i).$$

Thus, $\left(x^{N\setminus\{i\}}, up(x^i)\right)$ also satisfies the constraint, which contradicts the definition of the maximum. □

It follows that the maximum stable set of contracts can be obtained by Algorithm 1. This algorithm turns out to be the deferred acceptance algorithm of the worker's offer by Gale and Shapley (1962).

Algorithm 2 (Deferred acceptance algorithm)

1. $x^i := \max_{\succ^i} C^i$ for all $i \in N$;
2. $S := \{i | x^i \neq \min_{PD}(X^i)$ and $e^i(x^N) = \phi^i\}$;
3. if $S = \emptyset$ then stop;
4. $x^i := down(x^i)$ for all $i \in S$;
5. Jump to 2.

To discuss the minimum stable set of contracts, consider the following problem:

Problem 3

Input: the canonical strategic form game of a two-sided matching market.

Output: the minimum of x^N such that for all $i \in N$,

$$x^i \succsim^i e^i(x^N) \text{ or } x^i = \max_{\succ^i} X^i.$$

First, any stable set of contracts satisfies the constraint of Problem 3. Since G satisfies the punishment-dominance condition, the game G^* defined by

$$u^{*i}(x^N) := u^i(x^N) - x^i$$

also satisfies the punishment-dominance condition.

Proposition 5 If $x^N \in X^N$ is the minimum of Problem 3, then $D := \{x^i | i \in N\}$ is stable.

Proof. Let x^N be the minimum. If $x^i = \min_{\succ^i} X^i$, then, by the housework condition and the strong feasibility, $e^i(x^{N\setminus\{i\}}, x^i) \neq \phi^i$. In case of $x^i \neq \min_{\succ^i} X^i$, suppose to the contrary that $e^i(x^N) = \phi^i$. Then by the substitute property,

$$e^i(x^{N\setminus\{i\}}, down(x^i)) = \phi^i \text{ or } down(x^i).$$

Then, $(x^{N\setminus\{i\}}, down(x^i))$ also satisfies the constraint, which contradicts the definition of the minimum.□

Thus the minimum stable set of contracts can be also solved by Algorithm 1. One can easily see that it is equivalent to the National Intern Matching Program (NIMP) algorithm reported in Roth (1984a). Note that Step 4 is a result of acceleration.

Algorithm 3 (NIMP algorithm)

1. $x^i := \min_{\succ^i} C^i$ for all $i \in N$;
2. $S := \{i | x^i \neq \max_{\succ^i}(X^i) \text{ and } e^i(x^N) \succ^i x^i\}$;
3. if $S = \emptyset$ then stop;
4. $x^i := e^i(x^N)$ for all $i \in S$;
5. Jump to 2.

3.3 Monotonicity and Tarski's fixed point theorem

By Proposition 3, we can directly derive another characterization of the stable set of contracts in terms of the fixed point:

Claim 5 The set of contracts $D := \{x^i | i \in N\}$ is stable if and only if

$$\text{for all } i \in N, \;\; x^i = e^i(x^N)$$

A characterization of the stable two-sided matching by a fixed point of Tarski's monotonic function was first discussed by Adachi (2000). Hatfield and Milgrom (2005) used this method to derive generic results, while their model is not a strategic form game. Note that Tarski's fixed-point theorem is, however, not directly applicable to our model since e^i is not monotonic. To make the payoff function monotonic in the sense of Tarski, we define e^i_* as follows:

$$e^i_*(x^N) := \max_{\succ^i}\{c | c \in d(x^{N\setminus\{i\}}, y^i), y^i \in X^i, x^i \succsim^i y^i\}$$

Note that by the housework condition, e^i_* is well defined and $e^i_*(x^N) \neq \phi^i$ for all $x^N \in X^N$.

Claim 6 The following three statements are equivalent to each other:

$$1. e^i(x^N) \neq \phi^i, \qquad 2. e^i(x^N) = e^i_*(x^N), \qquad 3. e^i_*(x^N) \succsim^i x^i.$$

Claim 7 x^N is a fixed point of $(e^i)_{i\in N}$ iff it is a fixed point of $(e^i_*)_{i\in N}$.

By similar discussion, we have the following:

Proposition 6 (monotonicity) Under the substitute property, if $x^i \succsim^i y^i$ for all $i \in N$, then $e^j_*(x^N) \succsim^j e^j_*(y^N)$ for all $j \in N$.

Thus, by Tarski's fixed-point theorem, we derive the following theorem:

Proposition 7 The set of stable set of contracts is a nonempty complete lattice with respect to $(\succsim^i)_{i\in W}$.

4 Concluding Remarks

4.1 Discrete convex analysis

Murota (1996) initiated the theory of *discrete convex analysis*. He has proposed two notions of convexity, L-convexity and M-convexity, and define discrete convex functions: L-convex, $L^\natural$-convex, M-convex, and $M^\natural$-convex functions. In two-sided matching theory, Fujishige and Yang (2003) have proven the equivalence between *the gross substitute property* (Kelso and Clawford, 1982) of the demand function and the $M^\natural$-concavity of the utility function of the firms. A function with the substitute property is, however, not necessarily $M^\natural$-concave. This seems to be analogous to the fact that the convexity of TU game is independent of the punishment-dominant condition regardless of the striking properties of the condition (Masuzawa, 2004).

4.2 Single transferable voting

Recently, the single transferable voting with feedback counting has been analyzed by the punishment-dominance condition (Masuzawa 2012b). Obviously, the single transferable voting is a kind of many-to-one two-sided matching, while it has not ever been pointed out to my best knowledge. A voter is corresponding to a worker and has a preference over candidates, who are corresponding to firms. Meek (1969, 1970) initiated a method to transfer surplus of votes even to elected candidates, which is analogous to the deferred acceptance algorithm. Given a set of candidates and voters, one can construct a corresponding two-sided matching market, although it is out of the scope of this article.

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