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Abstract

We show that the dominant punishment strategy and coalitionally dominant strategy applied to TU coalitional strategic games give directly the saddle point expression for the coalitional aggregate payoff function, which is the condition for the equivalence of the strategic cores α and β. Also, we find that the existence of a saddle point together with the concavity of payoff functions is not sufficient for nonemptiness of the β-core.

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A Note on the α- and β-Cores in TU Coalitional Strategic Games

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In this note, we point out that the *dominant punishment strategy* (Nakayama [4]) and the *coalitionally dominant strategy* (Hirai et al. [2]) generate the equivalence of the cores α and β in TU coalitional strategic games. These strategies give directly the saddle point expression for the coalitional aggregate payoff function, which is the condition for the equivalence of the two cores. However, we find that the existence of a saddle point together with the concavity of payoff functions is *not* sufficient for nonemptiness of the TU β-core.

Let (N, v) be a TU coalitional game with $N = \{1, \ldots, n\}$ the set of players and v the real valued function on 2^N, called the *characteristic function*, satisfying $v(\emptyset) = 0$. Any nonempty subset $S \subseteq N$ is called a coalition, and $v(S)$ is usually assumed to describe the worth of S that S can obtain by itself.

The core $C(v)$ of game v is the set of payoff vectors given by

$$C(v) = \left\{\mu \in \Re^n \mid \sum_{i \in S} \mu_i \geq v(S) \quad \forall S \subseteq N, \quad \sum_{i \in N} \mu_i \leq v(N)\right\}$$

Now let $G = (N, (X^i, u_i)_{i \in N})$ be a game in strategic form, where for each $i \in N$, X^i is the strategy set and u_i is the transferable payoff function defined on $X := \Pi_{i \in N} X^i$. It will be assumed that for each $i \in N$, X^i is nonempty, compact and convex and u_i is continuous. For any nonempty subset $R \subseteq N$, let $X^R := \Pi_{i \in R} X^i$. Then the two TU characteristic functions v_α and v_β can be given as follows.

$$v_\alpha(S) = \max_{x^S \in X^S} \min_{x^{N \setminus S} \in X^{N \setminus S}} \sum_{i \in S} u_i(x) \quad \forall S \subseteq N \tag{1}$$

$$v_\beta(S) = \min_{x^{N \setminus S} \in X^{N \setminus S}} \max_{x^S \in X^S} \sum_{i \in S} u_i(x) \quad \forall S \subseteq N \tag{2}$$

Accordingly, $C(v_\alpha)$ is called the TU α- core, and $C(v_\beta)$, the TU β- core. Note that we always have $C(v_\beta) \subseteq C(v_\alpha)$. The following lemma, adapted from a well known result in two-person zero-sum games, is fundamental.

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Lemma 1. *$C(v_\alpha) = C(v_\beta)$ if and only if for any $S \subsetneq N$ with $S \neq \emptyset$ there exists a saddle point $(x^{*S}, x^{*N\setminus S}) \in X^S \times X^{N\setminus S}$ of $\sum_{i\in S} u_i(x)$; namely,*

$$\sum_{i\in S} u_i(x^S, x^{*N\setminus S}) \leq \sum_{i\in S} u_i(x^{*S}, x^{*N\setminus S}) \leq \sum_{i\in S} u_i(x^{*S}, x^{N\setminus S})$$
$$\forall x^S \in X^S,\ \forall x^{N\setminus S} \in X^{N\setminus S}.$$

Proof. The existence of the saddle point $(x^{*S}, x^{*N\setminus S}) \in X^S \times X^{N\setminus S}$ of the function $\sum_{i\in S} u_i(x)$ is equivalent to obtaining the equality

$$\max_{x^S\in X^S} \min_{x^{N\setminus S}\in X^{N\setminus S}} \sum_{i\in S} u_i(x^S, x^{N\setminus S}) = \min_{x^{N\setminus S}\in X^{N\setminus S}} \max_{x^S\in X^S} \sum_{i\in S} u_i(x^S, x^{N\setminus S}),$$

which by definition is equivalent to the equality $C(v_\alpha) = C(v_\beta)$. □

Fot the existence of a saddle point, we might resort to the theorem due to Sion [7] stating a sufficient condition that $\sum_{i\in S} u_i(x^S, x^{N\setminus S})$ be *both* quasi-concave in x^S and quasi-convex in $x^{N\setminus S}$. But, assuming this for all nonempty coalitions $S \subseteq N$ would considerably restrict the class of TU coalitional strategic games. Thus, we should instead seek conditions that would be compatible with ones in a more general class of games.

Zhao [8] has shown that if a game is *strongly separable* then the TU β-core coincides with a nonempty TU α-core under the concavity of payoff functions. The game is *strongly separable* if for each $S \subseteq N$, the strategy of $N \setminus S$ at the profile yielding the aggregate minimax payoff $v_\beta(S)$ is *simultaneously* yielding the minimum payoff to each member i of S separately. Namely, for each $S \subseteq N$,

$$v_\beta(S) = \sum_{i\in S} u_i(x^{*S}, x^{*N\setminus S}) \quad \text{implies that}$$
$$u_i(x^{*S}, x^{*N\setminus S}) \leq u_i(x^{*S}, x^{N\setminus S}) \quad \forall x^{N\setminus S} \in X^{N\setminus S} \qquad \forall i \in S.$$

It then immediately follows that the function $\sum_{i\in S} u_i(x)$ has a saddle point, since

$$\sum_{i\in S} u_i(x^S, x^{*N\setminus S}) \leq \sum_{i\in S} u_i(x^{*S}, x^{*N\setminus S}) \leq \sum_{i\in S} u_i(x^{*S}, x^{N\setminus S}) \quad \forall x \in X \tag{3}$$

which by our Lemma 1 yields the equivalence of $C(v_\alpha)$ and $C(v_\beta)$.

In the NTU context, we have shown that a strategy called the *dominant punishment strategy*, defined below, yields the same equivalence of cores in NTU games (Nakayama [4], see also Masuzawa [3]).

Definition 1. *Let $\emptyset \neq N \setminus S \subsetneq N$ and let $z^{N\setminus S} \in X^{N\setminus S}$. Then, $z^{N\setminus S}$ is called a dominant punishment strategy of $N \setminus S$ against S if*

$$u_i(x^S, z^{N\setminus S}) \leq u_i(x) \quad \forall x \in X \ \text{and} \ \forall i \in S \tag{4}$$

Among all strategies of $N \setminus S$ the dominant punishment strategy against S puts every member of S in the worst state irrespective of any strategy of S. Thus, it is obvious that the existence of a dominant punishment strategy $z^{N\setminus S} \in X^{N\setminus S}$ implies the strong separability for coalition S. Therefore, the dominant punishment strategy makes the two cores equivalent *also* in the TU context.

In generel, there will be other possibilities to obtain the saddle point property, among which is a strategy called the *coalitionally dominant strategy* defined by Hirai et al. [2] in the NTU context. A strategy of nonempty proper coalition $S \subsetneq N$ is *coalitionally dominant* if it Pareto dominates any other strategy of S irrespective of any strategy profile of $N \setminus S$. Namely,

Definition 2. *Let $\emptyset \neq S \subsetneq N$ and $y^S \in X^S$. Then, y^S is called the coalitionally dominant strategy of S if*

$$u_i(y^S, x^{N\setminus S}) \geq u_i(x) \quad \forall x \in X \quad and \quad \forall i \in S. \tag{5}$$

The coalitionally dominant strategy also makes the β-core equivalent to the α-core in NTU games, which is noted in Hirai et al. [2]. The coalitionally dominant strategy is, in a sense, *dual* to the dominant punishment strategy, but may be hard to obtain in general. Nevertheless, there is a natural example of the strategy in a game describing pure exchange of bads; and moreover, the β-core of this example is nonempty without the quasi-concavity of payoff functions (Hirai et al. [2]).

If $y^S \in X^S$ is the coalitionally dominant strategy of nonempty proper coalition $S \subsetneq N$, then clearly

$$\sum_{i\in S} u_i(x) \leq \sum_{i\in S} u_i(y^S, x^{N\setminus S}) \quad \forall x \in X. \tag{6}$$

Thus, these two strategies defined in NTU coalitional strategic games make the two cores equivalent also in the TU coalitional strategic games.

Proposition 1. *For each S with $\emptyset \neq S \subsetneq N$, assume that either $N \setminus S$ has a dominant punishment strategy against S, or S has a coalitionally dominant strategy. Then, we have that*

$$C(v_\alpha) = C(v_\beta).$$

Proof. The case with a dominant punishment strategy follows from (3). Suppose, next, that S has a coalitionally dominant strategy $d^S \in X^S$. Then, in view of (6), d^S is a best reply of S against any $x^{N\setminus S} \in X^{N\setminus S}$. Hence, letting $z^{N\setminus S}$ to be the minimizer of $\sum_{i\in S} u_i(d^S, \cdot)$, we see that the pair $(d^S, z^{N\setminus S})$ is a saddle point of $\sum_{i\in S} u_i(x)$. □

A typical example of a game with dominant punishment strategies is the pure exchange game defined in Scarf [5] and its TU version. Under the monotonicity of payoff functions, offering no goods to coalition S is a dominant punishment strategy of $N\setminus S$. The Cournot game considered in Zhao [8] is also one of the examples due to the decreasing inverse demand function, allowing maximum damaging production by $N \setminus S$ irrespective of the action of S.

The dominant punishment strategy is originally of an NTU concept, and the strong separability is also a condition imposed on each individual player *separately*. It appears therefore somewhat irrelevant or literally *too* strong to assume these strategies in TU games in which coalitions are concerned with aggregate payoffs by making unlimited side payments.

For any nonempty proper coalition $N \setminus S \subsetneq N$, we may now call the strategy $z^{N\setminus S} \in X^{N\setminus S}$ the *TU dominant punishment strategy* of $N \setminus S$ iff the aggregate version of (4) holds; namely,

$$\sum_{i\in S} u_i(x^S, z^{N\setminus S}) \leq \sum_{i\in S} u_i(x) \quad \forall x \in X.$$

Similarly $y^S \in X^S$ is the *TU coalitionally dominant strategy* iff it generates the inequality (6). Then, it will be natural and agree with the TU framework if these TU strategies, the TU dominant punishment strategy in particular, and concave payoff functions together lead to the nonempty β-core, just as in the NTU case where the dominant punishment strategy together with quasiconcave payoff functions is sufficient for the nonempty β-core (Nakayama [4]). But this idea will turn out to be unachievable in view of the fact that the *garbage game* due to Shapley and Shubik [6] has an empty core.

Consider the TU strategic pure exchange game of bads $G = \left(N, (X^i, u_i)_{i\in N}\right)$ defined by

$$X^i = \left\{ x^i = (x^{i1}, \ldots, x^{in}) \in \Re^n_+ \;\middle|\; \sum_{j\in N} x^{ij} = w^i \in \Re_{++} \right\} \quad \forall i \in N,$$

$$u_i(x) = -\sum_{j\in N} x^{ji} \quad \forall i \in N.$$

The TU coalitional game (N, v_α) derived from G is

$$v_\alpha(S) = \max_{x^S} \min_{x^{N\setminus S}} \Big(-\sum_{i\in S}\sum_{j\in N} x^{ji} \Big) = \begin{cases} -\sum_{i\in N\setminus S} w^i & \text{if } S \subsetneq N, \\ -\sum_{i\in N} w^i & \text{if } S = N. \end{cases}$$

Then, letting $w^i = 1$ for all $i \in N$, we have the Shapley-Shubik garbage game, the core of which is empty if $n \geq 3$; that is, $\emptyset = C(v_\alpha) \supseteq C(v_\beta) = \emptyset$.

Note that the strategic game G has a compact convex set X^i of strategies and a concave payoff function u_i over $X = \prod_{i\in N} X^i$ for each $i \in N$. It is obvious that any strategy $y^S \in X^S$ of a nonempty $S \subsetneq N$ satisfying

$$\sum_{i\in S}\sum_{j\in N\setminus S} y^{ij} = \sum_{i\in S} w^i$$

is coalitionally dominant in TU and NTU contexts.

On the other hand, any strategy $z^{N\setminus S} \in X^{N\setminus S}$ of $N \setminus S$ satisfying

$$\sum_{i\in S}\sum_{j\in N\setminus S} z^{ji} = \sum_{j\in N\setminus S} w^j$$

is a TU dominant punishment strategy of $N \setminus S$ against S. However, *none* of these strategies $z^{N\setminus S} \in X^{N\setminus S}$ is the original NTU dominant punishment strategy, nor the one that makes the game strongly separable.

This example indicates that as far as the nonemptiness of TU β-cores is concerned, the TU dominant punishment strategy is not sufficient. Thus, the NTU dominant punishment strategy cannot be weakened to the TU dominant punishment strategy for the nonemptiness of TU β-cores. This example can well be contrasted with the NTU version of the game having a nonempty β-core (Hirai et al. [2]).

Finally, as in Zhao [8], Hirai et al. [2], and in more recent Harada and Nakayama [1], the nonemptiness of the β-core in TU and NTU games appears generally to entail the equality to the α-core. Since the β-core is a subset of the α-core by definition, it would be interesting if we could find economic games, in TU or NTU, with a nonempty β-core not identical with the α-core.

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