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A Global Dynamics of Financial Integration under Capital Market
Imperfection -Delaying Financial Liberalization -

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We investigate the global dynamics of financial integration by constructing a two-country model of overlapping generations in the presence of financial market frictions. The inverted U-shaped relation between the interest rate and the capital stock brought about by imperfect enforcement becomes the source of the destabilizing behavior of the world financial market. We characterize the global dynamics that allow us to answer when and how the globalization magnifies and lessens symmetry-breaking. We observe symmetry-breaking accompanied by continued capital flows from the poor to the rich when there are stable asymmetric steady states. However, when the stable symmetric steady state coexists with asymmetric ones, we observe either convergence or symmetry-breaking, depending on developing stages when financial integration occurs. There are threshold levels of wealth only above which financial integration leads to convergence. Even when there is the unique stable steady state, we observe the short-run symmetry-breaking accompanied by the endogenous reversal of capital flows. We have cases when early financial integration leads to symmetry-breaking, but late one attains convergence. Emerging market countries can go on the convergent path by delaying the timing of opening capital accounts until they have reached some threshold level of wealth.

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A Global Dynamics of Financial Integration under Capital Market Imperfection -Delaying Financial Liberalization - *

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We investigate the global dynamics of financial integration by constructing a two-country model of overlapping generations in the presence of financial market frictions. The inversed U-shaped relation between the interest rate and the capital stock brought about by imperfect enforcement becomes the source of the destabilizing behavior of the world financial market. We characterize the global dynamics that allow us to answer when and how the globalization magnifies and lessens symmetry-breaking. We observe symmetry-breaking accompanied by continued capital flows from the poor to the rich when there are stable asymmetric steady states. However, when the stable symmetric steady state coexists with asymmetric ones, we observe either convergence or symmetry-breaking, depending on developing stages when financial integration occurs. There are threshold levels of wealth only above which financial integration leads to convergence. Even when there is the unique stable steady state, we observe the short-run symmetry-breaking accompanied by the endogenous reversal of capital flows. We have cases when early financial integration leads to symmetry-breaking, but late one attains convergence. Emerging market countries can go on the convergent path by delaying the timing of opening capital accounts until they have reached some threshold level of wealth.

Keywords: Financial integration, global dynamics, imperfect enforcement, inequality

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1. Introduction

Globalization in financial markets should promise greater prosperities through the channels of improved allocation of resources in the world. How global is the world economy in reality? The volume of capital flows is remarkably increasing within this decade, but capital still seems to flow from poor to rich countries (e.g., Prasad et al, 2007), which should be contrary to the prediction of the conventional theory.

Much literature has addressed that global financial markets are far from complete, and attempted to answer the question why capital does not flow from rich to poor countries, and sometimes even flow from poor to rich countries. The empirical literature includes Aizenman (2004), Reinhart and Rogoff (2004), Portes and Rey (2005), Alfaro et al (2004) (2008), and Braun and Raddatz (2007), Prasad et al (2007), and Gourinchas and Jeanne (2007).

At the same time, since the seminal work by Gertler and Rogoff (1990), a number of theoretical works have developed theoretical foundations that take the agency approach to corporate finance so as to explain perverse patterns of capital flows. The agency consideration emphasizes the role of financial market frictions that arise from asymmetric information and/or the immature institutional development as obstacles to smooth cross-border capital flows. Unfortunately, much of the literature predicts miserable consequences of financial integration. Models of the small-open-economy show that opening the capital account leads to the flight of capital and/or the crash of domestic financial market for developing countries (e.g., Caballero and Krishnamurty (2001) (2006), Aghion et al (2004), and Aoki, Benigno and Kiyotaki (2007), and others). Multi-country models explain the mechanism behind which financial integration gives rise to “symmetry-breaking” or “divergence” between countries, accompanied by the perverse pattern of capital flows (e.g., Boyd and Smith (1997), Sakuragawa and Hamada (2001), and Matsuyama (2004), and others). Policymakers and economic researchers might be pessimistic to the consequences of capital account liberalization on economic development and growth.

Nevertheless, how to progress the international financial architecture is a subject of
great importance so as to promote the allocation of capital internationally. Actually, capital flows smoothly among high-income countries. Even among middle-income countries, some East Asian countries, Japan, Korea, and Taiwan, historically provide interesting stories that may contradict with theoretical predictions of that literature. Despite of poorly developed financial markets, these countries attained miraculous growth. One of strategies for successful development for these countries was to delay the period of lifting capital control. Some economists advocate the importance of the gradual and/or late liberalization. McKinnon (1991) addressed the sequence of liberalization and objected to the early liberalization of capital accounts. Rodrik (2007) discusses negative aspects of globalization and recommend the gradual integration. Some empirical works, such as Bekaert et al, (2001) and Chinn and Ito (2006), also provide evidence suggesting the favor for the gradual and late liberalization.

The notion of timing for financial liberalization provides a hint for rethinking when and how financial integration should be conducted. Whether financial integration will eventually make every country better off? If so, is there a choice of either early or late integration so as to attain the successful goal? What determines the period for fast and successful financial integration from the prospect of the world economy? In order to reach our goal, we have to provide a model of the world economy that takes into account global dynamics.

In this paper we construct a two-country model of overlapping generations with two-period-lived agents when frictions arise in response to agency problems in financial markets. Our model is a version of Boyd and Smith (1997) and Matsuyama (2004) that exploit implications of financial integration on the eventual wealth and inequality. One distinguishable feature of our model from theirs is that entrepreneurs who need borrowing from others have no initial net worth for financing productive investment but can pledge up the future income (wage income in the model) as collateral. The forward-looking nature of collateral gives rise to the non-monotone, inversed U-shaped relation between the interest rate and the capital stock, which in turn generates the investment/wealth feedback followed by the destabilizing behavior of the world
financial market.

The small change in the setup drastically simplifies the dynamic analysis while preserving general features of financial market imperfections. The dynamical system is reduced to a one-dimensional map of the capital stock of either one country that allows us to characterize the global dynamics. Our model is one of the simplest versions that explain the financial accelerator mechanism that is brought about by the investment/collateral feedback (e.g., Kiyotaki and Moore, 1997).

We characterize equilibria globally so that we can describe the equilibrium path and the pattern of capital flows over time. We investigate when and under what conditions the globalization magnifies or lessens symmetry-breaking. The global dynamics exploits a number of interesting properties, some of which are not predictable from the local analysis. First, we trace out the story of symmetry-breaking over time when there are stable asymmetric steady states. This case describes the dynamic mechanism under which, when financial integration occurs, the investment/wealth feedback works to destabilize the world financial market accompanied by the continued capital flows from the poor to the rich, which in turn, makes the rich richer and the poor poorer.

Second, we show that symmetry-breaking does not necessarily occur, even when there are stable asymmetric steady states. When the stable symmetric steady state also coexists, either convergence or symmetry-breaking occurs, depending on the state when financial integration occurs. There are threshold levels of wealth only above which financial integration leads to convergence. This case could explain the evidence why capital flows smoothly among high-income countries but not for low-income countries. Policy implications are also striking. When developing and developed economies coexist, early financial integration leads to symmetry-breaking, but late one attains convergence. The poor can go on the convergent path by delaying the timing of opening capital accounts until having reached some threshold level of wealth.

Third, we show that when the steady state is unique, stable, and symmetric, convergence occurs surely, but symmetry-breaking occurs in the short run, accompanied by the endogenous reversal of capital flows. The poor first becomes poorer, with the
flight of capital, but later gains from the “trickle-down effect” from the rich that has exploits the gain of symmetry-breaking and comes to lend to the poor. Delaying financial integration avoids the short-run symmetry-breaking process and can lead to faster convergence.

This paper is closely related to the theoretical literature that explains the perverse pattern of capital flows by relying on the agency approach. The seminal work of Gertler and Rogoff (1990) demonstrated that, in their static model, the difference in borrowers’ ability to rely on external finance causes capital flows to go from the poor to the rich. Boyd and Smith (1997) developed the dynamic version of Gertler and Rogoff (1990), showed that financial integration gives rise to symmetric-breaking, even with identical technologies and institutional quality. Matsuyama (2004) developed a more tractable model than Boyd and Smith (1997), and provided conditions under which symmetric-breaking arises. The feedback between the borrowers’ wealth and the aggregate investment is a channel through which financial integration gives rise to symmetry-breaking, but at the same time makes the analysis so complicated that their scope turned out to be limited to the local analysis. Sakuragawa and Hamada (2001) are an exception that traces out the analysis globally, and show the capital flows from the poor to the rich in a two–country model with different institutions for contract enforcement, and with identical technological externalities in production.

This paper belongs also to the literature that studies international integration in terms of contract enforcement. Tornell and Velasco (1992) and Acemoglu and Zilibotti (1997) analyze perverse capital flows by focusing on the other aspects of the lack of enforcement than agency problem, such as poorly established property rights and incomplete risk sharing. Greif (1994) and Dixit (2003) investigate the effects of different internal enforcement systems on globalization. Tirole (2003) and Broner and Ventura (2007) provide mechanism in which the government’s enforceability to foreign

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1 Another research line has emphasized the cost of the governments’ time inconsistency problem associated with sovereign debt. The literature includes Bulow and Rogoff (1989), Eaton and Fernandes (1995), Tirole (2003). The corporate finance approach and the sovereign debt approach will be classified according to whether enforceability is exogenous or endogenous.
debts significantly affects the quality of the home financial markets and the performance of globalization.

This paper is organized as follows. Section 2 sets up the model. Section 3 characterizes the closed economy and Section 4 the world economy. Section 5 analyzes the dynamic properties of this model and provides policy implications for financial integration. Section 6 studies other related issues.

2. The Model

The model is based on an overlapping generation model that consists of two period lived agents, with no population growth. Time is discrete, and at each period \( t = 0, 1, \ldots \) a set of agents is born. Each generation consists of a continuum of agents of unit mass.

A single final good is produced using capital stock and labor as inputs. Let \( K_t \) denote the capital stock in period \( t \), and \( L_t \) denote the labor supply in period \( t \). Then, the output of the final good in period \( t \), \( Y_t \), is produced according to the production function given by \( Y_t = A_t F(K_t, L_t) \), where \( F(\cdot) \) exhibits the property of the CRTS production function and \( A_t \) is the TFP at period \( t \). We set \( A_t = 1 \) so long as unnecessary. We denote \( k_t = K_t / L_t \) and \( f(k_t) = F(k_t, 1) \), and assume that \( f(0) = 0 \), \( f'(k) > 0 > f''(k) \) and \( f'(0) = \infty \). It is also assumed that the factor markets are competitive. The firm’s profit maximizing leads to \( \rho_t = f'(k_t) \) and \( w_t = f(k_t) - k_t f'(k_t) \), where \( \rho_t \) is the rental rate on capital and \( w_t \) is the real wage rate. Let \( W(k_t) = f(k_t) - k_t f'(k_t) \), then \( f''(k) < 0 \) implies that \( W'(k_t) > 0 \) holds for all \( k > 0 \), and in addition we assume that

\[
(W')'(0) = \infty, \quad W''(k) < 0.
\]

For simplicity, capital is assumed to depreciate fully in one period.

Agents are risk neutral, and care only about old-age consumption. Within each generation, they are divided into two types, “investors” and “entrepreneurs”. A fraction \( \alpha \) \((0 < \alpha < 1)\) of agents are investors. Each of them is endowed with one unit of labor in the young age. Investors receive wages by supplying one unit of labor inelastically in the labor market in the young age, and lend all the earned income to others with a
(gross) interest rate $r_{t+1}$, which is endogenously determined in the model but exogenously given to agents.

The remaining fraction $1 - \alpha$ of agents are entrepreneurs. Each of them is endowed with one unit of labor not in the young age but in the old age, and in addition has access to a single indivisible investment project for converting one unit of the final good into $R$ units of capital goods after one period. Note that investment projects are not transferable among agents so that any one of them runs at most one project.

Entrepreneurs have no net worth for starting their investment and have to rely on outside finance for their projects. After having completed the production of capital goods, entrepreneurs rent the output and supply labor to the final-good firm.

Entrepreneurs receive the rental fee of capital good and wage from the output of the final good, and will repay their obligations to investors.

We assume an imperfect enforcement in the contractual arrangement made between creditors and debtors. Entrepreneurs can repudiate the obligation by hiding a fraction $1 - \lambda$ ($0 < \lambda \leq 1$) of the revenue generated from the investment project so that, in case of default, investors can seize a fraction $\lambda$ of the produced capital goods and all of the wage income.

We are now ready to look at the investment decision. If potential entrepreneurs start an investment project by financing one unit of the final good in the financial market with a (gross) interest rate $r_{t+1}$, their old-age consumption is equal to $\rho_{t+1} R + w_{t+1} - r_{t+1}$, while otherwise, their old-age consumption becomes $w_{t+1}$. Potential entrepreneurs are willing to produce capital if

$$R'f'(k_{t+1}) \geq r_{t+1}.$$  

We refer to this inequality as the **profitability constraint**. Even if (1) is satisfied, however, the enforcement problem may prevent entrepreneurs from financing their projects. If entrepreneurs repay their obligations honestly, they earn $R\rho_{t+1} - r_{t+1} + w_{t+1}$, while if they breach the contract, they would obtain $(1 - \lambda)R\rho_{t+1}$. Anticipating the entrepreneur’s strategic default, the investor supplies the fund only if the incentive compatibility is satisfied so that the entrepreneur can start the project only if
This inequality implies that the discounted pledgeable income of investors, 
\((\lambda R\rho_{t+1} + w_{t+1})/r_{t+1}\), should exceed unity, the amount required for starting the project. We shall thus call (2) the borrowing constraint. The parameter \(\lambda\) captures the country’s soundness of contract enforcement that will strongly influence the institutional development of the country’s financial system. Much literature has argued the importance of institutional quality as an important determinant of economic growth (e.g., North, 1981), Hall and Jones, 1999), Acemoglu et al, 2001 and others).2

The distinguishable feature of our model is that the future labor income is used as collateral. One might wonder if the assumption of full collateralization of wage income is restrictive, but our model will be one of the simplest versions that explain the financial accelerator mechanism that is brought about the investment/collateral feedback (e.g., Kiyotaki and Moore, 1997). When the value of collateral is attributed to human capital that is not transferrable across generations, financial accelerator mechanism is not accompanied by any asset price change.

Remark-1: We may mitigate the assumption of full collateralization of wage income in the following way. Suppose that in case of default, investors can seize a fraction \(\lambda\) of the produced capital good and a fraction \(\lambda_w(<1)\) of wage income. In addition, investors supply \(u_t\) units of labor in the young age, and entrepreneurs supply greater \(u_e(>u_t)\) units in the old age, probably as managers. As noted in footnote 7, general features are preserved also when \(\lambda_w < \lambda\).

Remark-2: We may provide the alternative specification. The final good is produced according to the CRTS technology that employs three inputs, capital, labor, and real estate. Real estate depreciates completely for one period. The rental fee of real estate becomes the increasing function of the capital stock under some plausible conditions of

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2 La Porta et al (1997, 1998), Levine (1998), and Beck et al(2000) have used indicators of legal protection for creditors and shareholders, and legal enforcement as instrumental variables for financial development so as to link economic and financial developments. La Porta et al (1997) find that countries with poor investor protections, measured by both the character of legal rules and the quality of law enforcement, have smaller capital markets. Levine (1998) shows that financial depth is closely linked with measures of legal treatment of outside creditors developed by theirs.
production. Entrepreneurs receive one unit of the final good when young that is transformed into one unit of real estate, a factory or a building, that is used as input for his/her own investment project and the production of the final good. In the second period of life, entrepreneurs rent the produced capital good and real estate to the final-good firm and receive the rental fees. While the aggregate capital stock is an increasing function of the number of entrepreneurs who have been funded for investment projects, the total amount of real estate is constant, $1 - \alpha$ so that the rental fee of real estate becomes the increasing function of the capital stock.

### 3. Closed Economy

In this section we study general features of the closed economy. The measure of domestic saving is $\alpha W(k_t)$, while the measure of maximum domestic investment is $1 - \alpha$. The measure of the maximum saving is $\alpha W((1 - \alpha)R)$ since the maximum of the attainable capital goods is $R(1 - \alpha)$. We simplify the analysis by restricting parameters under which the demand for investment always exceeds the supply of it. We impose the following assumption;

(A2) $\alpha W((1 - \alpha)R) < 1 - \alpha$.

Under (A2), the aggregate investment is constrained by $\alpha W(k_t)$. Since each investment supplies $R$ units of capital, the aggregate capital stock evolves as

$$k_{t+1} = R\alpha W(k_t).$$

Equation (3) gives the law of motion for the stock of capital. The assumption (A1) ensures that for any $k_0 > 0$, $k_t$ converges monotonically to a unique nontrivial steady state $k^*$ that satisfies $k^* = R\alpha W(k^*)$, as Figure 3-1 illustrates.

We are now ready to the determination of the real interest rate. The two constraints, (1) and (2), can be summarized as

$$r_{t+1} \leq \text{Min} \{ Rf'(k_{t+1}), \hat{\lambda}Rf'(k_{t+1}) + W(k_{t+1}) \}.$$ 

The market clearing in the financial market requires the inequality in (4) to be binding with equality, and thus (4) is rewritten as
\[
(5) \quad r_{t+1} = r(k_{t+1}) = \begin{cases} \lambda R'(k_{t+1}) + W(k_{t+1}) & \text{if } k_{t+1} < K(\lambda), \\ R'(k_{t+1}) & \text{if } k_{t+1} \geq K(\lambda), \end{cases}
\]

where \( K(\lambda) \) is the threshold defined implicitly by \( \lambda R'(K(\lambda)) + W(K(\lambda)) = R'(K(\lambda)) \). The parameter \( \lambda \) affects the interest rate, but not the dynamics of capital.

The measure of entrepreneurs who are able to fund the project is equal to \( \alpha W(k_t) \), less than \( 1 - \alpha \) under Assumption (A2). When the borrowing constraint is binding, entrepreneurs strictly prefer investment by borrowing, but all of them cannot have access to borrowing. The equilibrium allocation in the financial market has the property of credit rationing in the sense that some agents are denied borrowing although they would be willing to borrow at more unfavorable condition than the condition under which others have already received borrowing (e.g. Stiglitz and Weiss (1981) and Williamson (1986)). When the profitability constraint is binding, entrepreneurs are indifferent between borrowing and not borrowing.

The function \( R'(k_{t+1}) \) is monotonically decreasing, while \( \lambda R'(k_{t+1}) + W(k_{t+1}) \) is not. The latter is decreasing in \( k_{t+1} \) if \( k_{t+1} < \lambda R \) and increasing if \( k_{t+1} > \lambda R \). The two curves intersect at \( k_{t+1} = K(\lambda) \) satisfying \( R'(K(\lambda)) = \lambda R'(K(\lambda)) + W(K(\lambda)) \), that is decreasing in \( \lambda \), with \( K(0) > 0 \) and \( K(1) = 0 \). There exists a threshold \( \hat{\lambda} \in (0, 1) \) such that \( K(\lambda) > \lambda R \) holds if \( \lambda < \hat{\lambda} \). Figure 3-2 illustrates the case of \( \lambda \geq \hat{\lambda} \) when \( r(k_{t+1}) \) is decreasing. On the other hand, Figure 3-3 illustrates the case of \( \lambda < \hat{\lambda} \) when \( r(k_{t+1}) \) is first decreasing, later increasing, and finally again decreasing.\(^3\) At the low level of capital stock, the effect of decreasing returns to capital is dominant and thus \( r(k_{t+1}) \) is decreasing. As the level of capital increases beyond \( \lambda R \), the effect of mitigating the borrowing constraint begins to be dominant, and \( r(k_{t+1}) \) is increasing;

\(^3\) In case of Remark-1, \( \lambda R'(k_{t+1}) + u_E \hat{\lambda}_n W(k_{t+1}) \) attains the minimum at \( \lambda R/u_E \hat{\lambda}_n \), while we have \( k_{t+1} < R(1 - \alpha) \) by definition. Then if \( u_E > 1 \), there exists a region with \( r(k_{t+1}) \) being increasing even when \( \hat{\lambda}_n < \hat{\lambda} \).
as borrowers can pledge the greater amount of collateral, the borrowing constraint becomes less stringent, which drives the upward shift of the interest rate. Finally, as the level of capital increases beyond $K(\lambda)$, the borrowing constraint ceases to be binding; the effect of decreasing returns to capital only exists and $r(k_{t+1})$ is decreasing. The non-monotone feature is more likely to arise if $\lambda$ is small or if the share of capital is high. Assuming $f(k) = k^\beta$, the non-monotone property arises if $\lambda < \beta$.\(^4\)

4. World Economy with Financial Market Integration

Having investigated the closed economy, we now study a world economy that consists of two countries. Two countries, denoted as “N” and “S”, respectively, are inherently identical in production technology and institutional quality, but differ only in their initial levels of income. Letting $k_i^t$ denote the capital stock in country $i$ ($i = N, S$) in period $t$, we assume that country $S$ is initially poorer than country $N$, that is, $k_0^S \leq k_0^N$.

We consider the following integration. The final good is tradable, and thus the borrowing and lending of the final good is allowed across countries. The capital good is not tradable so that the capital good has to be used in the country where it is produced. Labor is immobile across countries. Assume that enforcement of any one country treats domestic and foreign people equally.\(^5\)

As a counterpart of (A2), we have $\alpha W(k_i^S) + \alpha W(k_i^N) < 2(1 - \alpha)$ for any $k_i^S$ and $k_i^N$.\(^6\) Under this inequality, the total supply of funds is always smaller than the total

\(^4\) The inversed-U shaped relation is not specific to the limited enforcement model, but rather arises from broader classes of agency models of corporate finance, including the costly-state-verification approach (e.g., Townsend (1979)), for example.

\(^5\) Enforceability is exogenous in our model. Actually, enforceability may depend on goods produced by different industries (e.g., Matsuyama, 2007), the distinction between traded and non-traded goods, the nationality, the distance between countries, and the state of the economy.

\(^6\) We have $\alpha W(k_i^S) \leq \alpha W((1 - \alpha)R) < 1 - \alpha$ from $k_i^S \leq (1 - \alpha)R$ by definition, and (A2).
demand for funds. The interest rate of each country then should evolve as

\[
 r_{t+1}^i = r(k_{t+1}^i) = \begin{cases} 
 \lambda Rf^*(k_{t+1}^i) + W(k_{t+1}^i) & \text{if } k_{t+1}^i < K(\lambda), \\
 Rf^*(k_{t+1}^i) & \text{if } k_{t+1}^i \geq K(\lambda), 
\end{cases}
\]

\((i = N, S)\). The market clearing in the world capital market requires the aggregate investments to be equal to the aggregate savings, given by

\[
 k_{t+1}^N + k_{t+1}^S = R\alpha W(k_{t+1}^N) + R\alpha W(k_{t+1}^S).
\]

In the world capital market, people of both countries face an identical world interest rate \(r_{t+1}\) so that we should have the no-arbitrage condition as

\[
 r_{t+1}^N = r_{t+1}^S = r_{t+1}.
\]

Accordingly, any of the equilibria with an integrated financial market is described as a sequence \(\{k_{t+1}^N, k_{t+1}^S, r_{t+1}\}_{t=0}^{\infty}\), satisfying (6), (7), and (8), given \(k_0^N > 0\) and \(k_0^S > 0\).

The dynamic behavior differs according to whether \(r(\cdot)\) is monotone or not. We first study the monotone case. Since \(r(\cdot)\) is not differentiable at \(K(\lambda)\), we have to analyze the correspondence from \(k_{t+1}^N\) to \(k_{t+1}^S\) by separating the whole interval into \([0, K(\lambda))\) and \([K(\lambda), +\infty)\) for \(k_{t+1}^S\) and \(k_{t+1}^N\), respectively. It follows from the monotonicity and the symmetric property that (8) is described as

\[
 \lambda Rf^*(k_{t+1}^N) + W(k_{t+1}^N) = \lambda Rf^*(k_{t+1}^S) + W(k_{t+1}^S) \quad \text{for } k_{t+1}^N < K(\lambda) \text{ and } k_{t+1}^S < K(\lambda),
\]

and

\[
 Rf^*(k_{t+1}^N) = Rf^*(k_{t+1}^S) \quad \text{for } k_{t+1}^N \geq K(\lambda) \text{ and } k_{t+1}^S \geq K(\lambda). \quad (7)
\]

It is easy to derive

\[
 k_{t+1}^N = k_{t+1}^S
\]

for any \(k_{t+1}^S \in [0, +\infty)\) and \(k_{t+1}^N \in [0, +\infty)\). Therefore, it follows from (7) and (9) that

\[
 k_{t+1}^i (i = N, S) \quad \text{evolves simply as } k_{t+1}^i = R\alpha W(k_i^i), \quad \text{with } k_i^N = k_i^S.
\]

Note that \(r(\cdot)\) is not differentiable at \(K(\lambda)\).
unique steady state is characterized by a pair \((k^*, k^*)\), satisfying \(k^N = k^S\) and the steady-state market clearing, given by

\[
k^N + k^S = R\alpha W(k^N) + R\alpha W(k^S).
\]

Figure 4-0 illustrates the dynamic configuration in the \((k^S_{t+1}, k^N_{t+1})\) space. The dotted curve going through \((k^*, 0), E\), and \((0, k^*)\) represents the steady state relationship for the market clearing in the world financial market, given by (10).\(^8\) The straight line \(PQ\) represents the state \((k^S_{t+1}, k^N_{t+1})\) that satisfies (7), the temporal market clearing in the world financial market, given \(k^S_t\) and \(N^S_t\), while the 45 degree line represents the no-arbitrage condition (9). Assume that levels of capital of two countries reach \(A'\) under autarky if integration does not happen. On integration, the economy jumps to \(A\), and afterward converges to the symmetric steady state \(E\). The convergence occurs no matter how the distribution of capital stock in the event of integration is.

We turn to the case when \(r(.)\) is not monotone decreasing. As Figure 4-1 depicts, we then separate nine regions for deriving the correspondence from \(k^S_{t+1}\) to \(k^N_{t+1}\). In each of regions (I), (V), and (IX), the intervals for \(k^S_{t+1}\) and \(k^N_{t+1}\) are the same, and we call them “symmetric regions”. On the other hand, in any of other regions, the intervals for \(k^S_{t+1}\) and \(k^N_{t+1}\) differ with each other, and we call them “asymmetric regions”.

In each of symmetric regions, it is easy to derive \(k^N_{t+1} = k^S_{t+1}\). We now go on asymmetric regions. In region (II), \(r(k^N_{t+1})\) is increasing while \(r(k^S_{t+1})\) is decreasing so

\[^8\] By differentiating (10), we obtain a slope of the curve given by the expression
\[
dk^N / dk^S \bigg|_{(10)} = \frac{[R\alpha W'(k^S) - 1]}{[1 - R\alpha W'(k^N)]}.\]
Therefore, \(dk^N / dk^S \bigg|_{(10)} = -1\) holds at \(E\).

Moreover, defining \(k\) by \(R\alpha W'(k) = 1\), \(dk^N / dk^S \bigg|_{(10)} \geq 0\) holds if and only if \((k^S, k^N)\in[0, k] \times [k^*, \infty]\) or \([k^*, \infty] \times [0, k]\).
that the no-arbitrage condition (8) requires that \( k_{r+1}^N \) be decreasing in \( k_{r+1}^S \). In region (III), both of \( r(k_{r+1}^N)'s \) are increasing so that (8) requires that \( k_{r+1}^N \) be increasing in \( k_{r+1}^S \). In region (VI), \( r(k_{r+1}^N) \) is decreasing while \( r(k_{r+1}^S) \) is increasing so that (8) requires that \( k_{r+1}^N \) be decreasing in \( k_{r+1}^S \). Regions (IV), (VII), and (VIII) are mirror images of (II), (III), and (VI), respectively. The formal derivation is left to Appendix. Therefore, the correspondence is composed of nine functions, each of which is continuous and differentiable on its distinct interval. The functions are summarized as

\[
\Psi(k_{r+1}^S) = \begin{cases} 
  k_{r+1}^S & \text{if } k_{r+1}^N < \lambda R \text{ and } k_{r+1}^S < \lambda R \\
  \mu(k_{r+1}^S) & \text{if } \lambda R \leq k_{r+1}^N < K(\lambda) \text{ and } k_{r+1}^S < \lambda R \\
  \phi(k_{r+1}^S) & \text{if } K(\lambda) \leq k_{r+1}^N \text{ and } k_{r+1}^S < \lambda R \\
  \mu^{-1}(k_{r+1}^S) & \text{if } k_{r+1}^N < \lambda R \text{ and } \lambda R \leq k_{r+1}^S < K(\lambda) \\
  \phi^{-1}(k_{r+1}^S) & \text{if } \lambda R \leq k_{r+1}^N < K(\lambda) \text{ and } K(\lambda) \leq k_{r+1}^S \\
  k_{r+1}^S & \text{if } K(\lambda) \leq k_{r+1}^N \text{ and } K(\lambda) \leq k_{r+1}^S 
\end{cases}
\]

where \( \mu(k_{r+1}^S) \), \( \mu^{-1}(k_{r+1}^S) \), \( \phi(k_{r+1}^S) \), and \( \phi^{-1}(k_{r+1}^S) \) are decreasing, \( \phi(k_{r+1}^S) \) and \( \phi^{-1}(k_{r+1}^S) \) are increasing, with \( \lim_{k_{r+1}^S \to \lambda R^-} \mu(k_{r+1}^S) = \lambda R = \lim_{k_{r+1}^S \to \lambda R^+} \mu^{-1}(k_{r+1}^S) \),

\[
\lim_{k_{r+1}^S \to \lambda R^-} \phi(k_{r+1}^S) = \lim_{k_{r+1}^S \to \lambda R^+} \phi(k_{r+1}^S) , \quad \lim_{k_{r+1}^S \to K(\lambda)^-} \mu^{-1}(k_{r+1}^S) = \lim_{k_{r+1}^S \to K(\lambda)^+} \phi^{-1}(k_{r+1}^S) , \\
\lim_{k_{r+1}^S \to K(\lambda)^-} \mu(k_{r+1}^S) = \lim_{k_{r+1}^S \to K(\lambda)^+} \phi(k_{r+1}^S) , \quad \lim_{k_{r+1}^S \to \phi^{-1}(\lambda)^-} \phi^{-1}(k_{r+1}^S) = \lim_{k_{r+1}^S \to \phi(\lambda)^+} \phi(k_{r+1}^S) , \\
\text{and } \lim_{k_{r+1}^S \to K(\lambda)^-} \phi(k_{r+1}^S) = K(\lambda) = \lim_{k_{r+1}^S \to K(\lambda)^+} \phi^{-1}(k_{r+1}^S) .
\]

Figure 4-1 illustrates the correspondence \( \Psi(\cdot) \), which is composed of both the 45 degree line and the connected six curves that we call ellipse.

For later reference, we define the steady state. Any of steady states is defined as a
state \( \{k^N, k^S\} \), satisfying (10) and

\[
(12) \quad k^N = \Psi(k^S).
\]

It finally follows from (7) and (11) that the system can be reduced to a one-dimensional map of \( k^S_t \), which satisfies

\[
(13) \quad k^S_{t+1} + \Psi(k^S_{t+1}) = R\alpha\{W(k^S_t) + W(\Psi(k^S_t))\}.
\]

Then there exists a correspondence from \( k^S_t \) to \( k^S_{t+1} \) that is continuous and differentiable except for at most countable points:

\[
(14) \quad k^S_{t+1} = \Phi(k^S_t),
\]

with

\[
(15) \quad \Phi'(k^S_t) = \frac{dk^S_{t+1}}{dk^S_t} = R\alpha \frac{W'(k^S_t) + W'(\Psi(k^S_t))\Psi'(k^S_t)}{1 + \Psi'(k^S_{t+1})},
\]

\( \Psi'(k^S_t) \neq -1 \), where \( \Psi'(.) = \mu' \) for \((k^S_{t+1}, k^N_{t+1}) \in [0, \lambda R] \times [\lambda R, K(\lambda)]\)

\( (k^S_t, k^N_t) \in [\lambda R, K(\lambda)] \times [K(\lambda), \infty) \), \( \Psi'(.) = \phi' \) for

\( (k^S_t, k^N_t) \in [\lambda R, K(\lambda)] \times [0, \lambda R] \), \( \Psi'(.) = \phi^{-1} \) for \((k^S_{t+1}, k^N_{t+1}) \in [K(\lambda), \infty) \times [0, \lambda R])\), and \( \Psi'(.) = \phi^{-1} \) for \((k^N_{t+1}, k^N_{t+1}) \in [K(\lambda), \infty) \times [\lambda R, K(\lambda))\). The function \( \Phi(k^S_t) \)

may not be differentiable at \( \mu^{-1}(K(\lambda)) = \phi^{-1}(K(\lambda)) \) or \( K(\lambda) \).

The remarkable feature is that this system can be reduced to a one-dimensional map

of \( k^S_t \) (or \( k^N_t \)), which allows us to trace out the equilibrium path globally. We describe

a relationship between \( k^S_t \) and \( k^S_{t+1} \) using the conventional 45 degree line analysis.

Before doing it, we have to comment on a possible singular case of \( \Psi'(k^S_{t+1}) = -1 \). We
cannot rule out a case when the system traverses \( \Psi'(.) = -1 \) in asymmetric regions. In order to avoid this troublesome case, we impose the following two technical conditions.

**Condition 1:** \( \varphi'(k_{i+1}^S) > -1 \)

**Condition 2:** \( \mu'(k_{i+1}^S) < -1 \)

Given the production function \( f(k) = k^\beta \), Condition 1 holds when \( \lambda \) is not too small relative to the capital income share \( \beta \) to satisfy \( \frac{\beta}{2-\beta} < \lambda < \beta \). Condition 2 is satisfied when the production function is the Cobb-Douglas form. The proof is left to Appendix. Note that we have \( \mu'(k_{i+1}^S) = -1 \) at \( (\lambda R, \lambda L) \). But on the 45 degree line, \( k_i^S \) evolves according to \( k_{i+1}^S = R\alpha W(k_i^S) \) so that the singular problem is innocent at the symmetric region.

We call the dynamic path **monotone** if the aggregate stock of capital available to two countries, \( \kappa_i \equiv k_i^S + k_i^N \), monotonically increases or decreases on the equilibrium path. This definition arises from the fact that \( k_i^S \) and \( k_i^N \) move very often in the opposite direction even when \( \kappa_i \) moves monotonically on the equilibrium path. Without loss of generality, we consider the equilibrium path for the region of \( k_i^S \leq k_i^N \).

We investigate first the dynamic behavior in symmetric regions (I), (V), and (IX), in which \( k_i^N = k_i^S \) holds. The system is expressed as \( k_{i+1}^i = R\alpha W(k_i^i) (i = N, S) \). The

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9 Condition 1 holds if \( \lambda > 0.053 \) for \( \beta = 0.3 \) and if \( \lambda > 0.109 \) for \( \beta = 0.4 \). Strictly speaking, Condition 1 holds if \( \max \{(\frac{\beta}{2-\beta})^{2-\beta}, 2\beta - 1\} < \lambda < \beta \). However, in reality, the capital income share \( \beta \) is typically smaller than 0.5 so that \( 2\beta < 1 \).

10 We apply l’Hôpital’s rule to obtain
\[
\lim_{k_{i+1}^i \to \lambda R} \mu'(k_{i+1}^i) = \lim_{k_{i+1}^i \to \lambda R} \{1/\mu'(k_{i+1}^i)\}
\]
from the fact that
aggregate stock of capital available to two countries becomes \( \kappa_t = 2R\alpha W(k^s_t) \), which monotonically converges to \( 2k^* \). We next go to the analysis in asymmetric regions. We obtain the following.

**Proposition 1**

Suppose that Condition 1 holds. Then there exists an equilibrium path that exhibits a monotone movement in regions (III) and (VI).

**Proof:** Consider first \( \Phi(.) \) in region (III), that is, for any

\[
k_t^S \in \Omega_{III} \equiv \{ k_t^S \mid K(\lambda) \leq \phi(k_t^S) \text{ and } k_t^S < \lambda R \}.
\]

The function \( \Psi(k_t^S) \) is increasing and thus \( \Phi(.) \) is increasing. Consider secondly \( \Phi(.) \) in region (VI), that is, for any

\[
k_t^S \in \Omega_{VI} \equiv \{ k_t^S \mid K(\lambda) \leq \phi(k_t^S) \text{ and } \lambda R \leq k_t^S < K(\lambda) \}.
\]

The function \( \Psi(.) \) is decreasing but its derivative is greater than \(-1\), as implied by Condition 1, \( \Phi(.) \) is again increasing since the derivative

\[
\Phi'(k_t^S) = R\alpha \frac{W'(k_t^S) + W'(k_t^S)\Psi'(k_t^S)}{1 + \Psi'(k_t^S)}
\]

is positive when \( W'(k_t^S) > W'(\Psi(k_t^S)) \) for \( k_t^S < k_t^N = \Psi(k_t^S) \). The equality

\[
\lim_{k_t^S \to \lambda R^-} \phi(k_t^S) = \lim_{k_t^S \to \lambda R^+} \phi(k_t^S) \text{ holds so that } \Phi(k_t^S) \text{ is continuous at } k_t^S = \lambda R.
\]

The function \( \Phi(.) \) is continuous and increasing for any \( k_t^S \in \Omega_{III} \cup \Omega_{VI} \). As Figure 4-2A illustrates in a 45 degree line diagram, for example, \( k_t^S \) exhibits a monotone movement. Finally, Condition 1 ensures that \( k_t^S + \Psi(k_t^S) = \kappa_t \) is increasing in \( k_t^S \).

Therefore, \( \kappa_t \) shows a monotone movement in regions (III) and (VI). Q.E.D.

When the economy is out of region (III) and (VI), how the economy behave is important to examine globally. The following lemma is useful.

\[
\lim_{k_t^S \to \lambda R} \mu(k_t^S) = \lambda R.
\]

Given that \( \mu(.) \) is decreasing, we have \( \lim_{k_t^S \to \lambda R} \mu'(k_t^S) = -1 \).
Lemma 0

When any point \((k^S_t, k^N_t)\) lies inside (outside) the curve (10), \(\kappa_t\) is increasing (decreasing), that is, \(\kappa_t < \kappa_{t+1} (\kappa_t > \kappa_{t+1})\).

Proof. Without loss of generality, we prove that when \((k^S_t, k^N_t)\) lies inside the curve (10),

\[ k^N_t + k^S_t < R\alpha W(k^N_t) + R\alpha W(k^S_t) \]

holds. This inequality is obvious when both \(k^N_t\) and \(k^S_t\) are less than \(k^*\), because \(R\alpha W(k) > k\) for any \(k < k^*\). Examine next the case where the capital stock of one country is greater than \(k^*\), while that of the other is less than \(k^*\). Consider a pair \((\tilde{k}^S_t, \tilde{k}^N_t)\), with \(\tilde{k}^S_t < k^* < \tilde{k}^N_t\), satisfying

\[ \tilde{k}^N_t + \tilde{k}^S_t = R\alpha W(\tilde{k}^N_t) + R\alpha W(\tilde{k}^S_t) . \]

Now letting \(\tilde{k}^N_t = \tilde{k}^N_t - \varepsilon\) for any \(\varepsilon \in (0, \tilde{k}^N_t - k^*)\) so that \((\tilde{k}^S_t, \tilde{k}^N_t)\) is inside (10), with \(\tilde{k}^N_t > k^*\), we have

\[ R\alpha W(\tilde{k}^S_t) - \tilde{k}^S_t = \tilde{k}^N_t - R\alpha W(\tilde{k}^N_t) > \tilde{k}^N_t - R\alpha W(\tilde{k}^N_t) , \]

where the last inequality follows from the fact that \(k - R\alpha W(k)\) is increasing in \(k\) for any \(k \geq k^*\). Thus we have

\[ \tilde{k}^N_t + \tilde{k}^S_t < R\alpha W(\tilde{k}^N_t) + R\alpha W(\tilde{k}^S_t) . \]

Inside (10), when combined with (7), we have

\[ \kappa_t = k^N_t + k^S_t < R\alpha W(k^N_t) + R\alpha W(k^S_t) = \kappa_{t+1} . \]

Q.E.D.

Figure 4-3 illustrates where the equilibrium path goes when it is apart from the steady state. When any point on the ellipse lies inside (outside) the dotted curve, the solid downward line \(\kappa_{t+1} = k^S_{t+1} + k^N_{t+1}\) is shifted upward (downward) toward the dotted downward line that goes on asymmetric steady states. Given this preparation, we obtain the following.
**Proposition 2**

Suppose that there is no steady state in region (II). In addition, Condition 2 holds. Then there exists an equilibrium path that increases monotonically and departs from region (II) after some finite periods.

*Proof:* Consider $\Phi(.)$ for any $k^S_i \in \Omega_{ii} \equiv \{k^S_i | \lambda R \leq \mu(k^S_i) < K(\lambda) \text{ and } k^S_i < \lambda R\}$. We have $\kappa_i < \kappa_{ir}$ from Lemma 0 since any pair $(k^S_i, \mu(k^S_i))$ that satisfies (11) lies inside the curve (10). When Condition 2 holds, the relation $k^S_i + \Psi(k^S_i) \equiv \kappa_i$ implies that a monotone rise in $\kappa_i$ should be associated with a monotone fall in $k^S_i$, and hence a monotone rise in $k^N_i (= \mu(k^S_i))$. However, since there is no steady state in region (II), as Figure 4-3 illustrates, any equilibrium path has to depart from region (II) after some finite periods. Q.E.D.

Propositions 1 and 2 jointly conjecture that, at least except for the case when both economies are borrowing-constrained at the steady state, we may trace out the dynamic analysis globally. We depict three typical possibilities generated by the dynamic system. Figure 4-4A illustrates one case for three steady states that occurs when $k^S < k^* < K(\lambda) < k^N$. There exists one symmetric steady state $E$, and two asymmetric steady states, $F$ and $G$. Both countries are borrowing-constrained in the symmetric steady state, while the poor country is borrowing-constrained, but the rich is not in the asymmetric steady states. Asymmetric steady states are stable, and the symmetric steady state $E$ is saddle-path stable. We see this by looking at Figure 4-2A again. The function $\Phi(.)$ is continuous and increasing, with single-crossing with the 45 degree line at $F$. In addition, as $k^S_i \to K(\lambda)$, we have $k^S_i < K(\lambda)$. Letting $k^S_F$ denote the

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11 As $k^S_i \to K(\lambda)$, $k^S_i + k^N_i = R\alpha\{W(k^S_i) + W(\phi(k^S_i))\} \to 2R\alpha W(K(\lambda))$ since

$$\lim_{k^S_i \to K(\lambda)} \phi(k^S_i) = K(\lambda).$$

It follows from (A1) and $k^* < K(\lambda)$ that $2R\alpha W(K(\lambda)) < 2K(\lambda)$. 

---
level of capital stock in country $S$ at $F$, $k^S_t$ is increasing (decreasing) for

\[ k^S_t < k^N_t \quad (k^N_t < k^S_t) \].

Combined with Proposition 2, any $k^S_t \in \Omega_{ii} \cup \Omega_{mi} \cup \Omega_{ni}$ converges monotonically to $F$. This case corresponds to the economy where Matsuyama (2004) intensively investigate and calls \textit{symmetry-breaking}.

Second, Figure 4-4B illustrates the case for five steady states that occurs when $k^S < K(\lambda) < k^* < k^N$. There exists one symmetric steady state $E$, and four asymmetric steady states. Both countries are not borrowing-constrained in the symmetric steady state, while in the asymmetric steady states the poor country is borrowing-constrained, but the rich is not. The symmetric steady state is stable. Among four asymmetric steady states, two are stable, and the other two are unstable. Consider again $\Phi(.)$ for any $k^S_t \in \Omega_{ii} \cup \Omega_{ni}$. The function $\Phi(.)$ is continuous and increasing, and crosses the 45 degree line twice. In addition, as $k^S_t \to K(\lambda)$, we have $k^S_{t+1} > K(\lambda)$.\footnote{As $k^S_t \to K(\lambda)$, $k^S_{t+1} + \phi(k^S_{t+1}) = R\alpha(W(k^S_t) + W(\phi(k^S_t))) \to 2R\alpha W(K(\lambda))$ since \[ \lim_{k^S_t \to K(\lambda)} \phi(k^S_t) = K(\lambda) \]. It follows from (A1) and $K(\lambda) < k^*$ that $2R\alpha W(K(\lambda)) > 2K(\lambda)$, which implies that as $k^S_t \to K(\lambda)$, $k^S_{t+1} + k^N_{t+1} > 2K(\lambda)$. However, no equilibrium pair $(k^S_{t+1}, k^N_{t+1})$ satisfies $k^S_{t+1} + k^N_{t+1} > 2K(\lambda)$, with $k^N_{t+1} > k^S_{t+1}$ when Condition 1 holds, which implies that $k^N_{t+1} = k^S_{t+1}$ has to be met. Thus the latter inequality directly leads to $k^S_{t+1} > K(\lambda)$.}

As Figure 4-2B depicts, $k^S_t$ is stable but $k^N_t$ is unstable, where $k^N_t$ denotes the level of capital stock in country $S$ at $H$. The steady state $H$ (or $J$) is the threshold only above which the stock of capital increases. The equilibrium path which lies at the right of $H$ will converge to $E$, whereas the one that lies at the left of $H$ to $F$.

Finally, Figure 4-4C illustrates the case with a unique steady state that is symmetric and stable. There are three paths that arrive at the steady state. One equilibrium path goes on the 45 degree line, while other two on the ellipse. Along the equilibrium path on
the ellipse, $k_t^S$ first decreases while $k_t^N$ increases, later $k_t^S$ and $k_t^N$ both increase, and finally $k_t^S$ increases while $k_t^N$ decreases.

We have to comment on another case for three steady states, in which the asymmetric steady state lies in regions (II) (or (IV)) so that both countries are borrowing-constrained at the steady state. In region (II), $\Psi(.)$ is decreasing with the derivative of less than $-1$, as implied by Condition 2, and $\Phi(.)$ may be increasing or decreasing. If $\Phi(.)$ is increasing, the equilibrium path moves monotonically, but if it is decreasing, the equilibrium path may generate endogenous fluctuations. Any equilibrium path that lies apart from region (II) goes monotonically toward region (II) (Lemma 0), but the monotone convergence is not guaranteed. If $\Phi(.)$ is decreasing around the steady state, the equilibrium path exhibits endogenous fluctuations over time.\textsuperscript{13}

5. Financial Integration and Global Analysis

Having studied dynamic properties of the model, we now turn to investigate the behavior of the equilibrium by relying on two equilibrium conditions, the temporal asset market clearing (7) and the correspondence derived from the no-arbitrage condition (11). We focus on three cases studied above and examine them in order.

In Figure 5-2A, we illustrate the case for three steady states and then trace out the whole process of symmetry-breaking. Two closed economies will arrive the symmetric steady state $E$. A small perturbation makes the economies away from $E$, but closed economies will soon recover stability. However, the world economy with integrated financial markets will lose stability if the economy departs from $E$. Consider a PQ line that goes through $B'$ that represents the state of would-be closed economies at period $T+1$ when the shock occurs at $T$. There exist three intersections between the PQ line

\textsuperscript{13} Boyd and Smith (1997) show the example of the damped cycles. Kikuchi and Stachurski (2008) show that the two-country version of the Matsuyama model generates endogenous fluctuations over time when two countries differ in the size of population.
and (11) \( k_{t+1}^N = \Psi(k_{t+1}^S) \). One intersection, \( B_2 \), lies on the 45 degree line, while other two, \( B_1 \) and \( B_3 \), lie on the ellipse.

Figure 5-2B illustrates the configuration of the world financial market, which is reminiscent of MacDougall (1960), Kemp (1962), and Hamada (1966). The length of interval from \( O_N \) to \( O_S \) stands for the aggregate savings, \( R\alpha(W(k_t^N) + W(k_t^S)) \). Any point in the horizontal axis stands for the allocation of capital between two countries.

The vertical axis stands for the interest rate. Each of the functions, \( r(k_{t+1}^i) \ (i = N, S) \), is illustrated from \( O_i \ (i = N, S) \). Both functions are non-monotone, and thus there are three intersections, \( B_1 \), \( B_2 \) and \( B_3 \), each of which corresponds to the same symbol in Figure 5-2A, respectively.

We specify the price adjustment rule in the market. We assume the Marshallian adjustment rule in which the demand and the supply react to the “excess demand price”. Investors will rationally anticipate that the autarkic interest rate is higher in country \( N \) than the country \( S \), that is, \( r(k_{T+1}^{N,A}) > r(k_{T+1}^{S,A}) \) at \( B' \) since \( B' \) is inside the ellipse, satisfying \( k_{i,t+1}^{N,A} > \Psi(k_{i,t+1}^{S,A}) \), where \( k_{i,t}^{i,A} \ (i = N, S) \) is the level of capital that attains under autarky. Following the Marshallian adjustment rule, capital begins to flow out of country \( S \) into country \( N \), until the interest rate differential disappears. The symmetric equilibrium loses stability, and the market adjustment drives the equilibrium to the asymmetric equilibrium \( B_1 \). After having arrived at \( B_1 \), the equilibrium approaches along \( k_{t+1}^N = \Psi(k_{t+1}^S) \) to the asymmetric steady state \( F \). By financial integration, the investment/wealth feedback works to destabilize the world financial market, accompanied by the capital flows from the poor to the rich, which in turn, makes the rich richer and the poor poorer (e.g., Boyd and Smith (1997) and Matsuyama (2004)). Although the genuine level of enforcement is the same, the small difference in the country’s wealth gives rise to further difference in the country’s wealth accompanied by
the difference in the institutional development. The world output is smaller in any equilibrium of symmetry-breaking than convergence. Financial integration is a device for the rich to increase the output at the sacrifice of the poor. This symmetry-breaking argument holds if two countries open capital accounts when the world savings are not too small but not too large to satisfy \( 2\lambda R < R\alpha \{W(k_T^N) + W(k_T^S)\} < 2K(\lambda) \).\(^{15}\)

Our model has the advantage of exploiting the pattern of capital flows not only at the steady state but also out of steady states. Without loss of generality we prove only for \( k^N_T > k^S_T \). In fact, we obtain the following.

**Lemma 1**

(a) Suppose that (7) intersects with (11) at \( (k_{r+1}^S, k_{r+1}^N) \in [0, \lambda R] \times [\lambda R, K(\lambda)) \). If a state of closed economies \( (k_{r+1}^{S,A}, k_{r+1}^{N,A}) \) satisfies \( k_{r+1}^{N,A} > (>)^\Psi(k_{r+1}^{S,A}) \), the equilibrium interest rates of each country satisfy \( r(k_{r+1}^{N,A}) > (>)r(k_{r+1}^{S,A}) \) under autarky.

(b) Suppose that (7) intersects with (11) at \( (k_{r+1}^S, k_{r+1}^N) \in [0, K(\lambda)) \times [K(\lambda), \infty) \). If a state of closed economies \( (k_{r+1}^{S,A}, k_{r+1}^{N,A}) \) satisfies \( k_{r+1}^{N,A} < (>)^\Psi(k_{r+1}^{S,A}) \), the equilibrium interest rates of each country satisfy \( r(k_{r+1}^{N,A}) > (>)r(k_{r+1}^{S,A}) \) under autarky.

**Proof:** We prove only part (a) to avoid the repetitious expression (readers may trace out

\(^{14}\) Note that poor countries then receive the gain of holding foreign assets so that the GNP gap is not so large as the GDP gap.

\(^{15}\) If financial integration occurs at the early developing stage, with \( k_{r+1}^N \) and \( k_{r+1}^S \) being so small to satisfy \( R\alpha \{W(k_{r+1}^N) + W(k_{r+1}^S)\} < 2\lambda R \) or at the stage, with \( k_{r+1}^N \) and \( k_{r+1}^S \) being great to satisfy \( 2K(\lambda) < R\alpha \{W(k_{r+1}^N) + W(k_{r+1}^S)\} \) (although the latter case is unlikely), the intersection is uniquely realized on the 45 degree line, and the equilibrium converges to \( E \). However, the symmetric steady state is saddle-path stable so that once the world economy has diverged from the symmetric equilibrium, it eventually falls into either one of the asymmetric equilibria.
the proof of part (b) by separating the case of \( k^{S}_{r+1} < \lambda R \) and \( \lambda R < k^{S}_{r+1} < K(\lambda) \).

Pick up a pair \((k^{S}_{r+1}, k^{N}_{r+1})\) satisfying \( k^{N}_{r+1} = \Psi(k^{S}_{r+1}) \) and \( k^{S}_{r+1} < \lambda R \leq k^{N}_{r+1} < K(\lambda) \).

We then pick up a pair under autarky \((k^{S,A}_{r+1}, k^{N,A}_{r+1})\) satisfying \( k^{N,A}_{r+1} > \Psi(k^{S,A}_{r+1}) \), by increasing \( k^{N}_{r+1} \), starting from \((k^{S}_{r+1}, k^{N}_{r+1})\), while fixing \( k^{S}_{r+1} \). We have

\[
\lambda Rf^*(k^{N}_{r+1}) + W(k^{N}_{r+1}) > \lambda Rf^*(\Psi(k^{S}_{r+1})) + W(\Psi(k^{S}_{r+1})) = \lambda Rf^*(k^{S}_{r+1}) + W(k^{S}_{r+1}) ,
\]

where the inequality arises from \( k^{N,A}_{r+1} > \Psi(k^{S,A}_{r+1}) \) and the fact that \( \lambda Rf^*(.) + W(.) \) is increasing in \( k^{N}_{r+1} \), and the equality arises from the definition of \( \mu(.) \) and \( k^{S,A}_{r+1} = k^{S}_{r+1} \). It finally follows that \( r(k^{N,A}_{r+1}) > r(k^{S,A}_{r+1}) \). Q.E.D.

We have the following from Lemma 1.

**Corollary 1**

If the state of closed economies \((k^{S,A}_{t+1}, k^{N,A}_{t+1})\) is inside (outside) the ellipse, equilibrium interest rates in two countries satisfy \( r(k^{N,A}_{t+1}) > (<)r(k^{S,A}_{t+1}) \) under autarky.

It is straightforward to have the following.

**Proposition 3**

Suppose that an equilibrium pair \((k^{S}_{r}, k^{N}_{r})\) satisfies \( k^{N}_{r} = \Psi(k^{S}_{r}) \). Then if the state of would-be closed economies at period \( t+1 \), \((k^{S,A}_{t+1}, k^{N,A}_{t+1})\), is inside (outside) the ellipse, capital flows from country S (N) to country N(S).
The position of the state of would-be closed economies at period $t+1$ relative to the ellipse determines the direction of capital flows. Looking at Figure 5-2C, we observe the pattern of capital flows on the path to the asymmetric steady state $F$. The arrows in solid line indicate where would-be closed economies will go next period from the state on the ellipse. All the dotted arrows indicate the dynamic behavior of closed economies that approach monotonically toward $E$. In Figure 5-2C, all the arrows are directed inside the ellipse around $F$. Proposition 3 predicts that capital flows out of country $S$ to $N$ on almost all the path to $F$ as well as the steady state.\footnote{The asymmetric steady state result follows simply from (A1) and the fact the level of capital in country $N$ ($S$) is larger (smaller) than $k^*$.}

Figure 5-3A illustrates the case for five steady states. We observe a story of bifurcation in which either symmetry-breaking or convergence occurs, depending on the state when financial integration arises. Now suppose, for example, that closed economies open their capital accounts at the state that is fairly poor, for example $C'$ in period $T+1$. When integration occurs, there exist three intersections, $C_1, C_3$, and $C_3$, between the $PQ$ line and (11). At $C', r(k_{N,A}^{N}) > r(k_{N,A}^{S})$ holds (Lemma 1(b)), and hence country $S$’s people have an incentive to invest in country $N$. The symmetric equilibrium loses stability, and the market adjustment drives the equilibrium to $C_1$. Having arrived at $C_1$, the equilibrium converges to the asymmetric steady state $F$. In Figure 5-3C, arrows are typically directed inside the ellipse at almost all the points on the path converging to $F$ so that symmetry-breaking is accompanied by the continued capital flows out of country $S$ to $N$.

However, we have a different consequence of financial integration when closed countries open their capital accounts at the sufficiently wealthy state $D'$, for example. At the instant of integration, \( r(k_{N,A}^{N}) > r(k_{N,A}^{S}) \) holds (Lemma 1(b)), and hence the capital outflow from the poor drives the equilibrium to $D_1$. Having arrived at $D_1$, the world economy goes on the ellipse toward $I$, converging finally to $E$. Now the positive feedback between wealth and investment works to increase savings, allowing
both functions, $r(k_{it}^N)$ and $r(k_{it}^S)$, to shift apart from each other, as Figure 5-3B illustrates. The symmetric equilibrium recovers stability. Figure 5-3C shows that on the path from $D_1$ to $I$, arrows are directed outside the ellipse, and thus capital tends to flow from the rich to the poor.

Whether the financial integration leads to symmetry-breaking or convergence depends on the state when financial integration arises. If the PQ line intersects with the ellipse at the left of the threshold $H$, the world economy will move to the asymmetric steady state $F$, while if it does at the right to the threshold, it will move to $E$. The level of wealth works as collateral to discipline the working of financial markets, and so there are threshold levels of wealth and hence institutional development only above which financial integration leads to convergence. In Figure 5-3D, the shaded region illustrates the pair of capital stocks of closed economies under which financial integration leads to convergence. The downward curve is a set of pair $(k_I^S, k_I^N)$, satisfying $k_I^S + k_I^N = R\alpha W(k_I^N) + R\alpha W(k_I^S)$, where $(k_{\mu I}^S, k_{\mu I}^N)$ is the pair corresponding to $H$. Convergence is more likely to arise if both countries are rich, and/or equal. Note that the shaded curve is convex to the origin, and thus any pair of capital stocks is more likely to lie in the shaded region as capital stocks are more equal. Financial integration is more likely to succeed in convergence as levels of wealth are close among countries.

When closed economies stay near $E$, the prompt integration will lead to convergence, but when they are apart from $E$, delaying integration will be necessary to attain the eventual convergence. In particular, the poor will be trapped into the poverty by opening capital accounts in their early stages of development. The poor can go on the convergent path when the timing of integration is delayed until the poor have reached some threshold level of capital stock, which depends also on the capital stock of the rich.

Episodes also abound. It was only in the 1980s that the governments of fast-growing East Asian countries, including Japan, Korea, and Taiwan, lifted capital
account completely. The Japanese government announced liberalization of capital control in 1979, but the Japanese liberalization was not genuine toward 1980s. In Taiwan, capital transactions were decontrolled only as recent as in 1987. In Korea, liberalization was gradually made toward 1980s, but restrictions on capital movements have not been completely removed.\textsuperscript{17} This story is contrasted sharply with experiences of Latin American countries that suffered from the flight of capital and economic stagnation in the 1980s.

Finally, Figure 5-4A illustrates the case for the unique steady state. Convergence will occur surely, but symmetry-breaking arises in the short run. Suppose that closed economies open their capital accounts at the fairly poor state $Z'$, for example. The market adjustment drives the equilibrium to $Z_i$. Having arrived at $Z_i$, the world economy goes on the ellipse toward $I$, converging finally to $E$. As Figure 5-4B illustrates, capital flows first from the poor, but later reverses the direction. The endogenous capital reversal arises since the increase in collateral enables the home financial market to work better and thus to pull foreign capital.

Financial integration in early stages gives rise to symmetry-breaking for some time. The poor first becomes poorer, accompanied by the flight of capital, but later gains from the “trickle-down effect” from the rich, with reversing the direction of capital flows. The rich that has exploited the gain of symmetry-breaking comes to lend to the poor, stimulating borrowing and investment of the poor.

Although the system has only one steady state, the room for the timing argument is left. Early integration tends to deter convergence, but late integration may gives rise to faster convergence. We verify this by examining the behavior of the poor. When capital accounts are opened at the fairly rich state $W'$, the market adjustment drives the

\textsuperscript{17} Some empirical evidence is also consistent with the late-liberalization view. Chinn and Ito (2006) find that financial openness promotes development of domestic financial markets only if a threshold level of legal development has been attained. Bekaert et al, (2001) find that financial liberalization tends to promote economic growth and the effect is greater for countries with high education levels in their sample of 30 emerging countries.
equilibrium to $W_i$, at which the poor runs capital outflows, cutting back the capital stock. Afterwards the world economy goes on the ellipse from $W_i$ to $I$, with the poor running capital inflows. Indeed, capital flows from the rich to the poor tends to speed up for convergence. If $W'$ is near $W_i$, the effect of fastening convergence tends to be greater.

We have some evidence of the reversal of capital flows. In the 1980s, a number of Latin American countries ran capital outflows, involving capital flights, but in the 1990s, ran capital inflows. The reversal arose from the decline in the world interest rate driven by the change in the global saving-investment balance, not by any domestic institutional change.\(^{19}\)

6: Other Related Problems

6-1: Change in Parameters and Equilibria

How configuration of steady states changes when parameters change is an interesting topic to study. The dotted curve (10) and the ellipse (12) characterize several configurations of equilibria.

The curve (10) depends on the technology level $A$, the productivity of investment project $R$ and the population of investors $\alpha$ that implies a measure of the global saving.\(^{20}\) As either of these parameters gets greater, the aggregate capital stock available to both countries get so great that the curve (10) tends to expand outwardly.

---

\(^{19}\) Bartolini and Drazen (1997) explain episodes of capital reversal for Italy, Spain, New Zealand, and Uruguay by emphasizing the signaling argument of capital account liberalization as a commitment to policy reforms. Their opening timing may have been coincident with the stage of capital reversal.

\(^{20}\) Consider the Diamond (1965)’s overlapping generation economy with two-period-lived agents. The preference of an agent is represented by $\log C_t^y + \beta \log C_t^o$, where $C_t^y(C_t^o)$ denote consumption when young (old), and $\beta$ is the subjective discount factor. The saving rate is then $\beta/(1 + \beta)$. The counterpart of (7) is described as $k_{i+1}^Y + k_{i+1}^X = \frac{\beta}{1 + \beta} \{W(k_i^Y) + W(k_i^X)\}$ so that $\alpha$ is interpreted as the direct measure of a country’s saving rate.
The ellipse (12) depends on $R$ and $\lambda$, but not on $A$. As $\lambda$ is smaller, the ellipse gets larger, with $\lambda R$ getting smaller and $K(\lambda)$ getting greater, put differently, the borrowing-constrained region being expanded. Conversely, as $\lambda$ is greater, the ellipse shrinks, and disappears at the threshold $\lambda^*$.21

As either $A$, $\lambda$, or $\alpha$ increases, the three-steady-state equilibria first emerges, second the five-steady-state equilibria, and finally the unique-steady-state equilibria. Symmetry-breaking is likely to arise if either of these parameters or all are small, whereas convergence is if they are great. In some parameter space both symmetry-breaking and convergence occur.

6-2: Sunspots and Capital Reversal with Leapfrogging

We have thus far focused on the case when the poor continues to be poorer than the rich, but our system does not exclude the case when the poor overtakes the rich. For example, in the case of three-steady-state equilibria, we argued that the equilibrium is settled down to $B_1$ as depicted in Figure 5-2A, but we may not exclude either $B_2$ or $B_3$, particularly $B_3$, to be chosen if investors react perversely to sunspot or confidence in the market ($B_2$ is unstable in many adjustment rule).

Suppose that the sunspot variable is a Markov chain with two states, $\{n, s\}$. When the state is $n$, all agents coordinate their expectations on the equilibrium with $k^N_i > k^S_i$. When the state is $s$, they do on the equilibrium with $k^N_i < k^S_i$. Let the transition probability in which the state $n$ (or $s$) occurs next period given that the current state is $n$ (or $s$) is $\delta$, with $\delta \to 1$. We may then approximate the sunspot economy by the deterministic model. We may have a case of leapfrogging in wealth between countries, with sudden capital reversal. The swing in the financial market leads to a sudden and great reversal of capital flows, as in the case of Asian financial crisis. However, except for that the equilibrium no longer inherits the initial difference in wealth, nothing

---

21 A rise in $R$ increases not only $\lambda R$ but also $K(\lambda)$ so that the effect of a change in $R$ on
essential is lost due to the symmetric property of the model. A forward-looking nature of collateral value can generate sunspot equilibria, but the predetermined global saving limits the choice to two solutions, \((k_a, k_b)\) and \((k_b, k_a)\) \((k_a \neq k_b)\). The rich and the poor may alternate with each other, but the dynamic evolution of the high and low capital stocks remains unchanged, and the emergence of sunspot equilibria does not influence when and under what conditions the globalization magnifies and lessens symmetry-breaking.

6-3: Political Economy of Financial Integration

One of important advantages of the global analysis is to provide a framework for studying the political equilibrium on financial integration. Interestingly, we may find how the preference for integration changes between the rich and the poor and over time.

Suppose that the aim of a country is to attain the higher steady state stock of capital as fast as possible. In addition, suppose that integration occurs only if either of the countries is at least better off given that other country is not worse off. As an example, we consider the five-steady-state case. In early stages of development, the rich prefers integration so as to exploit the gain of symmetry-breaking, but the poor incurs the loss by symmetry-breaking and dislikes it. Financial integration never occurs. In later stages, opening will promote faster convergence, with capital flows from the rich to the poor. Both the poor and the rich will prefer integration. Financial integration occurs in late stages as a result of political equilibrium.

Conclusion

We have explored the global dynamics of financial integration and provided a number of implications on when and how financial integration gives rise to successful outcomes. There are threshold levels of wealth only above which financial integration leads to convergence. This theoretical feature can explain why capital flows smoothly among high-income countries but not for low-income countries, and further why

the impact on (12) is ambiguous.
countries join the club of integration only when they become rich. Our model is expected to provide a useful benchmark for thinking the architecture of financial integration.

One promising direction of research is to enrich implications on the causal relationship between financial integration and institutional quality. We have narrowly captured the institutional development as the agency cost lessened by the improvement of borrowers’ balance sheets, but can broadly define institutional quality as influenced by enforcement, law, and governance that will be progressed as a greater number of skilled people can apply expertise and know-how of advanced countries to improve their institutional quality. One simple way to capture the latter idea is to model the enforcement parameter $\lambda$ to be an increasing function of $k_{t+1}$. The increasing $\lambda(.)$ function may then provide another route that generates the inversed-U relation between the interest rate and the capital stock, which turns out that the forward-looking nature of collateral will give richer collateral arguments for financial integration while preserving virtually the same implications as ours (e.g., Bartolini and Drazen, 1997, Kose et al, 2006, and Prasad and Rajan, 2008). Along this line, the idea that foreign capital itself becomes the collateral to discipline the policies and governance of borrowing countries is interesting.

The argument for the composition of capital flows will be promising. Particularly, the distinction between foreign direct investment (FDI) and other flows will be a promising avenue to extend this model since FDI is now a dominant vehicle for foreign finance to emerging market countries. FDI differs from other flows in that it involves the transfer of knowledge and technology, and is featured by the governance structure of the large equity participation associated with the transfer of the control right. The extended framework will explain the actual pattern and composition of capital flows and its relationship with financial integration.
References


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**Appendix A: Derivation and Some Properties of (11)**

We investigate six asymmetric regions in order. In region (II), the no-arbitrage condition (8) should satisfy

\[ \lambda R f'(k_{r+1}^N) + W(k_{r+1}^N) = \lambda R f'(k_{r+1}^S) + W(k_{r+1}^S), \]

where the LHS is increasing for \( \lambda R \leq k_{r+1}^N < K(\lambda) \), and the RHS is decreasing for \( k_{r+1}^S < \lambda R \). There exists a single-valued continuously differentiable function

\[ k_{r+1}^N = \mu(k_{r+1}^S) \]

that satisfies (A.1), with

\[ \frac{d\mu(k_{r+1}^S)}{dk_{r+1}^S} = \frac{f''(k_{r+1}^S)(\lambda R - k_{r+1}^S)}{f''(k_{r+1}^S)(\lambda R - \mu(k_{r+1}^S))} < 0 \]

since \( k_{r+1}^S < \lambda R < k_{r+1}^N = \mu(k_{r+1}^S) \).

In region (III), (8) should satisfy

\[ Rf'(k_{r+1}^N) = \lambda R f'(k_{r+1}^S) + W(k_{r+1}^S), \]
where both sides are decreasing for \( K(\lambda) \leq k_{r+1}^N \) and \( k_{r+1}^S < \lambda R \). There exists a single-valued continuously differentiable function \( k_{r+1}^N = \phi(k_{r+1}^S) \) that satisfies (A.2),

\[
\frac{d\phi(k_{r+1}^S)}{dk_{r+1}^S} = \frac{f''(k_{r+1}^S)}{Rf''(\phi(k_{r+1}^S))} > 0 \quad \text{since} \quad k_{r+1}^S < \lambda R .
\]

In region (IV), \( k_{r+1}^N \) and \( k_{r+1}^S \) should satisfy (A.1) again. The LHS of (A.1) is decreasing for \( k_{r+1}^N < \lambda R \), and the RHS is increasing for \( \lambda R \leq k_{r+1}^S < K(\lambda) \). From the mirror image of region (II), there exists a single-valued continuously differentiable function \( k_{r+1}^S = \mu(k_{r+1}^N) \) that satisfies (A.1). We have the inverse image of

\[
k_{r+1}^N = \mu^{-1}(k_{r+1}^S) , \quad \text{with} \quad \frac{d\mu^{-1}(k_{r+1}^S)}{dk_{r+1}^S} = \frac{f''(k_{r+1}^S)(\lambda R - k_{r+1}^S)}{f''(\mu^{-1}(k_{r+1}^S))(\lambda R - \mu^{-1}(k_{r+1}^S))} < 0 \quad \text{since} \quad \mu^{-1}(k_{r+1}^S) = k_{r+1}^N < \lambda R < k_{r+1}^S .
\]

In region (VI), (8) should satisfy (A.2). The LHS is decreasing for \( K(\lambda) \leq k_{r+1}^N \), and the RHS is increasing for \( \lambda R \leq k_{r+1}^S < K(\lambda) \). There exists a single-valued continuously differentiable function \( k_{r+1}^N = \phi(k_{r+1}^S) \) that satisfies (A.2), with

\[
\frac{d\phi(k_{r+1}^S)}{dk_{r+1}^S} = \frac{f''(k_{r+1}^S)(\lambda R - k_{r+1}^S)}{Rf''(\phi(k_{r+1}^S))} < 0 \quad \text{since} \quad \lambda R < k_{r+1}^S .
\]

In region (VII), (8) should satisfy

\[
(A.3) \quad \lambda R f''(k_{r+1}^N) + W(k_{r+1}^N) = R f'(k_{r+1}^N) ,
\]

where both sides are decreasing for \( k_{r+1}^N < \lambda R \) and \( K(\lambda) \leq k_{r+1}^S \). From the mirror image of region (III), there exists a single-valued continuously differentiable function \( k_{r+1}^S = \phi(k_{r+1}^N) \) that satisfies (A.3). By taking the inverse of \( \phi(\cdot) \), we have \( k_{r+1}^N = \phi^{-1}(k_{r+1}^S) \),

\[
\text{with} \quad \frac{d\phi^{-1}(k_{r+1}^S)}{dk_{r+1}^S} = \frac{f''(\phi^{-1}(k_{r+1}^S))(\lambda R - \phi^{-1}(k_{r+1}^S))}{Rf''(k_{r+1}^S)} > 0 \quad \text{since} \quad \phi^{-1}(k_{r+1}^S) = k_{r+1}^N < \lambda R .
\]
In region (VIII), (8) should satisfy (A.3) again. The LHS of (A.3) is increasing for \( \lambda R \leq k_{r+1}^N < K(\lambda) \), and the RHS is decreasing for \( K(\lambda) \leq k_{r+1}^S \). From the mirror image of region (VI), there exists a single-valued continuously differentiable function \( k_{r+1}^S = \varphi(k_{r+1}^N) \) that satisfies (A.3). We have the inverse image \( k_{r+1}^N = \varphi^{-1}(k_{r+1}^S) \), with

\[
\frac{d\varphi^{-1}(k_{r+1}^S)}{dk_{r+1}^S} = \frac{f''(\varphi^{-1}(k_{r+1}^S))(\lambda R - \varphi^{-1}(k_{r+1}^S))}{Rf''(k_{r+1}^S)} < 0 \quad \text{since} \quad \lambda R < k_{r+1}^N = \varphi^{-1}(k_{r+1}^S).
\]

Next we investigate several properties of the functions argued above.

**Lemma A.1** \( \lim_{k_{r+1}^N \to \lambda R^-} \mu(k_{r+1}^S) = \lim_{k_{r+1}^N \to \lambda R^+} \mu^{-1}(k_{r+1}^S) = \lambda R. \)

**Proof:** We prove first that \( \lim_{k_{r+1}^N \to \lambda R^-} \mu(k_{r+1}^S) = \lambda R. \) Since \( f'(k), W(k) \) are bounded and continuous for \( k \in (0, \lambda R) \), and so is \( \mu(k) \) from the definition of \( \mu(k) \), we have

\[
\lim_{k_{r+1}^N \to \lambda R^-} \left\{ \lambda Rf'(\mu(k_{r+1}^S)) + W(\mu(k_{r+1}^S)) \right\} = \lim_{k_{r+1}^N \to \lambda R^-} \left\{ \lambda Rf'(k_{r+1}^S) + W(k_{r+1}^S) \right\},
\]

and thus

\[
\lambda Rf' \left( \lim_{k_{r+1}^N \to \lambda R^-} \mu(k_{r+1}^S) \right) + W \left( \lim_{k_{r+1}^N \to \lambda R^-} \mu(k_{r+1}^S) \right) = \lambda Rf'(\lambda R) + W(\lambda R). \]

Note that \( \lambda Rf'(k) + W(k) \) is monotone decreasing for \( k \in (0, \lambda R) \) and minimized at \( k = \lambda R \). Then we have \( \lim_{k_{r+1}^N \to \lambda R^-} \mu(k_{r+1}^S) = \lambda R. \) Following the similar manner, we prove that

\[
\lim_{k_{r+1}^N \to \lambda R^+} \mu^{-1}(k_{r+1}^S) = \lambda R. \]

It is immediate that \( \lim_{k_{r+1}^N \to \lambda R^-} \mu(k_{r+1}^S) = \lim_{k_{r+1}^N \to \lambda R^+} \mu^{-1}(k_{r+1}^S) = \lambda R, \)

which establishes the lemma. Q.E.D.

**Lemma A.2** \( \lim_{k_{r+1}^N \to \lambda R^-} \phi(k_{r+1}^S) = \lim_{k_{r+1}^N \to \lambda R^+} \phi(k_{r+1}^S). \)

**Proof:** Since \( f'(k), W(k) \) are bounded and continuous for \( k \in (0, \lambda R) \), and so is \( \phi(k) \) from the definition of \( \phi(k) \),

\[
\lim_{k_{r+1}^N \to \lambda R^-} Rf'(\phi(k_{r+1}^S)) = \lim_{k_{r+1}^N \to \lambda R^-} \left\{ \lambda Rf'(k_{r+1}^S) + W(k_{r+1}^S) \right\}.
\]

This implies that \( Rf'(\lim_{k_{r+1}^N \to \lambda R^-} \phi(k_{r+1}^S)) = \lambda Rf'(\lambda R) + W(\lambda R). \) A similar argument reveals
that \( RF'(\lim_{k^S_{+1} \to \lambda R} \varphi(k^S_{+1})) = \lambda RF'(\lambda R) + W(\lambda R) \). Since \( f'(k) \) is monotone decreasing,

\[
\lim_{k^S_{+1} \to \lambda R^-} \varphi(k^S_{+1}) = \lim_{k^S_{+1} \to \lambda R^+} \varphi(k^S_{+1}) \]

must hold, which establishes the lemma. Q.E.D.

Lemma A.3

\[
\lim_{k^S_{+1} \to K(\lambda)^-} \mu^{-1}(k^S_{+1}) = \lim_{k^S_{+1} \to K(\lambda)^+} \phi^{-1}(k^S_{+1}).
\]

Proof: Since \( f'(k) \) and \( W(k) \) are bounded and continuous for \( k \in [\lambda R, K(\lambda)] \), and so is \( \mu(k) \), we have

\[
\lim_{k^S_{+1} \to K(\lambda)^-} \left\{ \lambda RF'(\mu^{-1}(k^S_{+1})) + W(\mu^{-1}(k^S_{+1})) \right\} = \lim_{k^S_{+1} \to K(\lambda)^-} \left\{ \lambda RF'(k^S_{+1}) + W(k^S_{+1}) \right\},
\]

and so

\[
\lambda RF' \left( \lim_{k^S_{+1} \to K(\lambda)^-} \mu^{-1}(k^S_{+1}) \right) + W \left( \lim_{k^S_{+1} \to K(\lambda)^-} \mu^{-1}(k^S_{+1}) \right) = \lambda RF'(K(\lambda)) + W(K(\lambda)) = RF'(K(\lambda)),
\]

where the last equality follows from the definition of \( K(\lambda) \). On the other hand, since \( f'(k) \) and \( W(k) \) are bounded and continuous for \( k \in [K(\lambda), +\infty) \), and so is \( \phi^{-1}(k) \),

\[
\lim_{k^S_{+1} \to K(\lambda)^+} \left\{ \lambda RF'(\phi^{-1}(k^S_{+1})) + W(\phi^{-1}(k^S_{+1})) \right\} = \lim_{k^S_{+1} \to K(\lambda)^+} RF'(k^S_{+1})
\]

holds. This implies that

\[
\lambda RF' \left( \lim_{k^S_{+1} \to K(\lambda)^-} \phi^{-1}(k^S_{+1}) \right) + W \left( \lim_{k^S_{+1} \to K(\lambda)^-} \phi^{-1}(k^S_{+1}) \right) = RF'(K(\lambda)).
\]

Then it follows that

\[
RF'(K(\lambda)) = \lambda RF' \left( \lim_{k^S_{+1} \to K(\lambda)^-} \mu^{-1}(k^S_{+1}) \right) + W \left( \lim_{k^S_{+1} \to K(\lambda)^-} \mu^{-1}(k^S_{+1}) \right)
\]

\[
= \lambda RF' \left( \lim_{k^S_{+1} \to K(\lambda)^+} \phi^{-1}(k^S_{+1}) \right) + W \left( \lim_{k^S_{+1} \to K(\lambda)^+} \phi^{-1}(k^S_{+1}) \right).
\]

Since both \( \lim_{k^S_{+1} \to K(\lambda)^+} \phi^{-1}(k^S_{+1}) \) and \( \lim_{k^S_{+1} \to K(\lambda)^-} \mu^{-1}(k^S_{+1}) \) are less than \( \lambda R \), and

\[
\lambda RF'(k) + W(k) \]

is decreasing for \( k < \lambda R \), \( \lim_{k^S_{+1} \to K(\lambda)^-} \mu^{-1}(k^S_{+1}) = \lim_{k^S_{+1} \to K(\lambda)^+} \phi^{-1}(k^S_{+1}) \)

must hold. Q.E.D.

Lemma A.4

\[
\lim_{k^S_{+1} \to \mu^-(K(\lambda)^-)} \mu(k^S_{+1}) = \lim_{k^S_{+1} \to \phi^-(K(\lambda)^+)} \phi(k^S_{+1}).
\]

Proof: From the symmetric property, it is straightforward from Lemma A.3.
Lemma A.5 \( \lim_{k_{t+1}^S \to \partial (-\lambda R)} \phi^{-1}(k_{t+1}^S) = \lim_{k_{t+1}^S \to \partial (+\lambda R)} \phi^{-1}(k_{t+1}^S). \)

Proof: From the symmetric property, it is straightforward from Lemma A.2.

Lemma A.6 \( \lim_{k_{t+1}^S \to K(\lambda)-} \phi(k_{t+1}^S) = K(\lambda) = \lim_{k_{t+1}^S \to K(\lambda)+} \phi^{-1}(k_{t+1}^S). \)

Proof: Since \( f'(k), W(k), \) and \( \phi(k) \) are bounded and continuous for \( k \in [\lambda R, K(\lambda)], \) we have \( \lim_{k_{t+1}^S \to K(\lambda)-} Rf'(\phi(k_{t+1}^S)) = \lim_{k_{t+1}^S \to K(\lambda)-} \left\{ \lambda Rf'(k_{t+1}^S) + W(k_{t+1}^S) \right\}, \) and

\[ Rf' \left( \lim_{k_{t+1}^S \to K(\lambda)-} \phi(k_{t+1}^S) \right) = \lambda Rf'(K(\lambda)) + W(K(\lambda)) = Rf'(K(\lambda)) \]

from the definition of \( \phi(k). \) Since \( f'(k) \) is monotone decreasing, \( \lim_{k_{t+1}^S \to K(\lambda)+} \phi(k_{t+1}^S) = K(\lambda) \) must hold. We also prove that \( \lim_{k_{t+1}^S \to K(\lambda)-} \phi^{-1}(k_{t+1}^S) = K(\lambda) \) in the similar manner. Q.E.D.

Appendix B: Configuration of (11) under the Cobb-Douglas Production Function

We show that when the production function is the Cobb-Douglas form,

\( \mu'(k_{t+1}^S) < -1 \) holds and \( \phi'(.) > -1 \) is satisfied for plausible parameters. First of all, we show that \( \mu'(k_{t+1}^S) < -1 \) holds when the production function is the Cobb-Douglas form.

Now let us define \( H(k) = \lambda R f'(k) + W(k). \) Remember that \( H(k) \) is decreasing for \( k < \lambda R, \) while increasing for \( k > \lambda R. \) Then for any \( k^1 \in (0, \lambda R), \) there exists a level of capital \( k^2 \in (\lambda R, \infty) \) that satisfies \( H(k^1) = H(k^2). \) This implies that \( k^2 = \mu(k^1). \)

Since \( \mu'(k^1) = H'(k^1) / H'(k^2), \) \( \mu'(k^1) < -1 \) is equivalent to \( -H'(k^1) > H'(k^2). \) Thus, below we derive the sufficient condition for \( -H'(k^1) > H'(k^2) \) to hold for any \( k^1 \in (0, \lambda R) \) and \( k^2 \in (\lambda R, \infty) \) that satisfies \( H(k^1) = H(k^2). \) and show that the sufficient condition is satisfied under the Cobb-Douglas production function. To do so, we first establish the following lemma.

Lemma B.1
For any $k^1 \in (0, \lambda R)$ and $k^2 \in (\lambda R, \infty)$ that satisfies $H(k^1) = H(k^2)$, $k^1 + k^2 > 2\lambda R$.

**Proof:** Since $W''(k) < 0$ implies that $f'''(k) > 0$, that is, $f'''(\cdot)$ is increasing,

$$-H'(\lambda R - \varepsilon) = -\varepsilon f''(\lambda R - \varepsilon) > -\varepsilon f''(\lambda R + \varepsilon) = H'(\lambda R + \varepsilon)$$

holds for any $\varepsilon \in (0, \lambda R)$.

Then for any $\varepsilon \in (0, \lambda R)$ we have

$$\int_0^\varepsilon -H'(\lambda R - \varepsilon)d\varepsilon > \int_0^\varepsilon H'(\lambda R + \varepsilon)d\varepsilon$$

by integration, and thus $H(\lambda R - \varepsilon) > H(\lambda R + \varepsilon)$. Now letting $k^1 = \lambda R - \varepsilon$, we can set $k^2$ to satisfy $H(k^2) = H(\lambda R - \varepsilon) > H(\lambda R + \varepsilon)$. Since $H(k)$ is increasing if $k > \lambda R$, $k^2 > \lambda R + \varepsilon$ should be met. Therefore, $k^1 + k^2 > 2\lambda R$ holds. Q.E.D.

By using this lemma, we prove the following lemma.

**Lemma B.2**

Suppose that $H'''(k) > H'''(\lambda R)$ for any $k \in (0, \lambda R)$, and that $H'''(k) < H'''(\lambda R)$ for any $k > \lambda R$. Then for any $k^1 \in (0, \lambda R)$ and $k^2 \in (\lambda R, \infty)$ that satisfies $H(k^1) = H(k^2)$, $-H'(k^1) > H'(k^2)$.

**Proof:** Suppose to the contrary that $-H'(k^1) \leq H'(k^2)$ for some $k^1$ and $k^2$ that satisfies $H(k^1) = H(k^2)$. Now define $\Gamma(\varepsilon) = -H'(k^1 + \varepsilon) - H'(k^2 - \varepsilon)$ for any $\varepsilon \in [0, \lambda R - k^1]$. Note from Lemma B.1 that $k^2 - (\lambda R - k^1) > \lambda R$, and thus $k^2 - \varepsilon > \lambda R$ for any $\varepsilon \in [0, \lambda R - k^1]$. Then under the condition of Lemma B.2, $\Gamma'(\varepsilon) = -H'''(k^1 + \varepsilon) + H'''(k^2 - \varepsilon) < 0$ for any $\varepsilon \in [0, \lambda R - k^1]$. Additionally, we are supposing that $\Gamma(0) = -H'(k^1) - H'(k^2) \leq 0$. Thus, for any $\varepsilon \in [0, \lambda R - k^1]$, we obtain

$$-H'(k^1 + \varepsilon) < H'(k^2 - \varepsilon), \text{ and } \int_{k^1}^{\lambda R - k^1} -H'(k^1 + \varepsilon)d\varepsilon < \int_{k^1}^{\lambda R - k^1} H'(k^2 - \varepsilon)d\varepsilon$$

by integration, which implies that $H(k^1) - H(\lambda R) < H(k^2) - H(k^2 - (\lambda R - k^1))$, or equivalently $H(\lambda R) > H(k^2 - (\lambda R - k^1))$ since $H(k^1) = H(k^2)$. The above inequality contradicts the fact that $k = \lambda R$ minimizes $H(k)$. This establishes the lemma. Q.E.D.

Hereafter we show that the condition of Lemma B.2 is satisfied when the production function is the Cobb-Douglas form. Assuming $f(k) = Ak^\beta$, with $\beta \in (0, 1)$, the simple algebra gives us $H(k) = A[\lambda R f(k) + (1 - \beta)k^\beta]$, $H'(k) = A\beta(\beta - 1)(\lambda R - k)k^{\beta-2}$, and
\[ H''(k) = A\beta(\beta - 1)[(1 - \beta)k - (2 - \beta)\lambda R]k^{\beta - 3}. \] Thus, \( H''(k) > 0 \) holds if \( k < \{(2 - \beta)/(1 - \beta)\}\lambda R \), while \( H''(k) < 0 \) holds if \( k > \{(2 - \beta)/(1 - \beta)\}\lambda R \). Note that \( H''(\lambda R) = A\beta(1 - \beta)(\lambda R)^{\beta - 2} > 0 \).

Moreover, we have \( H'''(k) = A\beta(\beta - 1)(\beta - 2)[(1 - \beta)k - (3 - \beta)\lambda R]k^{\beta - 4} < 0 \) if and only if \( k < \{(3 - \beta)/(1 - \beta)\}\lambda R \). Since \( \{(3 - \beta)/(1 - \beta)\}\lambda R > \{(2 - \beta)/(1 - \beta)\}\lambda R \), \( H''(k) \) is decreasing in \( k \) so long as \( H''(k) > 0 \). Thus, when \( f(k) = Ak^{\beta} \), \( H''(k) > H''(\lambda R) \) for any \( k \in (0, \lambda R) \) and \( H''(k) < H''(\lambda R) \) for any \( k > \lambda R \).

Next we show that for any \( k_{r+1} \in (\lambda R, K(\lambda)) \), \( \varphi(\cdot) > -1 \) is satisfied for plausible parameters when the production function is the Cobb-Douglas form. Assuming \( f(k) = Ak^{\beta} \), we have \( \varphi(k_{r+1}^S) = \left[ \lambda(k_{r+1}^S)^{\beta - 1} + \left(\frac{(1 - \beta)}{\beta \lambda R}\right)k_{r+1}^S \right]^{1/\beta - 1} \) for \( k_{r+1}^S \in (\lambda R, K(\lambda)) \). Note that \( \varphi(\cdot) < 0 \) holds for \( k_{r+1}^S > \lambda R \), \( \varphi''(\cdot) = 0 \) holds for \( k_{r+1}^S = 2\lambda R \), and \( \varphi''(\cdot) < 0 \) holds for \( k_{r+1}^S < 2\lambda R \), while \( \varphi''(\cdot) > 0 \) holds for \( k_{r+1}^S > 2\lambda R \). Thus, for \( \varphi(\cdot) > -1 \) to hold for any \( k_{r+1}^S \in (\lambda R, K(\lambda)) \), either (i) \( K(\lambda) \leq 2\lambda R \), or (ii) both \( 2\lambda R < K(\lambda) \) and \( \varphi(2\lambda R) > -1 \) must be satisfied, in addition to \( \lim_{k_{r+1}^S \to K(\lambda)^{-}} \varphi'(k_{r+1}^S) > -1 \). Since we have \( K(\lambda) = \left(1 - \lambda\right)\beta R / (1 - \beta) \), \( \lambda < \beta \) must be met for \( \lambda R < K(\lambda) \) to hold. We have \( \varphi'(k_{r+1}^S) = \frac{\lambda R - k_{r+1}^S}{R[(1 - \lambda)\beta R / (1 - \beta)^2]} \), which implies that \( \lim_{k_{r+1}^S \to K(\lambda)^{-}} \varphi'(k_{r+1}^S) = \varphi'(K(\lambda)) = \lambda - \beta(1 - \lambda)/(1 - \beta) \). Thus, for \( \lim_{k_{r+1}^S \to K(\lambda)^{-}} \varphi'(k_{r+1}^S) > -1 \) to hold, \( \lambda > 2\beta - 1 \) must hold. \( K(\lambda) \leq 2\lambda R \) is satisfied if \( \lambda \geq \beta(2 - \beta) \) holds. On the other hand, both \( 2\lambda R < K(\lambda) \) and \( \varphi(2\lambda R) > -1 \) are satisfied if \( \frac{\beta}{2 - \beta} < \lambda < 1 \). Thus, \( \varphi(\cdot) > -1 \) is satisfied for all \( k_{r+1}^S \in (\lambda R, K(\lambda)) \) if \( \max\{\frac{\beta}{2 - \beta}, 2\beta - 1\} < \lambda < 1 \) holds.
Figure 3-1

Figure 3-2

Figure 3-3
Figure 4-3

\[ \kappa_{t+1} = k_{t+1}^S + k_{t+1}^N \]

Figure 4-4A
Figure 5-3B

\[ R\alpha(W(k_T^N) + W(k_T^S)) \]

Figure 5-3C
Figure 5-4B