KEIO UNIVERSITY
KEIO/KYOTO MARKET QUALITY RESEARCH PROJECT
(Global Center of Excellence Project)

KEIO/KYOTO GLOBAL COE DISCUSSION PAPER SERIES

DP2008-019

On Auctions within a Ring of Collusive Bidders

Takayuki OISHI*

Abstract

Rings of collusive bidders at an English auction frequently distribute collusive gains among ring members via sequences of knockouts. This paper presents a model of sequences of knockouts. I investigate the relationship between the distributive outcome of each sequence and each solution of bidding ring games. I show that each sequence of knockouts yields an element of the core. In particular, a sequence of knockouts with the .nest nested rings yields the Shapley value. Also, a sequence of knockouts at each of which a ring can guarantee its advantage for itself yields the nucleolus.

*Takayuki OISHI
Institute of Economic Research, Kyoto University

KEIO/KYOTO MARKET QUALITY RESEARCH PROJECT
(Global Center of Excellence Program)

Graduate School of Economics and Graduate School of Business and Commerce,
Keio University
2-15-45 Mita, Minato-ku, Tokyo 108-8345 Japan

Kyoto Institute of Economics,
Kyoto University
Yoshida-honmachi, Sakyo-ku, Kyoto 606-8501 Japan

On Auctions within a Ring of Collusive Bidders

Takayuki OISHI*

February 22, 2009

Abstract

Rings of collusive bidders at an English auction frequently distribute collusive gains among ring members via sequences of knockouts. This paper presents a model of sequences of knockouts. I investigate the relationship between the distributive outcome of each sequence and each solution of bidding ring games. I show that each sequence of knockouts yields an element of the core. In particular, a sequence of knockouts with the finest nested rings yields the Shapley value. Also, a sequence of knockouts at each of which a ring can guarantee its advantage for itself yields the nucleolus.

JEL classification: C71

Keywords: Bidding rings; Knockout; Core; Shapley value; Nucleolus

1 Introduction

In many auctions, buyers compete with one another in order to purchase goods which a seller wants to sell to one of them. However buyers' competition may be eliminated by a bidding ring. A bidding ring is a group of collusive members who secretly agree not to compete against one another at an auction. Bidding rings have been operated throughout the world. For

*Institute of Economic Research, Kyoto University, Mita Toho Building 5th floor, 3-1-7, Mita, Minato-ku, Tokyo 108-0073, Japan. E-mail: oishi-t@kier.kyoto-u.ac.jp

example, Cassady (1967) reported bidding rings in many commodity fields, such as antique trades, fish trades and timber rights. Bidding rings reduce or eliminate buyers' competition at auctions and thus get an advantage over sellers. Therefore bidding rings perform price-influencing behavior in auctions.

Cassady (1967) explained that in an English auction, in practice, a ring allocates the object won at the auction via a re-auction. This re-auction is called a *knockout.* The knockout is an English auction among ring members. When bidders have different evaluations for the object being auctioned, *a sequence of knockouts* is produced. In order to explain sequences of knockouts, I will give a simple example as follows[1]. A ring at a main auction consists of five men. After the main auction, a sequence of knockouts is produced. Three of the five ring members may form another ring at the first knockout. In this example, one of the three ring members is the bidder whose evaluation for the object being auctioned is the highest among the five. Next, the second knockout is held. If the three bid competitively at the second knockout, then this sequence is finished and the bidder with the highest evaluation among them is the final owner of the object. Therefore this sequence composes of two knockouts. In practice, some sequences of knockouts are observed; for example, the ones in an antique trade reported by Cassady (1967) and in an auction of used commercial equipment (the case of *U.S. v. Seville Industrial Machinery Corp.*) explained in Deltas (2002).

This paper investigates the relationship between the distributive outcome of each sequence of knockouts and each of such well-known TU game solutions as the *core,* the *Shapley value* and the *nucleolus.* In order to deal with this problem, I formalize a generic model of sequences of knockouts mathematically and apply the model to the bidding ring game[2] defined by Graham *et al.* (1990). The main results in the present study are as the followings. First, the distributive outcome of each sequence of knockouts belongs to the core of the bidding ring game. Second, some sequences of knockouts yields one-point TU game solutions such as the Shapley value and the nucleolus. That is to say, a sequence of knockouts with the finest nested rings yields the Shapley value. Also, a sequence of knockouts at each of which a ring can guarantee its advantage for itself yields the nucleolus.

[1]In subsection 2.1, I will give a model of sequences of knockouts.

[2]Under a situation that the number of all bidders at a main auction is relatively small and the bidders have approximately complete information on the auction, bidding rings may be operated by cooperative behavior among the bidders.

Following observations in Cassady (1967), Graham *et al.* (1990) considered the relationship between the distributive outcome of a single sequence of knockouts and the Shapley value of the bidding ring game. However, there may exist many sequences of knockouts. The size of a ring at each knockout is, generally, variable and Graham *et al.* (1990) focused on only one case with respect to the size of a ring at each knockout. The model in the present study allows not only variability of the size of a ring at each knockout but also that of the number of participants of each knockout. Therefore, my model can generate various distributive outcomes including the one considered in Graham *et al.* (1990).

The remainder of this paper is organized as follows: in Section 2, I will give the model of sequences of knockouts and specify some sequential knockouts. Section 3 introduces bidding ring games and demonstrates the relationship between each distributive outcome and each solution of the bidding ring game. The paper closes in Section 4 with concluding remarks on two possible applications derived from the present study.

2 The model of Sequences of Knockouts

2.1 The sequence of knockouts

English auctions are oral auctions in which an auctioneer initially sets a bid at a low price and then gradually increases the price until one bidder remains active.

Suppose that there are n buyers in a single-object English auction. Let $v_1 > v_2 > \cdots > v_n > v_0 \geq 0$, where v_i is the evaluation of each buyer $i = 1, \cdots, n$ for the item being auctioned and v_0 is the reservation price of the only seller for his item.

There is possibility of a bidding ring at the main auction mentioned above. Suppose that n bidders form an initial ring $R_0 = N$, where $N = \{1, \cdots, n\}$. In the main auction, (i) buyer 1 remains active in the bidding up to v_1 and (ii) each of buyers except buyer 1 remains active in the bidding up to v_0 or does not participate. This is because the aim of operation of the initial ring R_0 is to reduce competition among the members of R_0. For this reason, the bidders except buyer 1 are phantom bidders in this auction. Buyer 1 wins the main auction and pays v_0 to the seller. Since buyer 1 is the representative of R_0, R_0 gets the ownership of the object won at the auction and the net gain

$v_1 - v_0$.

After the main auction, a sequence of knockouts is produced. The purpose of operating the sequence of knockouts is to allocate the object won at the main auction via distribution of the net gain $v_1 - v_0$ to the members of R_0. The sequence of knockouts has two key points. First, the size of a ring at each knockout shrinks consecutively. Second, at each knockout an *recursive* process on distribution is used. I will explain the details of the sequence of knockouts as follows.

(A) *A knockout:* Given the $j-1_{th}$ and the j_{th} rings, $R_{j-1} = \{1, 2, \cdots, m_{j-1}\} \supsetneq R_j = \{1, 2, \cdots, m_j\}$, where $m_j < m_{j-1}$ and $m_0 \equiv n$, the j_{th} knockout is a tuple

$$\left\langle K_j, (v_\alpha)_{\alpha \in K_j}, E \right\rangle.$$

Note that $K_j = \{1\} \cup \tilde{K}_j \cup (R_{j-1} \backslash R_j)$, where $\tilde{K}_j \subseteq R_j \backslash \{1\}$. K_j is the set of participants of the j_{th} knockout, $(v_\alpha)_{\alpha \in K_j}$ is the private values of the participants and E is the English auction rule in the knockout. I define $k_j = |K_j|$ as the number of the participants of the j_{th} knockout.

In the j_{th} knockout, (a) the members of R_j appoint buyer 1 to remain active in the bidding up to v_1 and (b) they appoint each of buyers except buyer 1 to remain active in the bidding up to v_{m_j+1} or not to participate. The reason for these (a) and (b) is the same as that of (i) and (ii) in operation of R_0. Each member of $R_{j-1} \backslash R_j$ bids competitively in the j_{th} knockout. Thus, the set of participants of the j_{th} knockout is given by $K_j = \{1\} \cup \tilde{K}_j \cup (R_{j-1} \backslash R_j)$, where $\tilde{K}_j \subseteq R_j \backslash \{1\}$. Note that $m_{j-1} - m_j + 1 \leq k_j \leq m_{j-1}$. The lower bound of k_j means that the members of R_{j-1} except $R_j \backslash \{1\}$ participate in the j_{th} knockout. The upper bound of k_j means that the members of R_{j-1} participate in the j_{th} knockout.

(B) *A sequence of knockouts:* A sequence of knockouts is given by

$$\left(\left\langle K_j, (v_\alpha)_{\alpha \in K_j}, E \right\rangle \right)_{j=1}^{t}, \text{ given } (R_j)_{j=0}^{t} \text{ such that } R_t = \{1\}.$$

This means that it is possible for the knockouts to continue in the above manner stated in (A) for $j = 1, 2, \cdots, t$ such that $R_t = \{1\}$. In the final knockout, buyer 1 is determined as the final owner of the object and the distributive outcome of the members of R_0 is determined. This distributive process will be explained in (C).

(C) *Distributive process:* Before giving an explanation on the distributive process generally, it may be useful to give a simple illustration on this process.

(C-I) Consider that there are three buyers and the only seller in a single-object English auction. Let $v_1 = \$120,\ v_2 = \$90,\ v_3 = \$70$ and $v_0 = \$10$.

In the main auction, $R_0 = \{1, 2, 3\}$ is formed. Then buyer 1 pays \$10 to the seller and R_0 gets the ownership of the object.

Suppose the existence of a ring center. The ring center has three roles. First, the ring center acts like a pawnbroker; he lends money to a ring at each knockout. Second, the ring center is an auctioneer of each knockout. Third, the ring center is an neutral distributor to the members of R_0. After the last knockout, the ring center has neither gain nor loss.

(1) Before the first knockout, R_0 exchanges the object for \$10 via the ring center; the ring center has the debt of \$10. R_0 gives this \$10 to buyer 1 in order to compensate him for his payment in the main auction. The ring center is an auctioneer of the first knockout and the reservation price of the ring center is \$10.

(2) Let $R_1 = \{1, 2\}$ and $K_1 = \{1, 2, 3\}$. In the first knockout, the ring center gradually increases the price \$10 through \$70. In this knockout, buyer 1 wins and pays \$70 to the ring center. Since buyer 1 is the representative of R_1, R_1 gets the ownership of the object. The ring center gets an advantage \$60[3] because of his debt of \$10. The ring center divides \$60 among the participants of the first knockout equally and pays the distribution \$20 to only the member $R_0 \backslash R_1 = \{3\}$. Then the ring center still has \$40. After getting \$20, buyer 3 exits immediately.

(3) Before the second knockout, R_1 exchanges the object for \$70 via the ring center. The ring center has the debt of \$30. This is because the ring center uses \$40 mentioned above in lending \$70 to R_1. R_1 gives \$70 to buyer 1 in order to compensate him for his payment in the first knockout. The ring center is an auctioneer of the second knockout and the reservation price of the ring center is \$70.

(4) Let $R_2 = \{1\}$ and $K_2 = \{1, 2\}$. In the second knockout, buyer 1 wins and pays \$90 to the ring center. Thus, buyer 1 gets the final ownership of the object and gets the net gain \$30. The ring center gets an advantage

[3] This advantage is eqal to the difference between the knockout gain of R_0 and the net gain of R_1. See (C-II).

\$60[4] because of his debt of \$30. The ring center divides this \$60 among the participants of the second knockout equally and pays the distribution \$30 to only the member $R_1 \backslash R_2 = \{2\}$. After getting \$30, buyer 2 exits immediately. Lastly, the ring center distributes the rest \$30 to buyer 1; he has neither gain nor loss. The distributive outcome of the sequence of these two knockouts is (\$60, \$30, \$20).

Next, I will give an explanation on the process generally.

(C-II) Consider an arbitrarily fixed j_{th} knockout. Buyer 1 wins the j_{th} knockout and pays v_{m_j+1} to the ring center. Since buyer 1 is the representative of R_j, R_j gets the ownership of the object and the net gain $v_1 - v_{m_j+1}$. The ring center divides the difference between the *knockout gain* of R_{j-1} (to be defined inductively) and the *net gain* of R_j, $v_1 - v_{m_j+1}$ equally divided among all participants of the j_{th} knockout. This equal distribution is expressed as

$$\frac{1}{k_j}\left((\text{the knockout gain of } R_{j-1}) - (v_1 - v_{m_j+1})\right).$$

As the result of the j_{th} knockout, each member of $R_{j-1} \backslash R_j$ gets the above equal distribution and exits. On the other hand, R_j gets the knockout gain: $v_1 - v_0$ minus the total amount of the distribution to the members of $R_0 \backslash R_j$.

I will give the definition of the knockout gain. Given $N = R_0 \supsetneqq R_1 \supsetneqq \cdots \supsetneqq R_j$ and $\left(\left\langle K_i, (v_\alpha)_{\alpha \in K_i}, E\right\rangle\right)_{i=1}^{j}$, the *knockout gain* of R_j is defined by the followings:

(i) the knockout gain of R_0 is $v_1 - v_0$ and

(ii) the knockout gain of R_j is $v_1 - v_0 - \sum_{i=1}^{j} |R_{i-1} \backslash R_i| \Lambda_i$, where Λ_i is $((\text{the knockout gain of } R_{i-1}) - (v_1 - v_{m_i+1}))/k_i$.

It is possible for the knockouts to continue to use the above distributive manner for $j = 1, 2, \cdots, t$ such that $R_t = \{1\}$. In the final knockout, buyer 1 is the final owner of the object and the distributive outcome of the members of R_0 is determined. Figure 1 is an illustration on how to produce the sequence of knockouts. Thus, the distributive process has two features. The first is that the final ownership of the object being auctioned belongs to only buyer 1 by a nested ring structure $(R_j)_{j=0}^{t}$ such that $N = R_0 \supsetneqq R_1 \supsetneqq \cdots \supsetneqq R_t = \{1\}$. The

[4] This \$60 is equal to the difference between the knockout gain of R_1 and the net gain of R_2. See (C-II).

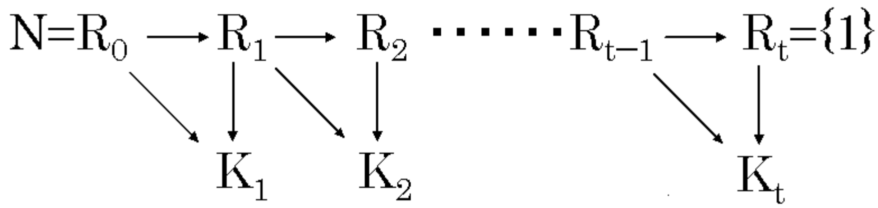

Figure 1: A sequence of knockouts

second is that at each j_{th} knockout, the distributive outcome of each member of $R_{j-1} \backslash R_j$ is determined recursively.

If an arbitrary sequence of knockouts is fixed, the distributive outcome of the sequence of knockouts is defined as follows:

Definition 1 *Let $(z_i)_{i \in N}$ be a distributive outcome of the sequence of knockouts. Let $R_j = \{1, 2, \cdots, m_j\}$ where $m_j < m_{j-1}$ and let the number of participants of the j_{th} knockout be given by k_j where $m_{j-1} - m_j + 1 \leq k_j \leq m_{j-1}$. Given $N = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_t = \{1\}$ and $(k_j)_{j=1}^t$ where $k_t = m_{t-1}$, z_i is defined inductively by $z_1 = v_1 - v_2 + z_t$ and*

$$z_i = \frac{v_{m_j+1} - v_0 - \sum_{l=m_{j-1}+1}^{n} z_l}{k_j} \quad \textit{for} \quad m_j + 1 \leq i \leq m_{j-1},$$

beginning with $m_0 \equiv n$, $\sum_{l=m_0+1}^{n} z_l \equiv 0$ and continuing for $j = 1, \cdots, t$.

2.2 Specification of the sequential knockouts

There may be many cases of the sequential knockouts. This is because the size of a ring and the number of participants at each knockout are variable. I will characterize especially two distinct cases of the sequential knockouts.

2.2.1 The sequential knockouts with the finest nested rings

A simple case is a sequence of knockouts with the finest nested rings. The finest nested rings means that the nested ring structure is the finest. I will formulate the sequential knockouts with the finest nested rings as follows:

Definition 2 *The sequential knockouts with the finest nested rings are* $\left(\left\langle K_j, (v_\alpha)_{\alpha \in K_j}, E \right\rangle\right)_{j=1}^{n-1}$ *satisfying* $R_j = \{1, 2, \cdots, n-j\}$ *and* $K_j = R_{j-1}$ *consecutively for* $j = 1, 2, \cdots, n-1$.

Then the distributive outcome of the sequential knockouts with the finest nested rings is given by the following proposition[5]. The proof of this proposition will be omitted since it is a matter of calculation derived from Definition 1 and Definition 2.

Proposition 1 *Let* $(\hat{x}_i)_{i \in N}$ *be a distributive outcome of the sequential knockouts with the finest nested rings. Then* $\hat{x}_i$ *is given by*

$$\begin{aligned} \hat{x}_i &= \sum_{j=i}^{n-1} \frac{v_j - v_{j+1}}{j} + \frac{v_n - v_0}{n} \quad \textit{for } i \leq n-1 \quad \textit{and} \\ \hat{x}_n &= \frac{v_n - v_0}{n}. \end{aligned}$$

2.2.2 The sequential knockouts with the nested rings with guaranteed surplus

Let us consider a sequence of knockouts satisfying that the difference between the knockout gain of R_j and the net gain of R_j is minimal at each j_{th} knockout. The *surplus of a ring* R_j is defined as the knockout gain of R_j minus the net gain of R_j. The net gain of R_j is the worth that R_j can get by itself. The knockout gain of R_j is the worth that R_j can get via the distributive process of the j_{th} knockout. Therefore the surplus of R_j is regarded as its advantage via the j_{th} knockout.

[5] Graham *et al.* (1990) deal with only the sequential knockouts with the finest rings. They consider another distributive process, by which equal division of the difference between the net gain of R_{j-1} and that of R_j is distributed among the members of R_{j-1} consecutively for $j = 1, \cdots, n-1$. Therefore Proposition 1 is not derived from Graham *et al* (1990).

By the recursively distributive process in subsection 2.1, the surplus of R_j depends only on R_j and K_j, given $(R_l)_{l=0}^{j-1}$ and $(K_l)_{l=1}^{j-1}$ where $K_0 \equiv \emptyset$. The minimal surplus of R_j means that the surplus of R_j is minimal at R_j and K_j, given $(R_l)_{l=0}^{j-1}$ and $(K_l)_{l=1}^{j-1}$.

Definition 3 *The sequential knockouts with the nested rings with guaranteed surplus are* $\left(\left\langle K_j, (v_\alpha)_{\alpha \in K_j}, E\right\rangle\right)_{j=1}^{t}$ *satisfying that the surplus of* R_j *is minimal consecutively for* $j = 1, 2, \cdots, t$ *such that* $R_t = 1$.

This is a sequence of knockouts satisfying that at each knockout a ring can guarantee its advantage for itself. The following proposition gives the distributive outcome of this sequential knockouts.

Proposition 2 *Let* $(x_i^*)_{i \in N}$ *be a distributive outcome of the sequential knockouts with the nested rings with guaranteed surplus. Let* $M_i = v_i - v_0$ *for each* $i \in N$*. Then* x_i^* *is given by* $x_i^* = \Lambda_t^*$ *and* $x_1^* = v_1 - v_2 + \Lambda_{t'}^*$ *for* $m_{t-1}^* - k_t^* + 2 \le i \le m_{t-1}^*$ *and* $1 \le t \le t'$ *such that* $m_{t'}^* = 1$*, where* Λ_t^**,* k_t^* *and* m_t^* *are defined inductively by*

$$\Lambda_t^* = \min_{k=2,\cdots,m_{t-1}^*} \left\{ \frac{M_{m_{t-1}^*-k+2} - \sum_{i=0}^{t-1}(k_i^* - 1)\Lambda_i^*}{k} \right\},$$

k_t^* *denotes the largest value of* k *for which the above expression attains its minimum and* $m_t^* = m_{t-1}^* - k_t^* + 1$ *(beginning with* $m_0^* \equiv n$*,* $\Lambda_0^* \equiv 0$ *and continuing for* $t = 1, \cdots, t'$ *such that* $m_{t'}^* = 1$*).*

Proof. The proof will be completed by the following two steps.

Step 1 : In the 1st knockout, for any fixed $k \in R_0 \backslash \{1\}$, let $|R_1| = n - k + 1$. The knockout gain of R_0 is M_1 and the net gain of R_1 is $v_1 - v_{n-k+2}$. If the number of participants of R_1 at the 1st knockout is f, then the surplus of R_1 is given by

$$M_{n-k+2} - \frac{M_{n-k+2}}{(k-1)+f}(k-1) = \frac{1}{\frac{k-1}{f}+1} \cdot M_{n-k+2}.$$

If the surplus of R_1 is minimal for any fixed k, then $f = 1$. Therefore the minimal surplus of R_1 is given by

$$\Lambda_1 = \min_{k=2,\cdots,n} \left\{ \frac{M_{n-k+2}}{k} \right\}.$$

Each member of $R_0 \backslash R_1$ gets Λ_1 as his distributive outcome. Take k_1 as a k giving Λ_1. Let $R_1 = \{1, \cdots, m_1\}$. Then $m_1 = n - k_1 + 1$.

In the 2nd knockout, for any fixed $k \in R_1 \backslash \{1\}$, let $|R_2| = m_1 - k + 1$. The knockout gain of R_1 is $M_1 - (k_1 - 1)\Lambda_1$ and the net gain of R_2 is $v_1 - v_{m_1-k+2}$. If the number of participants of R_2 at the 2nd knockout is g, then the surplus of R_2 is given by

$$\frac{1}{\frac{k-1}{g} + 1} \cdot \left(M_{m_1-k+2} - (k_1 - 1)\Lambda_1\right).$$

If the surplus of R_2 is minimal for any fixed k, then $g = 1$. Therefore the minimal surplus of R_2 is given by

$$\Lambda_2 = \min_{k=2,\cdots,m_1} \left\{ \frac{M_{m_1-k+2} - (k_1 - 1)\Lambda_1}{k} \right\}.$$

Each member of $R_1 \backslash R_2$ gets Λ_2 as his distributive outcome. Take k_2 as a k giving Λ_2. Let $R_2 = \{1, \cdots, m_2\}$. Then $m_2 = m_1 - k_2 + 1$. Similarly, the same procedure can be repeated until the distributive outcome of each member of R_0 is determined.

Step 2 : Next, I will show that at each i_{th} knockout we can take $k_i = k_i^*$ such that k_i^* is the largest k giving Λ_i^* in order to decrease the number of the knockouts. (That is to say, the distributive outcome of the sequential knockouts with the nested rings with guaranteed surplus is independent on the number of the knockouts.)

Let k_p and k_q ($k_q < k_p$) be two distinct ks giving Λ_1^*, namely $\Lambda_1^* = \frac{M_{n-k_p+2}}{k_p} = \frac{M_{n-k_q+2}}{k_q}$. Take k_q as a k giving Λ_1^* in the 1st knockout. Let $s_1 = \Lambda_1^*(k_q - 1)$, $m_1 = n - k_q + 1$ and $\Lambda_2^* = \min_{k=2,\cdots,m_1} \left\{ \frac{M_{m_1-k+2-s_1}}{k} \right\}$. I will prove (i) $\frac{M_{n-k_p+2}}{k_p} = \frac{M_{m_1-k'+2-s_1}}{k'}$, where $k' = k_p - k_q + 1$ and (ii) for any $k = 2, \cdots, m_1$, $\frac{M_{n-k''+2}}{k''} \leq \frac{M_{m_1-k+2-s_1}}{k}$, where $k'' = k + k_q - 1$. It suffices to show (i) and (ii) for the purpose of Step 2 since the same procedure can be repeated until the t'_{th} knockout.

Proof of (i):

$$\begin{aligned}
\frac{M_{m_1-k'+2}-s_1}{k'} &= \frac{M_{n-k_q+1-(k_p-k_q+1)+2}-s_1}{k_p-k_q+1} \\
&= \frac{M_{n-k_p+2}-\frac{M_{n-k_p+2}}{k_p}\left(k_q-1\right)}{k_p-k_q+1} \\
&= \frac{M_{n-k_p+2}\left(1-\frac{k_q-1}{k_p}\right)}{k_p-k_q+1} = \frac{k_p-k_q+1}{k_p}\cdot\frac{M_{n-k_p+2}}{k_p-k_q+1} \\
&= \frac{M_{n-k_p+2}}{k_p},
\end{aligned}$$

which completes the proof of (i).

Proof of (ii): For any $k = 2,\cdots,m_1$, it suffices to prove $k''\left(M_{m_1-k+2}-s_1\right) - k\cdot M_{n-k''+2}\geq 0$. For any $k=2,\cdots,m_1$,

$$\begin{aligned}
& k''\left(M_{m_1-k+2}-s_1\right)-k\cdot M_{n-k''+2} \\
=\ & (k+k_q-1)\left(M_{n-k_q+1-k+2}-\Lambda_1^*\left(k_q-1\right)\right)-k\cdot M_{n-(k+k_q-1)+2} \\
=\ & (k_q-1)M_{n-k_q-k+3}-(k+k_q-1)\Lambda_1^*\left(k_q-1\right) \\
=\ & (k_q-1)(k+k_q-1)\left(\frac{M_{n-k_q-k+3}}{k+k_q-1}-\Lambda_1^*\right) \\
=\ & (k_q-1)(k+k_q-1)\left(\frac{M_{n-k''+2}}{k''}-\Lambda_1^*\right) \\
\geq\ & 0,
\end{aligned}$$

since $\frac{M_{n-k''+2}}{k''}\geq\Lambda_1^*$ from the definition of Λ^*. This completes the proof of (ii).

Therefore, at each i_{th} knockout we can take k_i^* as the largest value of k for which the formula of Λ_i^* attains its minimum and we have $m_i^*=m_{i-1}^*-k_i^*+1$, which completes the proof. ■

Remark 1 *The distributive outcome of the sequential knockouts with the nested rings with guaranteed surplus x^* satisfies $x_1^*\geq x_2^*\geq\cdots\geq x_n^*$ owing to Step 2 in the proof of Proposition 2. In other words, this distributive outcome has order-preserving in the sense that $x_1^*\geq x_2^*\geq\cdots\geq x_n^*$ if $v_1>v_2>\cdots>v_n$. Also, obviously, the distributive outcome of the sequential knockouts with the finest nested rings has order-preserving.*

The following example gives the distributive outcome of the sequential knockouts with the nested rings with guaranteed surplus of a 7 buyers case.

Example 1 ***(A 7-buyers case)***

$$v_1 = 30, v_2 = 25, v_3 = 21, v_4 = 15, v_5 = 14, v_6 = 11, v_7 = 10, v_0 = 5.$$

The 1st knockout:

$$\Lambda_1^* = \min\left\{\frac{M_7}{2}, \frac{M_6}{3}, \cdots, \frac{M_2}{7}\right\} = \min\left\{\frac{5}{2}, \frac{6}{3}, \frac{9}{4}, \frac{10}{5}, \frac{16}{6}, \frac{20}{7}\right\} = 2$$
$$k_1^* = 5, \quad x_i^* = \Lambda_1^* = 2 \quad (i = 4, 5, 6, 7)$$
$$m_1^* = n - k_1^* + 1 = 3$$

The 2nd knockout:

$$\Lambda_2^* = \min\left\{\frac{M_3 - (k_1^* - 1)\Lambda_1^*}{2}, \frac{M_2 - (k_1^* - 1)\Lambda_1^*}{3}\right\} = \min\left\{\frac{8}{2}, \frac{12}{3}\right\} = 4$$
$$k_2^* = 3, \quad x_i^* = \Lambda_2^* = 4 \quad (i = 2, 3)$$
$$m_2^* = m_1^* - k_2^* + 1 = 1 = m_{t'}^*$$
$$x_1^* = v_1 - v_2 + \Lambda_{t'}^* = v_1 - v_2 + \Lambda_2^* = 9$$

The distributive outcome of the sequential knockouts with the nested rings with guaranteed surplus is given by $x^* = (9, 4, 4, 2, 2, 2, 2)$.

3 Solutions of bidding ring games

3.1 Bidding ring games

A bidding ring game at an English auction under complete information is a TU game defined by Graham *et al.* (1990).

Let $N = \{1, \cdots, n\}$ be the finite set of buyers, and let $S \subseteq N$ be a coalition. Let us consider an English auction with the possibility of a bidding ring among buyers of the auction. This situation can be described as the TU game (N, v) satisfying

$$v(S) = \begin{cases} v_1 - \max_{j \notin S} v_j & \text{if} \quad 1 \in S \\ 0 & \text{if} \quad 1 \notin S, \end{cases}$$

where $\max_{j \notin N} v_j \equiv v_0$. We call this game the bidding ring game.

The characteristic function $v(S)$ denotes the net gain that S can get by itself. This function is based on the followings.

First, under the English auction rule, it is a dominant strategy for each bidder to remain active until bidding reaches his evaluation. Second, any coalition including buyer 1 can win the auction with the net gain $v_1 - \max_{j \notin S} v_j$ by making the sole bidder in the coalition. This is because the coalition S including buyer 1 can get the maximal net gain by eliminating of competition among the members of S. Lastly, in any coalition not including buyer 1, the coalition does not win the auction, hence the net gain is 0.

Remark 2 *The bidding ring game is convex (See Graham et al. (1990)). Therefore the core of this game is nonempty. Furthermore, Shapley value and the nucleolus belong to the core*[6].

The following subsections 3.2 and 3.3 discuss the relationship between each distributive outcome of the sequential knockouts and each solution of bidding ring games. I will state here that well-known one point solutions such as the Shapley value and the nucleolus are yielded by the sequential knockouts introduced in subsection 2.2 respectively.

3.2 The core and the Shapley value

An n-dimensional vector x of the bidding ring game is a payoff vector if it satisfies $\sum_{i \in N} x_i = v(N)$. Then the *core* of this game is a set of payoff vectors x satisfying the stability conditions $\sum_{i \in S} x_i \geq v(S)$ for all $S \subseteq N$.

The *Shapley value* $\phi(v)$ of this game is a payoff vector given by the formula

$$\phi_i(v) = \sum_{S \subseteq N,\ i \notin S} \frac{|S|!(n - |S| - 1)!}{n!} \left(v(S \cup i) - v(S)\right) \quad \text{for all } i \in N.$$

Proposition 3 *Each distributive outcome of the sequential knockouts belongs to the core.*

[6] The definitions of the core, the Shapley value and the nucleolus will be expleined in subsection 3.2 and 3.3.

Proof. Let $z \in \mathbb{R}^n$ be the distributive outcome of the sequential knockouts. By Definition 1, z is a payoff vector. If $1 \notin S$, then $\sum_{i \in S} z_i > 0 = v(S)$. If $1 \in S$, it suffices to show $\sum_{i \in S} z_i \geq v(S)$ such that $S = R_j (= \{1, \cdots, m_j\})$, since $z_i > 0$.

By Definition 1, we have

$$\begin{aligned}
\sum_{i \in S} z_i - v(S) &= v_1 - v_0 - \sum_{i \in N \setminus R_j} z_i - \left(v_1 - v_{m_j+1}\right) \\
&= \left(1 - \frac{m_{j-1} - m_j}{k_j}\right) \left(v_{m_j+1} - v_0 - \sum_{i \in N \setminus R_{j-1}} z_i\right) \\
&> 0,
\end{aligned}$$

which completes the proof. ■

Proposition 4 *The distributive outcome of the sequential knockouts with the finest nested rings is the Shapley value.*

Proof. By Theorem 2 in Graham *et al.* (1990) and Proposition 1 in the present paper, the proof is completed. ■

3.3 The nucleolus

A payoff vector x is called an imputation if x satisfies the individual rationality conditions $x_i \geq v(\{i\})$ for all $i \in N$. Let X be the set of all imputations of the bidding ring game. Obviously, $X \neq \emptyset$. Given an imputation $x \in X$, the *excess* of a coalition S with respect to x is defined as the number $v(S) - \sum_{i \in S} x_i$. Let $e(x)$ be the $(2^n - 2)$ dimensional vector, the components of which are the excesses of every coalition $S \neq N, \emptyset$ with respect to x, arranged in the non-increasing order.

The *nucleolus* of v is defined as a set of imputations such that the vector $e(x)$ is *lexicographically minimal* over X. It is well known that the nucleolus is never empty and a singleton (Schmeidler (1969)).

Let v' be *the 0-normalization* of v, that is, $v'(S) = v(S) - \sum_{i \in S} v(\{i\})$ for every coalition S. It is well known that an imputation z is the nucleolus of the game v *if and only if* an imputation z' satisfying $z'_i = z_i - v(\{i\})$ for each $i \in N$ is the nucleolus of the game v'.

Proposition 5 *The distributive outcome of the sequential knockouts with nested rings with guaranteed surplus is the nucleolus.*

The following proof is basically the same as that of the theorem of Littlechild (1974). But as the present case is more complicated, I will give the proof of it.

Proof. Let v' be the 0-normalization of the bidding ring game and let $M_i = v_i - v_0$ for each $i \in N$. Then,

$$v'(S) = \begin{cases} M_2 & \text{if} \quad S = N \\ M_2 - \max_{i \notin S} M_i & \text{if} \quad 1 \in S \subsetneq N \\ 0 & \text{if} \quad 1 \notin S. \end{cases}$$

In the followings, I will focus on the game v' stated above.

Let us consider how to divide the net gain of R_0 minus $v_1 - v_2$ among all members of R_0. By Proposition 2, we can give a normalized distributive outcome as follows:

(P): *Let $(x_i'^*)_{i \in N}$ be a normalized distributive outcome of the sequential knockouts with nested rings with guaranteed surplus. Let $M_i = v_i - v_0$ for each $i \in N$. Then $x_i'^*$ is given by $x_i'^* = -\lambda_t^*$ and $x_1'^* = -\lambda_{t'}^*$ for $m_{t-1}^* - k_t^* + 2 \leq i \leq m_{t-1}^*$ and $1 \leq t \leq t'$ such that $m_{t'}^* = 1$, where λ_t^*, k_t^* and m_t^* are defined inductively by*

$$\lambda_t^* = \max_{k=2,\cdots,m_{t-1}^*} \left\{ -\frac{M_{m_{t-1}^*-k+2} + \sum_{i=0}^{t-1}(k_i^* - 1)\lambda_i^*}{k} \right\},$$

k_t^* denotes the largest value of k for which the above expression attains its maximum and $m_t^* = m_{t-1}^* - k_t^* + 1$ (beginning with $m_0^* \equiv n$, $\lambda_0^* \equiv 0$ and continuing for $t = 1, \cdots, t'$ such that $m_{t'}^* = 1$).

It suffices to show the distributive outcome stated in (P) is the nucleolus of v'. The proof is based on Kopelowitz$'$ algorithm[7]. Initially consider the

[7] For example, see Littlechild (1974).

optimal solution to the following problem (I) as

$$\begin{aligned}
&\min \quad \lambda \\
s.t. \quad &\sum_{i\in S} x_i \geq -\lambda \quad \text{for all } S \subsetneq N \text{ such that } 1 \notin S, \\
&\sum_{i\in S} x_i \geq M_2 - \max_{i\notin S} M_i - \lambda \quad \text{for all } S \subsetneq N \text{ such that } 1 \in S, \\
&\sum_{i\in N} x_i = M_2 \quad \text{and} \\
&x_i \geq 0 \quad \text{for all } i \in N.
\end{aligned}$$

λ is bounded below. For $\lambda = 0$, an n-dimensional vector x satisfying $x_1 = M_2$ and $x_i = 0$ for $i = 2, 3, \cdots, n$ is feasible. Hence, the optimal solution to problem (I) exists and $\lambda \leq 0$. If $1 \notin S$, then the corresponding constraint is dominated by the constraints

$$x_i \geq -\lambda$$

for the coalitions $\{i\}$ where all $i \in S$. If $1 \in S$, then the corresponding constraint is dominated by

$$\sum_{i=1}^{j(S)-1} x_i \geq M_2 - M_{k+1} - \lambda$$

for the coalition $\{1, 2, \cdots, j(S) - 1\}$ where $j(S) = \min\{j | j \notin S, 1 \in S\}$. Problem (I) may, therefore, be simplified to the following problem (I′) as

$$\begin{aligned}
&\min \quad \lambda \\
s.t. \quad &x_i \geq -\lambda \quad \text{for all } i \in N, \\
&\sum_{i=1}^{k} x_i \geq M_2 - M_{k+1} - \lambda \quad \text{for } k = 2, \cdots, n-1, \\
&\sum_{i\in N} x_i = M_2 \quad \text{and} \\
&x_i \geq 0 \quad \text{for all } i \in N.
\end{aligned}$$

I now claim that the optimal value to problem (I′) is

$$\lambda_1^* = \max\left\{-\frac{M_n}{2}, -\frac{M_{n-1}}{3}, \cdots, -\frac{M_2}{n}\right\},$$

which is as defined in (P). Since it is trivial that $\lambda \geq \max\left\{-\frac{M_n}{2}, \cdots, -\frac{M_2}{n}\right\}$, it suffices to show $\lambda = \max\left\{-\frac{M_n}{2}, \cdots, -\frac{M_2}{n}\right\}$. Let x' be an n-dimensional vector satisfying $x'_1 = -\lambda + a_1$, $x'_2 = -\lambda + a_2$, $x'_i = -\lambda$ for $i = 3, 4, \cdots, n$, $a_1 \geq 0$, $a_2 \geq 0$ and $a_1 + a_2 = M_2 + n\lambda$. Then it will be shown that x' is feasible for $\lambda = \max\left\{-\frac{M_n}{2}, \cdots, -\frac{M_2}{n}\right\}$. It is trivial that $x'_i \geq -\lambda$ for $i = 1, \cdots, n$, $\sum_{i \in N} x'_i = M_2$, and $x'_i \geq 0$ because $\lambda \leq 0$. Moreover, for $k = 2, \cdots, n-1$, we have

$$\begin{aligned}\sum_{i=1}^{k} x'_i - (M_2 - M_{k+1} - \lambda) &= (n-k+1)\lambda + M_{k+1} \\ &\geq (n-k+1)\left(-\frac{M_{k+1}}{n-k+1}\right) + M_{k+1} = 0\end{aligned}$$

because $\lambda \geq \max\left\{-\frac{M_n}{2}, \cdots, -\frac{M_2}{n}\right\}$. Therefore, for $\lambda = \max\left\{-\frac{M_n}{2}, \cdots, -\frac{M_2}{n}\right\}$, x' is feasible. This establishes the claim.

This claim implies that

$$\begin{aligned}x'^*_i &= -\lambda^*_1 \quad \text{for } i = n - k^*_1 + 2, \cdots, n \quad \text{and} \\ \sum_{i=1}^{n-k^*_1+1} x'^*_i &= M_2 - M_{n-k^*_1+2} - \lambda^*_1,\end{aligned}$$

where λ^*_1 and k^*_1 are as defined in (P). If $k^*_1 \neq n$, variables $x'^*_{m^*_1+1}, \cdots, x'^*_n$, where m^*_1 is as defined in (P), may be eliminated from problem (I$'$), together with the additional equality constraints involving the set of coalitions S whose excess is equal to λ^*_1 for all optimal solutions to (I$'$). Therefore the second linear programme may be written as Problem (II) as follows:

$$\begin{aligned}&\min \quad \lambda \\ s.t. \quad & x_i \geq -\lambda \quad \text{for } i = 1, \cdots, m^*_1, \\ & \sum_{i=1}^{k} x_i \geq M_2 - M_{k+1} - \lambda \quad \text{for } k = 2, \cdots, m^*_1 - 1, \\ & \sum_{i=1}^{m^*_1} x_i = M_2 + (k^*_1 - 1)\lambda^*_1 \quad \text{and} \\ & x_i \geq 0 \quad i = 1, \cdots, m^*_1.\end{aligned}$$

This is of precisely the same form as problem (I$'$), so the same procedure may be repeated for $t = 1, \cdots t'$ such that $m^*_{t'} = 1$. Therefore the proof is completed. ∎

4 Concluding Remarks

I will give concluding remarks by mentioning two possible applications derived from the present study.

(1) Let v be *the monotonic game*, that is, $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$. For each coalition S, let the value of $v(N) - v(N \setminus S)$ be *the marginal contribution of* S to the grand coalition N. Note that the marginal contribution of each coalition to the grand coalition is non-negative because of the monotonicity of v. For every singleton $S = \{i\}$, the value of $v(N) - v(N \setminus S)$ is especially called *the marginal contribution of player* i to N. Oishi (2006) defined a monotonic game as follows:

v is a game with collectively contributing coalitional leaders if it satisfies that

$$v(N) - v(N \backslash S) = \max_{i \in S} (v(N) - v(N \backslash \{i\})) \quad \text{for all } S \subseteq N.$$

The game v stated above describes a situation that for each coalition S, the marginal contribution of S to N is made by a player with the highest marginal contribution to N among all the members in S. Then the player whose marginal contribution to N is the highest in S is *a collectively contributing coalitional leader*.

Oishi (2006) demonstrated that the bidding ring game is a game with collectively contributing coalitional leaders. Therefore the results in the present study may be generalized to games with collectively contributing coalitional leaders. As an application along this line, we can calculate the nucleolus of the sewerage system game, which deals with the benefit allocation problem of cities sharing a sewer and a sewage plant on a river (See Oishi (2006)). This implies that the result of the present study may apply to various benefit allocation problems defined on the line graph.

(2) Let (N, v) be a coalitional game. A game (N, v) will be abbreviated to v. Oishi and Nakayama (forthcoming) have defined the anti-dual of v to be the dual of $(-v)$. As an economic meaning of the anti-dual, the anti-dual game may be considered as a cost game when we regard the original game as a profit game. A good example of the anti-dual is the relation between the airport game due to Littlechild (1974) with one aircraft in each type and the bidding ring game. Oishi and Nakayama (forthcoming) showed that

solutions such as the core, the Shapley value and the nucleolus of anti-dual games are obtained straightforwardly from original games.

The anti-dual of the bidding ring game is the simple airport game mentioned above. Therefore the distributive rule considered in the present study may lead to the core, the Shapley value and the nucleolus of the simple airport game.

Acknowledgements

The author would like to thank Youngsub Chun, Shinsuke Nakamura, Mikio Nakayama, Shigeo Muto and Makoto Yano for various helpful comments on an earlier draft of this paper. He also would like to thank Toshiyuki Hirai and Shin Sakaue for their helpful comments.

References

[1] Cassady, R.Jr., 1967. Auctions and Auctioneering. Berkeley, University of California Press.

[2] Deltas, G., 2002. Determining damages from the operation of bidding rings: An analysis of the post-auction knockout sale. Economic Theory 19, 243-269.

[3] Graham, D.A., Marshall, R.C., 1987. Collusive bidder behavior at single-object second-price and English auctions. Journal of Political Economy 95, 1217-1239.

[4] Graham, D.A., Marshall, R.C., Richard, J.-F., 1990. Differential payments within a bidder coalition and the Shapley value. The American Economic Review 80, 493-510.

[5] Littlechild, S.C., 1974. A simple expression for the nucleolus in a special case. International Journal of Game Theory 3, 21-29.

[6] Oishi, T., 2006. The nucleolus in a game with collectively contributing coalitional leaders. DP2005-040, KUMQRP Discussion Paper Series.

[7] Oishi, T., Nakayama, M., forthcoming. The anti-dual of economic coalitional TU games. The Japanese Economic Review, doi: 10.1111/j.1468-5876.2008.00451.x.

[8] Schmeidler, D., 1969. The nucleolus of a characteristic function game. SIAM Journal on Applied Mathematics 17, 1163-1170.