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The Strategic Cores α, β, γ and δ

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Abstract

In a strategic cooperative game, we consider four cores α, β, γ and the one we call δ which is essentially the same to conjectural cooperative equilibria due to Currarini and Marini [2]. We show that if every player has a dominant strategy, the β-core includes the γ-core, and therefore that the four cores refine themselves in the greek alphabetical order. Two examples will be examined to see how the refinement is realized. While no strict refinement is attained at all in the pure exchange game, a radical reduction of the α-core is obtained in the commons game, a simple version of the Cournot game, bringing about a single strategy profile as the δ-core.

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The Strategic Cores α , β , γ and δ

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In a strategic cooperative game, we consider four cores $\alpha, \beta, \gamma,$ and the one we call δ which is essentially the same to *conjectural cooperative equilibria* due to Currarini and Marini [2]. We show that if every player has a dominant strategy, the $\beta-$core includes the $\gamma-$core, and therefore that the four cores refine themselves in the greek alphabetical order. Two examples will be examined to see how the refinement is realized. While no strict refinement is attained at all in the pure exchange game, a radical reduction of the $\alpha-$core is obtained in the commons game, a simple version of the Cournot game, bringing about a single strategy profile as the $\delta-$core.

Keywords: strategic cores, dominant strategy, $S-$Pareto Nash equilibria, $\gamma-$core, $\delta-$core, pure exchange game, commons game

1 Introduction

The $\alpha-$ and $\beta-$ cores defined in Aumann and Peleg [1] are the most classical cooperative solution concepts in games of strategic form. These cores can be viewed as generalizations of the 'maxmin' and 'minimax' payoffs, respectively, to coalitional multi-person games. One of the most important development on these concepts is the existence result of the $\alpha-$core due to Scarf [7]. But, it is also true that the 'maxmin' behavior of a coalition is too conservative in situations where the conflict between the members and the non-members of a coalition is not very hard. In the literature, there can be found several alternative core concepts other than the $\alpha-$ and $\beta-$cores. The $\gamma-$*core* defined by Chander and Tulkens [3], and the *conjectural cooperative equilibrium* defined by Currarini and Marini [2] are the typical examples.

These new cores are defined in an attempt to obtain more appropriate formulations of how a coalition can improve its state in the 'non zero-sum-like' situations. The $\gamma-$core due to Chander and Tulkens [3] is based on the idea that each of the players outside a coalition simply takes the coalition as given, thereby just best replying each other, yielding a Nash equilibrium including the coalition as a player.

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If no coalition can be made better off by such a Nash equilibrium, then the $\gamma-$core is obtained. They applied this concept to a specific game describing an economy with detrimental externalities, and proved its existence for the game.

The conjectural cooperative equilibrium is still another reformulation of behavior of players in and ouside a coalition. It differs from the $\gamma-$notion only in that any outsider of a coalition always best replies each other *given* any strategy choice of the coalition. Thus, the equilibrium amounts to a 'multi-person Stackelberg equilibrium' where a coalition is the leader and each of the players outside the coalition is the follower. Currarini and Marini [2] proved its existence for symmetric supermodular games (Topkis [8]), and presented an analysis for an environmental Cobb-Douglas economy to show how the existence of the equilibrium is affected by the parameters of the model.

In this paper, we shall put these four cores together in a strategic game and show that they are in the appealing consecutive inclusion relation. It is well-known and straightforward by definition that $\alpha-$core includes the $\beta-$core; and, it is also immediately shown by definition that the $\gamma-$core includes the $\delta-$core. But, whether or not the $\beta-$core includes the $\gamma-$core is not known so far in the literature. If this inclusion is obtained, the four cores α, β, γ and δ refine themselves in this order. We will show that the existence of a *dominant strategy equilibrium* is sufficient for the inclusion.

Two examples of economic games will be examined to see if the refinement is strict. One is the pure exchange game discussed in Scarf [7], in which the four cores turn out to be equal with no strict refinement at all; and the other is a simple 'commons game', a version of the Cournot game, that reveals the strict refinement in the order of β, γ and δ, leading to a unique strategy profile.

2 Preliminaries

In this paper, a *game in strategic form*, or simply, a *strategic game* is a list $G = (N, (X^i)_{i\in N},\ (u_i)_{i\in N})$ where N is a finite set of players, X^i is a nonempty compact set of strategies of player $i \in N$ and u_i is a real-valued continuous payoff funtion from $\prod_{i\in N} X^i$. We assume for each $i \in N$ that u_i is *quasiconcave* in $x^i \in X^i$ in order to guarantee the existence of a Nash equilibrium.

Any nonempty set $S \in 2^N$ will be called a *coalition*. For each coalition S, let $X^S := \prod_{i\in S} X^i$ be the set of strategies available to coalition S. We put $X := X^N$, and identify X^i with $X^{\{i\}}$.

Given $x \in X$ and any coalition S, we say that S can *deviate* at x if there is a $y^S \in X^S$ such that $u_i(y^S, x^{N\setminus S}) > u_i(x)\ \ \forall i \in S$. When N cannot deviate at x, the strategy profile x is said to be *weakly Pareto efficient*. If no coalition S can deviate at x, x is called a *strong Nash equilibrium*. Recall also that a strategy profile $x \in X$ is a *dominant strategy equilibrium* if for each $i \in N$, x^i is a best reply against *any* $y^{N\setminus\{i\}} \in X^{N\setminus\{i\}}$.

Given $x \in X$, we say that coalition S can *α-improve* at x if there exists a $y^S \in X^S$ such that for all $z \in X$, $u_i(y^S, z^{N\setminus S}) > u_i(x)$, $\forall i \in S$. Strategy profile $x \in X$

is said to be an α*-core strategy profile* if no coalition can α-improve at x. The set of all $\alpha-$core strategy profiles will be called the $\alpha-$*core*.

Similarly, given $x \in X$, we say that coalition S can β*-improve* at x if for any $z \in X$, there exists a $y^S \in X^S$ such that $u_i(y^S, z^{N\setminus S}) > u_i(x)$, $\forall i \in S$. Strategy profile $x \in X$ is said to be a β*-core strategy profile* if no coalition can β-improve at x. The set of all $\beta-$core strategy profiles will be called the $\beta-$*core*. By definition, the β-core is a subset of the α-core.

3 Strategic Cores and Their Relations

As solutions of a strategic game G, we now define the $\gamma-$core and the $\delta-$core in an appropriate way, and state general relations among these four core concepts.

Definition 1. *Given a coalition $S \subseteq N$, strategy profile $y = (y^S, y^{N\setminus S}) \in X$ is said to be an $S-$Pareto Nash equilibrium if for every $j \in N \setminus S$ and S there is no deviation at y.*

Note that when $|S| = 1$, the $S-$Pareto Nash equilibrium is a usual Nash equilibrium, whereas when $S = N$, it is simply a weakly Pareto efficient strategy profile.

Definition 2. *Given $x \in X$, we say that coalition S can γ-improve at x if there exists a strategy profile $y = (y^S, y^{N\setminus S}) \in X$ such that*

1. *y is an $S-$Pareto Nash equilibrium,*
2. $u_i(y) > u_i(x)$, $\forall i \in S$.

Following Chander and Tulkens [3], the $\gamma-$improvement formalizes the situation that when coalition S forms at x, each of the outsiders simply takes the coalition as given; and therefore, S and each of the outsiders play the noncooperative game yielding an $S-$Pareto Nash equilibrium. If coalition S can thereby make each of its members better off, S can $\gamma-$ improve at x.

Definition 3. *Strategy profile $x \in X$ is said to be a γ-core strategy profile if no coalition can γ-improve at x. The set of all $\gamma-$core strategy profiles is said to be the $\gamma-$core.*

Given any strategy profile $y \in X$ and any coalition S, the subgame $G(S|y^{N\setminus S})$ of G is defined to be the game $(S, (X^i)_{i\in S}, (u_i(\cdot,\ y^{N\setminus S}))_{i\in S})$.

Definition 4. *Given $x \in X$, we say that coalition S can δ-improve at x if there exists a strategy profile $y = (y^S, y^{N\setminus S}) \in X$ such that*

1. *$y^{N\setminus S}$ is a Nash equilibrium in the subgame $G(N \setminus S\,|\,y^S)$,*
2. $u_i(y) > u_i(x)$, $\forall i \in S$.

Similar to the $\gamma-$improvement, the $\delta-$improvement describes the situation in which each of the outsiders takes the coalition as given. But unlike the $\gamma-$ improvement, the coalition here is in a stronger position just like the Stackelberg leader in an oligopolistic market.

Definition 5. *Strategy profile $x^\dagger$ is said to be a δ- core strategy profile if no coalition can δ-improve at $x^\dagger$. The set of all $\delta-$core strategy profiles is said to be the $\delta-$core.*

The $\delta-$core strategy profile as defined above is a weaker version of the conjectural cooperative equilibrium given by Currarini and Marini [2].

The importance of these new two cores lies in the relation to the classical cores, which can be stated as follows.

Theorem 3.1.

1. *$\delta-$core $\subseteq \gamma-$core $\bigcap$ $\alpha-$core.*
2. *If every player has a dominant strategy, then we have that*

$$\gamma - core \subseteq \beta - core,$$

and therefore that $\delta - core \subseteq \gamma - core \subseteq \beta - core \subseteq \alpha - core.$

Proof. To show 1, let us suppose that coalition S can γ-improve at x. Then, there is an $S-$Pareto Nash equilibrium $y = (y^S, y^{N\setminus S})$ such that $u_i(y) > u_i(x)$ for all $i \in S$. Since $y^{N\setminus S}$ is a Nash equilibrium in the subgame $G(N \setminus S \mid y^S)$, coalition S can δ-improve at x. Moreover it immediately follows from the fact that $\alpha-$improvement at x amounts to an improvement at x against any Nash equilibrium in $G(N \setminus S \mid y^S)$.

To show 2, assume that S can $\beta-$improve at x. Take the dominant strategy profile $x^* \in X$. Then, S has a strategy $z^S \in X^S$ such that

$$u_i(z^S, x^{*N\setminus S}) > u_i(x), \quad \forall i \in S.$$

Since, for any $y \in X$, $x^{*N\setminus S}$ is a Nash equilibrium in the subgame $G(N \setminus S \mid y^S)$, the strategy profile $(z^S, x^{*N\setminus S})$ is an $S-$Pareto Nash equilibrium by taking $z^S \in X^S$ to be weakly Pareto efficient in the subgame $G(S \mid x^{*N\setminus S})$. Hence, S can $\gamma-$improve at x. $\square$

Existence of a dominant strategy equilibrium is, of course, not always assured. However, the pure exchange game appeared in Scarf [7], which is one of the typical economic strategic games, does have a dominant strategy equilibrium. We now turn to the analysis of this game, the first example.

4 The Pure Exchange Game

The pure exchange game due to Scarf [7] is given by $G = (N, (X^i)_{i \in N}, (u_i)_{i \in N})$ where

$$X^i = \Big\{x^i \in \Re^{nm}_+ \mid x^i = (x^{i1}, \dots, x^{in}),$$
$$x^{ij} = (x^{ij}_1, \dots, x^{ij}_m);\ \sum_{j \in N} x^{ij}_h = w^i_h,\ h = 1, \dots m\Big\},$$

is the strategy set of player $i \in N$. Here, $w^i = (w^i_1, \dots, w^i_m) \in \Re^m_+$ is the initial endowments to player i. For any strategy profile $x \in X$, the payoff $u_i(x)$ to each player $i \in N$ is given by

$$u_i(x) = v_i(\sum_{j \in N} x^{ji}),$$

where $v_i(\cdot)$ is assumed to be a continuous, quasiconcave and strictly monotone increasing function.

A vector $\xi = (\xi^1, \dots, \xi^n) \in \Re^{nm}_+$ with $\xi^i = \sum_{j \in N} x^{ji}$ for some $x \in X$ is called an $S-$ *feasible allocation* if it satisfies for the given $S \subseteq N$ that

$$\sum_{i \in S} \xi^i_h \leq \sum_{i \in S} w^i_h, \quad h = 1, \dots, m.$$

Due to the strict monotonicity of utility function $v_i(\cdot)$, the strategy profile $x^\circ \in X$ describing no exchange at all, defined for all $i \in N$ by

$$x^{\circ\ ii} = w^i, \ \text{ and } \forall j \neq i,\ x^{\circ\ ij} = 0 \in \Re^m_+$$

is a dominant strategy equilibrium which is the *only* Nash equilibrium in the game.

We can now state that

Proposition 4.1. *Let G be the pure exchange game. Then,*

$$\emptyset \neq \delta - core\ = \gamma - core\ = \beta - core\ = \alpha - core.$$

Proof. Since there is a dominant strategy equilibrium, it follows from Theorem 3.1 that $\gamma - core \subseteq \beta - core$.

Now, take any $\alpha-$core strategy profile $x^* \in X$ and let ξ^* be the allocation generated by x^*. Then, for any $S \subseteq N$ there exists no $S-$feasible allocation ζ^S such that

$$v_i(\zeta^i) > v_i(\xi^{*i}) = u_i(x^*) \quad \forall i \in S.$$

For all $S \subsetneq N$ and $x^S \in X^S$, the dominant strategy profile $x^{\circ\ N \setminus S}$ is the unique Nash equilibrium in the subgame $G(N \setminus S\ |x^S)$. But, no strategy profile $(x^S, x^{\circ\ N \setminus S})$ can satisfy that

$$u_i(x^S, x^{\circ\ N \setminus S}) > u_i(x^*) \quad \forall i \in S,$$

since for any $x^S \in X^S$, $(x^S, x^{\circ\ N \setminus S})$ generates an $S-$feasible allocation. Hence, no coalition can $\delta-$improve at x, implying that x is a $\delta-$core strategy profile. The equality of all cores therefore follows from Theorem 3.1.

Finally, since $v_i(\cdot)$ is quasiconcave, $u_i(\cdot)$ is also quasiconcave in $x \in X$, which guarantees the existence of $\alpha-$core strategies by Scarf's theorem in [7]. □

Thus, in the pure exchange game, any of the four cores generates the same set of strategy profiles at which no coalition can improve. In this game, if the initial state is not weakly Pareto efficient, only cooperative solutions, the cores in particular, can generate Pareto efficient exchange of goods. This is because the strong Nash equilibrium does not exist when x° is not weakly Pareto efficient (see [4]), and any other noncooperative strategic equilibrium including the *coalition-proof Nash equilibrium* leads to the strategy profile x° describing no exchange at all.

5 The Commons Game

The cores of the pure exchange game exhibit no strict refinement. In this section, however, we can show a definite refinement with the $\delta-$ core being a singleton set. The game is a special case of the Cournot game, called the 'commons game'.

The *commons game* is a strategic game $G = (N, (X^i)_{i\in N}, (u_i)_{i\in N})$ such that

(i) $X^i = \Re_+$ for all $i \in N$

(ii) $u_i(q^1, \dots, q^n) := q^i P\Big(\sum_{k\in N} q^k\Big)$ for all $i \in N$, where

- $q^i \in X^i$ for all $i \in N$
- $P(\sum_{k\in N} q^k) = \max\Big(0,\ a - \sum_{k\in N} q^k\Big)$; and $a > 0$

We write $q(S)$ for $\sum_{k\in S} q^k$, $\forall S \subseteq N$. We may interpret the game as follows. There is a pasture land (commons) where every farmer can use commonly to put their domestic animals, say goats, for grazing. The strategy of each farmer q^i is the number of goats put into pasture land. The profit per one goat from the grazing in pasture land is the decreasing function of the sum of all goats, denoted by $P(q(N))$, which means that an increase in the number of goats results in the decrease of the unit profit.

The unique Nash equilibrium $q^* = (q^{*1}, \dots, q^{*n})$ and the profits at the equilibrium are easily computed as $q^{*i} = \frac{a}{n+1}$ for all $i \in N$, and $u_i(q^*) = \frac{a^2}{(n+1)^2}$ for all $i \in N$, respectively.

As for the α- and β-cores, the worst state of any coalition S is the zero payoff to each of its members independently of any strategy of S. Thus, taking $|S| = 1$ in particular, we see that any strategy profile $x \in X$ is *individually rational*; that is, $u_i(x) \geq 0$ for all $i \in N$. The result in this section is stated as follows.

Proposition 5.1. *Let G be the commons game. Then,*

1. *α−core = β−core = WP,*
   where WP is the set of all weakly Pareto efficient strategy profiles.
2. *δ − core* $= \Big\{\Big(\dfrac{a}{2n}, \dots, \dfrac{a}{2n}\Big)\Big\}.$
3. *δ − core $\subsetneq$ γ − core $\subsetneq$ β − core.*

Proof. 1. Straightforward.

2. Note first that the strategy profile $z := (\frac{a}{2n}, \ldots, \frac{a}{2n})$ is Pareto efficient. This is so because z maximizes the sum $\sum_{i\in N} u_i(\cdot)$ of payoffs feasible for N.

Second, given $x \in X$ and subgame $G(N \setminus S|x^S)$ with $S \neq N$, let $E_{N\setminus S} : X^S \twoheadrightarrow X^{N\setminus S}$ be defined by

$$E_{N\setminus S}(x^S) = \{y^{N\setminus S} \in X^{N\setminus S} \mid y^{N\setminus S} \text{ is a Nash equilibrium in } G(N \setminus S|x^S)\}.$$

In our commons game, it is clear that $E_{N\setminus S}(\cdot)$ is a function. We then show that

$$\max_{q^i\in X^i} u_i\Big(q^i, E_{N\setminus\{i\}}(q^i)\Big) = u_i\Big(\frac{a}{2}, E_{N\setminus\{i\}}(\frac{a}{2})\Big) = \frac{a^2}{4n} \tag{1}$$

If this is proved, we will have that z is the only $\delta-$core strategy profile whenever $(u_i(z))_{i\in N} = \Big(\frac{a^2}{4n}, \ldots, \frac{a^2}{4n}\Big)$ is a $\delta-$core utility profile. This is so because $(u_i(z))_{i\in N}$ is Pareto efficient and

$$u_i(x) = (a - x(N))x^i = \frac{a^2}{4n},\ \forall i \in N \Longrightarrow x^i = \frac{a}{2n} = z^i,\ \forall i \in N.$$

To show (1), let $i \in N$ and $\overline{q}^i \in X^i$ be fixed. Then, it follows by the standard calculation that

$$E_{N\setminus\{i\}}(\overline{q}^i) = \left(\frac{a - \overline{q}^i}{n}\right)_{j\in N\setminus S}. \tag{2}$$

Hence, the player i's payoff is given by

$$u_i(q^i, E_{N\setminus\{i\}}(q^i)) = \Big(a - \frac{(n-1)(a-q^i)}{n} - q^i\Big)q^i,$$

the maximum of which is attained at $\hat{q}^i = \frac{a}{2} \in X^i$. Then, by (2), we have $E_{N\setminus\{i\}}(\hat{q}^i) = \Big(\frac{a}{2n}, \ldots, \frac{a}{2n}\Big)$, and $u_i(\hat{q}^i, E_{N\setminus\{i\}}(\hat{q}^i)) = \frac{a^2}{4n}$, which proves (1).

It remains to show that no coalition S with $|S| < n$ can δ-improve at z. By equation (1), any 1-person coalition cannot δ-improve at z. Take any coalition S with $1 < |S| < n$. Then, it must be shown that for any $q^S \in X^S$,

$$\sum_{i\in S} u_i(z) > \sum_{i\in S} u_i(q^S, E_{N\setminus S}(q^S)) \tag{3}$$

If this is proved, it then follows that no coalition S with $1 < |S| < n$ has a strategy $q^S \in X^S$ satisfying

$$u_i(z) < u_i(q^S, E_{N\setminus S}(q^S)),\quad \forall i \in S,$$

which implies that no coalition S with $1 < |S| < n$ can $\delta-$improve at z. But, letting $s := |S|$ and noting that $u_i(z) = \frac{a^2}{4n}$, and that

$$E_{N\setminus S}(q^S) = \left(\frac{a - q(S)}{n - s + 1}\right)_{j\in N\setminus S},$$

inequality (3) can be shown by direct calculation, which is therefore omitted.

3. We first show that the $\gamma-$core has a strategy profile x other than the $\delta-$core strategy profile z. By (3) and (1), we have for any coalition S with $1 \leq |S| < n$ and any $S-$Pareto Nash equilibrium $y \in X$, that

$$\sum_{i \in S} u_i(z) > \sum_{i \in S} u_i(y) \tag{4}$$

Then, by the continuity of payoff functions, there can be found, in a neighborhood of z, a strategy profile $x \neq z$ such that for any S with $1 \leq |S| < n$,

$$\sum_{i \in S} u_i(x) \geq \sum_{i \in S} u_i(y).$$

Since any neighborhood of z contains weakly Pareto efficient strategy profiles, we can take the above x to be weakly Pareto efficient Hence, x is a $\gamma-$core strategy profile since no coalition $S \subseteq N$ can $\gamma-$improve at x.

Finally, since any $\{i\}-$Pareto Nash equilibrium $q^* \in X$ satisfies that $u_i(q^*) = \frac{a^2}{(n+1)^2} > 0$, any $\gamma-$core strategy profile $x \in X$ must satisfy that $u_i(x) \geq u_i(q^*) > 0$ for all $i \in N$, whereas the $\beta-$core contains a strategy profile $x' \in X$ such that $u_i(x') = 0$ for some $i \in N$. This completes the proof.

□

The profit $\frac{a^2}{4n}$ is the one obtainable by any player i *if player i could act as the unique Stackelberg leader* with all the rest being the followers, which we may call the $\delta-$*individually rational boundary*. The individual rationality usually guarantees each player $i \in N$ the utility level obtainable by i's own effort. But here, the $\delta-$individual rationality just specifies an *opportunity* open equally to every player. In this sense, the $\delta-$core strategy profile of this game might be understood as a *compromise* among the players guaranteeing each other the amount that could be obtained on the equal opportunity of taking the role of the unique Stackelberg leader.

In the commons game, the $\alpha-$ and $\beta-$cores both impose no restrictions on the strategy profiles except for the weak Pareto efficiency. The assumption that any coalition should prepare for its worst state when forming the coalition will be inappropriate in such situations with reactive actions by the outsiders being too costly to the outsiders themselves. The $\gamma-$improvement, on the other hand, seems more appropriate in that the reactive actions may not cause a heavily detrimental state over the outsiders.

In this game, the improvement in terms of γ and δ by any proper coalition with at least two players is so weak that, when dividing the Pareto efficient proceeds among the players, they may resort to the individual rationality. Then, since the $\delta-$individually rational boundary is generally at least as great as the $\gamma-$individually rational boundary, which is just the Nash equilibrium payoff, players may arrive at the compromise on the $\delta-$individual rationality. The unique $\delta-$core strategy profile could thus be attained in this game.

6 Concluding Remarks

The two examples of games considered in this paper are *not* in the class of the super modular games; nevertheless, they each have a nonempty $\delta-$core. This may imply that its existence can be assumed more widely than is established generally.

The $\gamma-$core as well as the $\delta-$core can be empty. In the pure exchange game of *bads* with *strictly decreasing* payoff functions (see, Hirai et al. [4]), it can be shown that the $\gamma-$core is empty when there are at least three players, whereas the $\beta-$core is always nonempty and is equal to the $\alpha-$core. It is also well-understood since Scarf [7] that the $\beta-$core can be empty even if the $\alpha-$core is nonempty.

The existence of the $\beta-$core appears to entail the equivalent $\alpha-$core in many cases as the examples in this paper indicate. Jentzsch [5] called such a game *clear*. As a matter of fact, the *dominant punishment strategy* (see, Masuzawa [6]) generally makes the $\alpha-$improvement equivalent to the $\beta-$improvement. In the pure exchange game, for example, any nonempty proper subset $N \setminus S$ has a strategy $y^{N\setminus S}$ prescribing no offers to S from $N \setminus S$, which is the dominant punishment strategy of $N \setminus S$ against S. Then, if coalition $S \subsetneq N$ can $\beta-$improve at $x \in X$, S can also $\alpha-$improve at x by playing the strategy $z^S \in X^S$ such that $u_i(z^S, y^{N\setminus S}) > u_i(x)$ for all $i \in S$. Thus, by the monotonicity of u_i, coalition S can $\alpha-$improve at x with z^S. Alternatively, the *dominant strategy of a coalition* also makes them equivalent (Hirai et al. [4]).

We do not know whether it is at all possible for the $\beta-$core to effectively refine the $\alpha-$core in meaningful economic games.

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