KEIO UNIVERSITY
KEIO/KYOTO MARKET QUALITY RESEARCH PROJECT
(Global Center of Excellence Project)

KEIO/KYOTO GLOBAL COE DISCUSSION PAPER SERIES

DP2008-018

A change in a competitive economy with indivisible commodities

Takayuki OISHI*
Shin SAKAUE**

Abstract
We consider the relationship between a traditional competitive market and a competitive market with middlemen for trading indivisible commodities. We demonstrate that existence of many homogeneous middlemen leads to a change from the market with middlemen to the bilateral market.

*Takayuki OISHI
Institute of Economic Research, Kyoto University
**Shin SAKAUE
Graduate School of Economics, Keio University

KEIO/KYOTO MARKET QUALITY RESEARCH PROJECT
(Global Center of Excellence Program)

Graduate School of Economics and Graduate School of Business and Commerce,
Keio University
2-15-45 Mita, Minato-ku, Tokyo 108-8345 Japan

Kyoto Institute of Economics,
Kyoto University
Yoshida-honmachi, Sakyo-ku, Kyoto 606-8501 Japan

A change in a competitive economy with indivisible commodities

Takayuki OISHI*and Shin SAKAUE†

February 23, 2009

We consider the relationship between a traditional competitive market and a competitive market with middlemen for trading indivisible commodities. We demonstrate that existence of many homogeneous middlemen leads to a change from the market with middlemen to the bilateral market.

Keywords: Middlemen; Competitive equilibrium; Partitioning linear program

JEL classification: C62; D41; D50

1 Introduction

In the literature, many economists have analyzed a traditional competitive market with indivisible commodities: a market in which there are sellers and

*Corresponding author: Institute of Economic Research, Kyoto University, Mita Toho Building 5th floor, 3-1-7, Mita, Minato-ku, Tokyo 108-0073, Japan. E-mail: oishi-t@kier.kyoto-u.ac.jp

†Graduate School of Economics, Keio University, 2-15-45, Mita, Minato-ku, Tokyo, 108-8345, Japan. E-mail: shins@gs.econ.keio.ac.jp

buyers and indivisible commodities are traded between them, e.g., Shapley and Shubik (1972) and Yang (2003). On the other hand, there can be found the literature describing various markets that middlemen play the role of matchmakers, who purchase indivisible commodities from sellers and sell them to buyers., e.g., Rubinstein and Wolinsky (1987). Although these markets with indivisible commodities have been investigated independently, what is the relationship between the bilateral competitive market and the competitive market with middlemen is still an open question. Our paper presents an answer to this question: many homogeneous middlemen lead to a change from the market with middlemen to the bilateral market.

This paper is organized as follows: in Section 2, we will introduce the model of competitive markets with middlemen. Section 3 gives our main result. The paper closes in Section 4 with concluding remarks on the present study.

2 The model of a market with middlemen

First, we introduce some notations of a model of a market with middlemen (N_1, N_2, N_3). Let $N_1 = \{i_1, i_2, \ldots, i_{n_1}\}$, $N_2 = \{j_1, j_2, \ldots, j_{n_2}\}$ and $N_3 = \{k_1, k_2, \ldots, k_{n_3}\}$ $(n_1, n_2, n_3 \in \mathbb{N})$ be the set of sellers, the set of middlemen and the set of buyers respectively. Let $N = N_1 \cup N_2 \cup N_3$ be the set of all agents. There are n_1 kinds of indivisible goods and they are exchanged for money. Each seller $i \in N_1$ owns only one unit of indivisible goods initially,

namely $\omega_i = 1$. Each middleman and each buyer own no unit of goods initially. Let us denote the demand and supply of this market as follows.

Sellers' side: Each seller i sells one unit of her goods to a middleman or consumes it by herself. Let x_i be the consumption of seller i, namely $x_i \in \{0, 1\}$.

Middlemen's side: Each middleman j wants to sell at most one unit of goods to only a buyer. For this purpose, each middleman purchases at most one unit of goods from only a seller. Let $\tilde{x}_{ij}$ be the supply of seller i to middleman j, namely $\tilde{x}_{ij} \in \{0, 1\}$. Note that each middleman consumes no unit of goods which she purchases from the seller.

Buyers' side: Each buyer k purchases at most one unit of goods from only a middleman in order to consume it by herself. Let x_{ijk} be the consumption of buyer k, namely $x_{ijk} \in \{0, 1\}$. This means that in the case of $x_{ijk} = t$ buyer k demands t unit of goods which middleman j purchases from seller i. Moreover, let $\tilde{x}_{ijk}$ be the supply of middleman j, namely $\tilde{x}_{ijk} \in \{0, 1\}$. This means that in the case of $\tilde{x}_{ijk} = \tilde{t}$ middleman j supplies to buyer k $\tilde{t}$ unit of goods which she purchases from seller i.

From the above explanation, each of **A1**, **A2** and **A3** gives the set of feasible allocations of sellers, middlemen and buyers respectively.

A1 : For all $i \in N_1$, $X_i \equiv \{(x_i, (\tilde{x}_{ij})_{j \in N_2}) \in \mathbb{Z}_+^{1+n_2} : x_i + \sum_{j \in N_2} \tilde{x}_{ij} = \omega_i = 1\}$.
A2 : For all $j \in N_2$, $X_j \equiv \{(\tilde{x}_{ijk})_{i \in N_1, k \in N_3} \in \mathbb{Z}_+^{n_1 n_3} : \sum_{i \in N_1} \sum_{k \in N_3} \tilde{x}_{ijk} \leq 1\}$.
A3 : For all $k \in N_3$, $X_k \equiv \{(x_{ijk})_{i \in N_1, j \in N_2} \in \mathbb{Z}_+^{n_1 n_2} : \sum_{i \in N_1} \sum_{j \in N_2} x_{ijk} \leq 1\}$.

Each seller and each buyer have a utility function on consumption. The utility functions on consumption are measured in terms of money and given by $U_i : \mathbb{Z}_+ \to \mathbb{R}$ for each $i \in N_1$ and $U_k : \mathbb{Z}_+^{n_1 n_2} \to \mathbb{R}$ for each $k \in N_3$. We assume that for each $l \in N_1 \cup N_3,\ U_l(\cdot)$ is non-decreasing and $U_l(\mathbf{0}) = 0$. Each middleman j purchases at most one unit of goods at the price $p_i \in \mathbb{R}_+$ from seller i. Also, middleman j sells at most one unit of i's initial goods at the price $q_{ij} \in \mathbb{R}_+$ to buyer k. Let two distinct price lists be given by $p = (p_i)_{i \in N_1} \in \mathbb{R}_+^{n_1}$ and $q = (q_{ij})_{i \in N_1, j \in N_2} \in \mathbb{R}_+^{n_1 n_2}$ respectively.

Then each utility function is given by the followings.

· For all $i \in N_1,\ U_i(x_i) + p_i(\sum_{j \in N_2} \tilde{x}_{ij})$.

· For all $j \in N_2,\ -\sum_{i \in N_1} p_i(\sum_{k \in N_3} \tilde{x}_{ijk}) + \sum_{i \in N_1} q_{ij}(\sum_{k \in N_3} \tilde{x}_{ijk})$.

· For all $k \in N_3,\ U_k((x_{ijk})_{i \in N_1, j \in N_2}) - \sum_{i \in N_1} \sum_{j \in N_2} q_{ij} x_{ijk}$.

We will give an interpretation only of the utility function of each middleman. This utility function consists of two components. The first is the total cost of purchasing goods from a seller. The second is revenue produced by her selling goods to a buyer.

Since in each market demand equals supply at any equilibrium, **A4** and **A5** give the market-clearing conditions.

A4 : For all $(i, j) \in N_1 \times N_2,\ \sum_{k \in N_3} \tilde{x}_{ijk} = \tilde{x}_{ij}$.

A5 : For all $(i, j, k) \in N_1 \times N_2 \times N_3,\ x_{ijk} = \tilde{x}_{ijk}$.

Next, let us introduce a competitive equilibrium (price) and a competitive outcome in the market (N_1, N_2, N_3) respectively.

A tuple $(\hat{p}, \hat{q}, \hat{x}) = ((\hat{p}_i)_{i\in N_1}, (\hat{q}_{ij})_{i\in N_1, j\in N_2}, ((\hat{x}_i)_{i\in N_1}, (\hat{x}_{ijk})_{i\in N_1, j\in N_2, k\in N_3}) \in \mathbb{R}_+^{n_1} \times \mathbb{R}_+^{n_1 n_2} \times \mathbb{Z}_+^{n_1+n_1 n_2 n_3}$ is called a competitive equilibrium if $(\hat{p}, \hat{q}, \hat{x})$ satisfies

$(\mathbf{I})$: for all $i \in N_1,\ U_i(\hat{x}_i) + \hat{p}_i(\omega_i - \hat{x}_i)$

$=\max_{(x_i, (\tilde{x}_{ij})_{j\in N_2})\in X_i} \left[U_i(x_i) + \hat{p}_i(\sum_{j\in N_2} \tilde{x}_{ij}) \right],$

$(\mathbf{II})$: for all $j \in N_2,\ -\sum_{i\in N_1} \hat{p}_i(\sum_{k\in N_3} \hat{x}_{ijk}) + \sum_{i\in N_1} \hat{q}_{ij}(\sum_{k\in N_3} \hat{x}_{ijk})$

$= \max_{(\tilde{x}_{ijk})_{i\in N_1, k\in N_3}\in X_j} \left[-\sum_{i\in N_1} \hat{p}_i(\sum_{k\in N_3} \tilde{x}_{ijk}) + \sum_{i\in N_1} \hat{q}_{ij}(\sum_{k\in N_3} \tilde{x}_{ijk}) \right],$

$(\mathbf{III})$: for all $k \in N_3,\ U_k((\hat{x}_{ijk})_{i\in N_1, j\in N_2}) - \sum_{i\in N_1} \sum_{j\in N_2} \hat{q}_{ij} \hat{x}_{ijk}$

$= \max_{(x_{ijk})_{i\in N_1, j\in N_2}\in X_k} \left[U_k((x_{ijk})_{i\in N_1, j\in N_2}) - \sum_{i\in N_1} \sum_{j\in N_2} \hat{q}_{ij} x_{ijk} \right]$ and

$(\mathbf{IV})$: for all $i \in N_1,\ \hat{x}_i + \sum_{j\in N_2} \sum_{k\in N_3} \hat{x}_{ijk} = \omega_i (= 1)$.

Note that (IV) is equivalent to **A1** through **A5**. This means the equilibrium conditions for all goods.

We call $(\hat{p}, \hat{q})$ a competitive equilibrium price if there exists a competitive equilibrium $(\hat{p}, \hat{q}, \hat{x})$. If there exists a competitive equilibrium $(\hat{p}, \hat{q}, \hat{x})$, then a competitive outcome $(\hat{u}, \hat{v}, \hat{w}) \in \mathbb{R}^{n_1+n_2+n_3}$ is given by the followings.

· For all $i \in N_1,\ \hat{u}_i = U_i(\hat{x}_i) + \hat{p}_i(\sum_{j\in N_2} \sum_{k\in N_3} \hat{x}_{ijk})$.

· For all $j \in N_2,\ \hat{v}_j = -\sum_{i\in N_1} \hat{p}_i(\sum_{k\in N_3} \hat{x}_{ijk}) + \sum_{i\in N_1} \hat{q}_{ij}(\sum_{k\in N_3} \hat{x}_{ijk})$.

· For all $k \in N_3,\ \hat{w}_k = U_k((\hat{x}_{ijk})_{i\in N_1, j\in N_2}) - \sum_{i\in N_1} \sum_{j\in N_2} \hat{q}_{ij} \hat{x}_{ijk}$.

Assume that for each $(i, k) \in N_1 \times N_3,\ U_k(e^{ij}) = U_k(e^{ij'})$ for all $j, j' \in N_2$ such that $j \neq j'$, where e^{ij} is the $n_1 n_2$-dimensional vector such that

$e^{ij}_{ij} = 1$ and $e^{ij}_{i''j''} = 0$ for all $(i'', j'') \neq (i, j)$. This assumption means that for an arbitrarily fixed $i \in N_1$ and an arbitrarily fixed $k \in N_3$, one unit of goods of i traded by each middleman yields the same utility to buyer k. We call this type of middlemen *homogeneous middlemen.*

3 The main result

In this section we will establish a theorem that implies a relationship between a bilateral competitive market and a competitive market with middlemen. Let π be a subset of 2^N satisfying $\{i\} \in \pi$ for all $i \in N$ and $y = (y_T)_{T \in \pi} \in \mathbb{R}^{|\pi|}$. Let a set of numbers $\{a_T\}_{T \in \pi}$ be given. *The partitioning linear program* (Quint, 1991b), in short PLP, is given by

$$
\begin{aligned}
(PLP) \qquad & \max_{(y_T)_{T \in \pi}} \sum_{T \in \pi} a_T y_T \\
& s.t. \sum_{T \in \pi, T \ni i} y_T = 1 \quad \text{for all } i \in N \\
& \qquad y_T \geq 0 \quad \text{for all } T \in \pi.
\end{aligned}
$$

The PLP can describe linear programs associated with various markets with indivisible goods, e.g., Quint (1991a,b). Let $\pi = \{\{i\} | i \in N\} \cup \{\{i, j, k\} | i \in N_1, j \in N_2, k \in N_3\}$ and $a = (a_T)_{T \in \pi} \in \mathbb{R}^{n+n_1 n_2 n_3}_+$ satisfying (i) $a_{\{i\}} = U_i(\omega_i)$ for all $i \in N_1$, (ii) $a_{\{j\}} = a_{\{k\}} = 0$ for all $j \in N_2$ and all $k \in N_3$ and (iii) $a_{\{i,j,k\}} = U_k(e^{ij})$. We call the PLP-III as the PLP associated with the market with homogeneous middlemen. Thus the PLP-III is given by the above form. Also the PLP associated with the bilateral market between the sellers in N_1 and the buyers in N_3, the PLP-II, is given by the above form

under $\pi' \equiv \{\{i\}|i \in N_1 \cup N_3\} \cup \{\{i,k\}|i \in N_1, k \in N_3\}$ and $a' = (a'_T)_{T\in\pi'} \in \mathbb{R}_+^{n_1+n_3+n_1n_3}$ satisfying (i) $a'_{\{i\}} = a_{\{i\}}$ for all $i \in N_1$ and (ii) by homogeneity of middlemen, $a'_{\{i,k\}} = a_{\{i,j,k\}}$ for all $i \in N_1$, all $j \in N_2$ and all $k \in N_3$.

We are now ready to establish the main theorem.

Theorem If $n_2 \geq \min\{n_1, n_3\}$, then there always exists a competitive equilibrium of the market with homogeneous middlemen. Moreover, the competitive outcome of each middleman is 0 if $n_2 > \min\{n_1, n_3\}$. ■

Proof. Step 1: First, we will show that the PLP-III always has an integral solution. By Quint (1991b), the PLP-II always has an integral solution.

Let $y' = (y'_T)_{T\in\pi'} \in \mathbb{R}^{n_1+n_3+n_1n_3}$ be an integral solution of the PLP-II. Let $\{(i'_s, k'_s)\}_{s=1}^m$ is a set of all pairs $(i', k') \in N_1 \times N_3$ such that $y'_{\{i',k'\}} = 1$. Note that $0 \leq m \leq n_2$ since $n_2 \geq \min\{n_1, n_3\}$. Define $y^* = (y^*_T)_{T\in\pi} \in \mathbb{R}^{n+n_1n_2n_3}$ such that $y^*_{\{i'_s,j_s,k'_s\}} = y'_{\{i'_s,k'_s\}}$ for all $s = 1, \ldots, m$, $y^*_{\{i\}} = 1 - \sum_{k\in N_3} y'_{\{i,k\}}$ and $y^*_{\{i,j,k\}} = 0$ otherwise. Then $\sum_{T\in\pi} a_T y^*_T = \sum_{T\in\pi'} a'_T y'_T$.

Let $y = (y_T)_{T\in\pi} \in \mathbb{R}^{n+n_1n_2n_3}$ be a vector which satisfies the constraints of the PLP-III. Define $y'' = (y''_T)_{T\in\pi'} \in \mathbb{R}^{n_1+n_3+n_1n_3}$ such that $y''_{\{i,k\}} = \sum_{j\in N_2} y_{\{i,j,k\}}$ for all $i \in N_1$ and all $k \in N_3$, $y''_{\{i\}} = y_{\{i\}}$ for all $i \in N_1$. Then $\sum_{T\in\pi} a_T y_T = \sum_{T\in\pi'} a'_T y''_T$. Since y' is an integral solution of the PLP-II and y'' satisfies the constraints of the PLP-III $\sum_{T\in\pi} a_T y_T = \sum_{T\in\pi'} a'_T y''_T \leq \sum_{T\in\pi'} a'_T y'_T = \sum_{T\in\pi} a_T y^*_T$. Therefore y^* is an integral solution of the PLP-III.

Step 2: Let $\hat{y}$ be a vector in $\mathbb{Z}_+^{n_1+n_1n_2n_3}$ such that $\hat{y}_i \equiv y^*_{\{i\}}$ for all $i \in N_1$

and $\hat{y}_{ijk} \equiv y^*_{\{i,j,k\}}$ for all $\{i,j,k\} \in \pi$. S_P is the set of all $\hat{y}$. Let C be given by the set of utility vectors $(\bar{u}, \bar{v}, \bar{w}) \in \mathbb{R}^n$ satisfying

(1): *the π-partition efficiency conditions*: for $\hat{y} \in \{0,1\}^{n_1+n_1n_2n_3}$,
$\bar{u}_i + \bar{v}_j + \bar{w}_k = a_{\{i,j,k\}}$ if $\hat{y}_{ijk} = 1$;
$\bar{u}_i = a_i$ if $\hat{y}_i = 1$;

(2): *the stability conditions*:
$\bar{u}_i + \bar{v}_j + \bar{w}_k \geq a_{\{i,j,k\}}$ if $\{i,j,k\} \in \pi$;
$\bar{u}_i \geq a_{\{i\}}$ if $i \in N_1$;
$\bar{v}_j \geq a_{\{j\}} = 0$ if $j \in N_2$;
$\bar{w}_k \geq a_{\{k\}} = 0$ if $k \in N_3$.

We can show that the set C is not empty by Step 1, the theorem in Quint (1991b) and the complementary slackness condition in Dantzig (1963, pp.135-136). The proof is omitted. Choose any $(\bar{u}, \bar{v}, \bar{w})$ in the set C. We set a price list $(\hat{p}, \hat{q})$ satisfying $\hat{p}_i = \bar{u}_i$ for all $i \in N_1$ and $\hat{q}_{ij} = \bar{u}_i + \bar{v}_j$ for each $i \in N_1$ and all $j \in N_2$. We will show that $(\hat{p}, \hat{q}, \hat{y})$ is a competitive equilibrium, which yields a competitive outcome $(\bar{u}, \bar{v}, \bar{w})$. Let $\hat{y}_i = x^*_i$, $\hat{y}_{ijk} = x^*_{ijk} = \tilde{x}^*_{ijk}$ and $\sum_{k \in N_3} \hat{y}_{ijk} = \tilde{x}^*_{ij}$.

Substep 2-1: We will show that $((x^*_i)_{i \in N_1}, (\tilde{x}^*_{ij})_{i \in N_1, j \in N_2})$ satisfies **A1**, $(\tilde{x}^*_{ijk})_{i \in N_1, j \in N_2, k \in N_3}$ satisfies **A2** and $(x^*_{ijk})_{i \in N_1, j \in N_2, k \in N_3}$ satisfies **A3**. The proof will be omitted since it is a matter of calculation under $\hat{y} \in S_P$.

Substep 2-2: Let $\hat{y} \in S_P$. Then

$$\bar{v}_j = 0 \text{ if there exists } j \in N_2 \text{ such that } \hat{y}_{i'jk'} = 0 \text{ for all } (i', k') \in N_1 \times N_3;$$

$$\bar{w}_k = 0 \text{ if there exists } k \in N_3 \text{ such that } \hat{y}_{i'j'k} = 0 \text{ for all } (i', j') \in N_1 \times N_2.$$

Let $J \equiv \{j \in N_2 : \hat{y}_{ijk} = 0 \text{ for all } i \in N_1 \text{ and all } k \in N_3\}$ and $K \equiv \{k \in$

$N_3 : \hat{y}_{ijk} = 0$ for all $i \in N_1$ and all $j \in N_2\}$. Since $(\bar{u}, \bar{v}, \bar{w})$ in C, $\bar{v}_j \geq 0$ for all $j \in J$ and $\bar{w}_k \geq 0$ for all $k \in K$.

Then, by the theorem in Quint (1991b), we have

$$\sum_{j \in J} \bar{v}_j + \sum_{k \in K} \bar{w}_k = 0,$$

which implies $\bar{v}_j = 0$ for all $j \in J$ and $\bar{w}_k = 0$ for all $k \in K$.

Substep 2-3: We will show $(x_i^*, (\tilde{x}_{ij}^*)_{j \in N_2})$ is a maximal solution for all $i \in N_1$, $(\tilde{x}_{ijk}^*)_{i \in N_1, k \in N_3}$ is a maximal solution for all $j \in N_2$, and $(x_{ijk}^*)_{i \in N_1, j \in N_2}$ is a maximal solution for all $k \in N_3$.

Subsubstep 2-3-1: Consider a seller $i \in N_1$. The seller i can gain $a_{\{i\}}$ by consuming her own goods or gain $\hat{p}_i$ by selling her own goods to a middleman.

Case 1: $x_i^* = 1$ and $\tilde{x}_{ij}^* = 0$ for all $j \in N_2$. In this case, $a_{\{i\}} \geq \hat{p}_i$ must be satisfied. In fact, $a_{\{i\}} = \bar{u}_i = \hat{p}_i$.

Case 2: $x_i^* = 0$ and there is $j \in N_2$ such that $\tilde{x}_{ij}^* = 1$ and $\tilde{x}_{ij'}^* = 0$ for all $j' \in N_2 \setminus \{j\}$. In this case, $\hat{p}_i \geq a_{\{i\}}$ must be satisfied. In fact, $\hat{p}_i = \bar{u}_i \geq a_{\{i\}}$.

Subsubstep 2-3-2: Next, consider a middleman $j \in N_2$. The middleman j can gain $\hat{q}_{ij} - \hat{p}_i$ by buying one unit of goods from a seller i and selling it to a buyer, or gain 0 by doing nothing.

Case 1: $\tilde{x}_{ijk}^* = 0$ for all $i \in N_1$ and all $k \in N_3$. In this case, $0 \geq \hat{q}_{ij} - \hat{p}_i$ must be satisfied. In fact, $0 = \bar{v}_j = (\bar{u}_i + \bar{v}_j) - \bar{u}_i = \hat{q}_{ij} - \hat{p}_i$. Note that the first equality holds since Substep 2.2.

Case 2: There is $(i,k) \in N_1 \times N_3$ such that $\tilde{x}^*_{ijk} = 1$ and $\tilde{x}^*_{i'jk'} = 0$ for all $(i',k') \in N_1 \times N_3$ such that $(i',k') \neq (i,k)$. In this case, $\hat{q}_{ij} - \hat{p}_i \geq \hat{q}_{i'j} - \hat{p}_{i'}$ for all $i' \in N_1 \setminus \{i\}$ and $\hat{q}_{ij} - \hat{p}_i \geq 0$ must be satisfied. In fact, $\hat{q}_{ij} - \hat{p}_i = (\bar{u}_i + \bar{v}_j) - \bar{u}_i = \bar{v}_j = (\bar{u}_{i'} + \bar{v}_j) - \bar{u}_{i'} = \hat{q}_{i'j} - \hat{p}_{i'}$ and $\bar{v}_j \geq 0$.

Subsubstep 2-3-3: Finally, consider a buyer $k \in N_3$. The buyer k can gain $a_{\{i,j,k\}} - \hat{q}_{ij}$ by buying one unit of goods of a seller i from a middleman j, or gain 0 by consuming nothing.

Case 1: $x^*_{ijk} = 0$ for all $i \in N_1$ and all $j \in N_2$. In this case, $0 \geq a_{\{i,j,k\}} - \hat{q}_{ij}$ must be satisfied. In fact, $0 = \bar{w}_k \geq a_{\{i,j,k\}} - (\bar{u}_i + \bar{v}_j) = a_{\{i,j,k\}} - \hat{q}_{ij}$. Note that the first equality holds since Substep 2.2.

Case 2: There is $(i,j) \in N_1 \times N_2$ such that $x^*_{ijk} = 1$ and $x^*_{i'j'k} = 0$ for all $(i',j') \in N_1 \times N_2$ such that $(i',j') \neq (i,j)$. In this case, $a_{\{i,j,k\}} - \hat{q}_{ij} \geq a_{\{i',j',k\}} - \hat{q}_{i'j'}$ for all $(i',j') \in N_1 \times N_2$ such that $(i',j') \neq (i,j)$ and $a_{\{i,j,k\}} - \hat{q}_{ij} \geq 0$ must be satisfied. In fact, $a_{\{i,j,k\}} - \hat{q}_{ij} = (\bar{u}_i + \bar{v}_j + \bar{w}_k) - (\bar{u}_i + \bar{v}_j) = \bar{w}_k \geq a_{\{i',j',k\}} - (\bar{u}_{i'} + \bar{v}_{j'}) = a_{\{i',j',k\}} - \hat{q}_{i'j'}$. and $\bar{w}_k \geq 0$.

Substep 2-4: $(\hat{p}, \hat{q}, \hat{y})$ is a competitive equilibrium. It is clear by Substep 2-1 and 2-3. Therefore a competitive outcome at $(\hat{p}, \hat{q}, \hat{y})$ is $(\bar{u}, \bar{v}, \bar{w})$.

Step 3: Lastly, we will show $\bar{v}_j = 0$ for all $j \in N_2$ if $n_2 > \min\{n_1, n_3\}$. Let y^* be an integral solution of the PLP-III. Let $\hat{y}^*$ be a vector in S_p such that $\hat{y}^*_i \equiv y^*_{\{i\}}$ for all $i \in N_1$ and $\hat{y}^*_{ijk} \equiv y^*_{\{i,j,k\}}$ for all $\{i,j,k\} \in \pi$. By constraints of the PLP-III, $\sum_{i \in N_1} \sum_{j \in N_2} \sum_{k \in N_3} \hat{y}^*_{ijk} = \sum_{i \in N_1} \sum_{j \in N_2} \sum_{k \in N_3} y^*_{\{ijk\}} = \sum_{i \in N_1} (1 -$

$y^*_{\{i\}}) = n_1 - \sum_{i \in N_1} y^*_{\{i\}} \leq n_1$ and $\sum_{k \in N_3} \sum_{i \in N_1} \sum_{j \in N_2} \hat{y}^*_{ijk} = \sum_{k \in N_3} \sum_{i \in N_1} \sum_{j \in N_2} y^*_{\{ijk\}} = \sum_{k \in N_3} (1 - y^*_{\{k\}}) = n_3 - \sum_{k \in N_3} y^*_{\{k\}} \leq n_3$. Thus $\sum_{i \in N_1} \sum_{j \in N_2} \sum_{k \in N_3} \hat{y}^*_{ijk} \leq \min\{n_1, n_3\} < n_2$.

Suppose that for each $j \in N_2$ there exists $(i, k) \in N_1 \times N_3$ such that $\hat{y}^*_{ijk} = 1$. Then $\sum_{j \in N_2} \sum_{i \in N_1} \sum_{k \in N_3} \hat{y}^*_{ijk} \geq \sum_{j \in N_2} 1 = n_2$, which is a contradiction. Therefore there exists $j \in N_2$ such that $\hat{y}^*_{i'jk'} = 0$ for all $(i', k') \in N_1 \times N_3$. This leads to $\bar{v}_j = 0$ for all $j \in N_2$ by Substep 2-2. ■

An interpretation of our theorem: Consider existence of many homogenous middlemen. There always exists a competitive equilibrium and utility of each middlemen is zero. In the market with middlemen welfare produced by this market is distributed to only sellers and buyers. This implies that all middlemen no longer play a role of matchmaker; they vanish in the market. Thus the structure of the market with middlemen will be the same as that of the traditional market.

4 Concluding remarks

We assume homogeneity of middlemen in the present study. If we assume heterogeneity of middlemen, a competitive equilibrium does not necessarily exist in our model. The following example shows this statement.

Example Let $N_1 = \{i_1, i_2\}$, $N_2 = \{j_1, j_2\}$ and $N_3 = \{k_1, k_2\}$. Let the utility functions be given by

(i) $U_{i_1}(x_{i_1}) = U_{i_2}(x_{i_2}) = 0$;

(ii) $U_{k_1}(e^{i_2j_2}) = U_{k_2}(e^{i_1j_2}) = U_{k_2}(e^{i_2j_1}) = U_{k_2}(e^{i_2j_2}) = 1$;

(iii) $U_{k_1}(e^{i_1j_1}) = U_{k_1}(e^{i_1j_2}) = U_{k_1}(e^{i_2j_1}) = U_{k_2}(e^{i_1j_1}) = 0$. ■

By Quint (1991a,b), one can easily check that no integral solution exists in this example.

Oishi and Sakaue (2009) gives a necessary and sufficient condition for existence of a competitive equilibrium in a market with middlemen. This condition can be characterized by the existence of an integral solution of the PLP. Therefore no competitive equilibrium exists in the above example.

References

[1] Dantzig, G.B., 1963, Linear Programming and Extensions. (Princeton University Press, Princeton, NJ).

[2] Oishi, T. and S. Sakaue, 2009, Competitive equilibria in a market for indivisible commodities with middlemen, mimeo.

[3] Quint, T., 1991a, The core of an m-sided assignment game, Games and Economic Behavior 3, 487-503.

[4] Quint, T., 1991b, Necessary and sufficient conditions for balancedness in partitioning games, Mathematical Social Sciences 22, 87-91.

[5] Rubinstein, A. and A. Wolinsky, 1987, Middlemen, Quarterly Journal of Economics 102, 581-593.

[6] Shapley, L.S. and M. Shubik, 1972, The assignment game I: The core, International Journal of Game Theory 1, 111-130.

[7] Yang, Z., 2003, A competitive market model for indivisible commodities, Economics Letters 78, 41-47.