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On the Core of a Market for Indivisible Commodities with Middlemen

Takayuki OISHI*

Abstract

This study provides su¢ cient conditions for non-emptiness of the core of a market for indivisible commodities with middlemen. This market is formulated as a three-sided assignment game without sidepayments. Our conditions are characterized by special properties of three-sided assignment games respectively and these conditions are equivalent to the balancedness condition(Scarf, 1967).

*Takayuki OISHI
Institute of Economic Research, Kyoto University

KEIO/KYOTO MARKET QUALITY RESEARCH PROJECT
(Global Center of Excellence Program)

Graduate School of Economics and Graduate School of Business and Commerce,
Keio University
2-15-45 Mita, Minato-ku, Tokyo 108-8345 Japan

Kyoto Institute of Economics,
Kyoto University
Yoshida-honmachi, Sakyo-ku, Kyoto 606-8501 Japan

On the Core of a Market for Indivisible Commodities with Middlemen

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This study provides sufficient conditions for non-emptiness of the core of a market for indivisible commodities with middlemen. This market is formulated as a three-sided assignment game without sidepayments. Our conditions are characterized by special properties of three-sided assignment games respectively and these conditions are equivalent to the balancedness condition (Scarf, 1967).

1 Introduction

Shapley and Shubik (1972) formulated a market game of indivisible goods, called the assignment game. This game deals with a market called an assignment market. In the assignment market, each seller owns one unit of indivisible goods initially and wants to sell it, and each buyer wants to purchase at most one unit of the indivisible goods. Indivisible goods are exchanged for money. Shapley and Shubik (1972) showed that the core of this game is never empty.

However, the assignment game may be inapplicable to a three-sided housing market in which there are sellers, buyers and middlemen namely real estate brokers. This is because the situation underlying the two-sided assignment market does not assume a middlemen's role such as a matchmaker, who buys houses from sellers and sells them to buyers.

*Institute of Economic Research, Kyoto University, Mita Toho Building 5th floor, 3-1-7, Mita, Minato-ku, Tokyo 108-0073, Japan. E-mail: oishi-t@kier.kyoto-u.ac.jp

In this paper, we shall consider the core of three-sided assignment games without sidepayments. In the literature, the core of two-sided assignment games without sidepayments has been investigated, e.g., Kaneko (1982). Also, the core of three-sided assignment games with sidepayments has been investigated, e.g., Quint (1991a,b). However, few studies have been caught a light on the core of three-sided assignment games without sidepayments. In three-sided assignment games with sidepayments, income effect is ignored in the case of three-sided housing markets. Therefore it is necessary to analyze the core of three-sided assignment games without sidepayments.

We formulate a three-sided assignment game without sidepayments as a partitioning game defined by Kaneko and Wooders (1982). We present sufficient conditions for non-emptiness of the core of the three-sided assignment game without sidepayments. These conditions are equivalent to the balancedness condition (Scarf, 1967). Our conditions are characterized by special properties of three-sided assignment games. For example, one of our conditions is characterized by a generalization of an approach of Shapley and Scarf (1974) to three-sided assignment games. Shapley and Scarf (1974) introduced a line stochastic matrix representation in considering non-emptiness of the core of a pure exchange market of indivisible goods without sidepayments. In the present study, we introduce a plane stochastic matrix representation in considering non-emptiness of the core of an assignment market with middlemen.

This paper is organized as follows: in Section 2, we will explain the three-sided assignment games. Section 3 introduces a new balancedness underlying the structure of the games. Section 4 introduces plane stochastic matrixes and a special property underlying these matrixes. In this section we will present equivalent conditions to the balancedness condition (Scarf, 1967). The paper closes in Section 5 with concluding remarks on the present study.

2 The three-sided assignment games

First, we will introduce some notations for giving the definition of a three-sided assignment game without sidepayments. Let $N_1 = \{i_1, i_2, \ldots, i_{n_1}\}$, $N_2 = \{j_1, j_2, \ldots, j_{n_2}\}$ and $N_3 = \{k_1, k_2, \ldots, k_{n_3}\}$ be the set of sellers, the set of middlemen and the set of buyers respectively. Note that $n_i \in \mathbb{N}$ for $i = 1, 2, 3$.

In the market there are n_1 kinds of indivisible goods. Each seller $i \in N_1$ owns only one unit of indivisible goods initially. Each middleman $j \in N_2$ and each buyer $k \in N_3$ own no unit of goods initially.

The situation underlying this assignment market deals with a middlemen's role such as a matchmaker, who buys one unit of goods from a seller and sells it to a buyer. Therefore our game is a simple model of various assignment markets with middlemen, such as three-sided housing markets and intermediate labor markets.

We will define the three-sided assignment game as the followings.

Let $N \equiv N_1 \cup N_2 \cup N_3$ be the finite set of players and let S be a nonempty subset of N. S demotes a coalition. Let π be a class of nonempty coalitions satisfying

$$\pi \equiv \{\{i\}|i \in N\} \cup \{\{i, j, k\}|i \in N_1, j \in N_2, k \in N_3\}.$$

For any nonempty $S \subseteq N$, we call $p_S = \{T_1, \cdots, T_k\}$ a π-partition of S iff $T_t \in \pi$ for all $t = 1, \cdots, k$ and p_S is a partition of S. Let $P(S)$ be the set of all π-partitions of S.

Let $\bar{V}$ be a function on π to a class of subsets of $\mathbb{R}^N$ such that for all $S \in \pi$:

(i) $\bar{V}(S)$ is a closed set in $\mathbb{R}^n$;
(ii) if $x \in \bar{V}(S)$ and $y \in \mathbb{R}^n$ with $y_i \leq x_i$ for all $i \in S$, then $y \in \bar{V}(S)$;
(iii) $\text{Pro}_S\left[\bar{V}(S) \setminus \cup_{i \in S} \text{int}\bar{V}(\{i\})\right]$ is nonempty and bounded. Note that $\text{Pro}_S X = \{(x_i)_{i \in S} : x \in X\}$ for $S \subseteq N$ and $X \subset \mathbb{R}^n$.

Condition (iii) means that for each $S \subseteq N$, there exists a payoff vector in $\bar{V}(S)$ that guarantees each member in S at least as much as he can get by himself.

We define *a three-sided assignment game without sidepayments* (N, V) as

$$V(S) = \bigcup_{p_S \in P(S)} \bigcap_{T \in p_S} \bar{V}(T) \quad \text{for all nonempty } S \subseteq N.$$

This game is a partitioning game defined by Kaneko and Wooders (1982).

3 The core

We can say that a non-empty coalition S can *improve upon* a vector $x \in V(N)$ iff there is a vector $y \in V(S)$ such that $y_i > x_i$ for all $i \in S$. The *core* of (N, V) is the set of all vectors in $V(N)$ which cannot be improved upon by any non-empty coalition.

A family γ of nonempty coalitions of N is said to be balanced iff the system of equations $\sum_{S:\ S \ni j} \delta_S = 1$ for all $j \in N$, has a nonnegative solution $\delta = (\delta_S)_{S \in 2^N \setminus \{\emptyset\}}$ such that $\delta_S = 0$ iff $S \notin \gamma$. The solution δ is called a balancing weight vector.

Our three-sided assignment game (N, V) is said to be *balanced* iff

$$\bigcap_{S \in \gamma} V(S) \subseteq V(N) \ \text{ for any balanced family } \gamma.$$

It is well known that Scarf (1967) demonstrates that every balanced game has a nonempty core. Note that this Scarf's result does not give a necessary condition for nonempty cores.

In the following examples, we will examine cores of three-sided assignment games. The first example shows that a three-sided assignment game is not balanced and has a nonempty core. The second example shows that another three-sided assignment game has an empty core.

(1) Let me begin with an example of three-sided assignment games with nonempty cores and non-balancedness. This game (N, V) satisfies that $|N_1| = |N_2| = |N_3| = 2$ and V is derived from the following 6×14 *matrix* C.

A 6×14 *matrix* $C = (C_{pq})_{p \in N,\ q \in \{1, \cdots, 14\}}$ is given by

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
i_1	0	M	M	M	M	M	1	0.8	1.2	0.7	M	M	M	M
i_2	M	0	M	M	M	M	M	M	M	M	0.7	1.1	0.8	0.6
j_1	M	M	0	M	M	M	1	0.9	M	M	0.8	1	M	M
j_2	M	M	M	0	M	M	M	M	0.6	0.8	M	M	0.7	1.4
k_1	M	M	M	M	0	M	1	M	1.2	M	1.5	M	0.4	M
k_2	M	M	M	M	M	0	M	1.3	M	0.6	M	0.9	M	1.4

Note that M is a sufficiently large number. Also note that each column stands for a corner of each coalition in π. For example, the 10th column stands for a corner of $S = \{i_1, j_2, k_2\}$.

Take a submatrix D from C. For example, let D be given by

	7	10	12	13
i_1	1	0.7	M	M
i_2	M	M	1.1	0.8
j_1	1	M	1	M
j_2	M	0.8	M	0.7
k_1	2	M	M	0.4
k_2	M	0.6	0.9	M

Define u_i to be the minimum of the ith row of this submatrix D, then

$$u = (0.7, 0.8, 1, 0.7, 0.4, 0.6).$$

The fact that u can*not* be blocked by any coalition is clear from the matrix C. We can also verify that $u \notin V(N)$. This is because all corners of $V(N)$ are

$$\begin{pmatrix} 1 \\ 0.6 \\ 1 \\ 1.4 \\ 2 \\ 1.4 \end{pmatrix}, \begin{pmatrix} 0.8 \\ 0.8 \\ 0.9 \\ 0.7 \\ 0.4 \\ 1.3 \end{pmatrix}, \begin{pmatrix} 1.2 \\ 1.1 \\ 1 \\ 0.6 \\ 1.2 \\ 0.9 \end{pmatrix} \text{ and } \begin{pmatrix} 0.7 \\ 0.7 \\ 0.8 \\ 0.7 \\ 1.5 \\ 0.6 \end{pmatrix}.$$

Therefore this game is not balanced.

Next, take another submatrix D^* from C. For example, let D^* be given by

	7	14
i_1	1	M
i_2	M	0.6
j_1	1	M
j_2	M	1.4
k_1	2	M
k_2	M	1.4

Define u_i^* to be the minimum of the ith row of this submatrix D^*, then

$$u^* = (1, 0.6, 1, 1.4, 2, 1.4).$$

The fact that u^* can*not* be blocked by any coalition is clear from the matrix C. We can also verify that $u^* \in V(N)$. Therefore u^* is in the core.

(2) Another 6×14 *matrix* $C' = (C'_{pq})_{p \in N,\ q \in \{1, \cdots, 14\}}$ is given by

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
i_1	0	M	M	M	M	M	1	1.9	1.1	0.8	M	M	M	M
i_2	M	0	M	M	M	M	M	M	M	M	1.3	1.8	1.2	1
j_1	M	M	0	M	M	M	1	0.1	M	M	0.9	0	M	M
j_2	M	M	M	0	M	M	M	M	1.1	1.7	M	M	1.1	1
k_1	M	M	M	M	0	M	1	M	1.4	M	0.8	M	0.7	M
k_2	M	M	M	M	M	0	M	1	M	0.5	M	0.6	M	1

From this example, it is clear that the core is empty.

According to the above examples, the class of three-sided assignment games is not necessarily balanced and does not necessarily have cores. Therefore we will cast a light on a subclass of three-sided assignment games with nonempty cores.

It is necessary to introduce the following concepts in order to lead one of our results. A family γ^* of nonempty coalitions of N is said to be balanced π-family iff the system of equations $\sum_{S \in \pi:\ S \ni j} \delta_S = 1$ for all $j \in N$, has a nonnegative solution $\delta = (\delta_S)_{S \in 2^N \setminus \{\emptyset\}}$ such that $\delta_S = 0$ iff $S \notin \gamma^*$.

Our game (N, V) is said to be π^*-balanced iff

$$\bigcap_{S \in \gamma^*} V(S) \subseteq V(N) \quad \text{for any balanced } \pi\text{-family } \gamma^*.$$

Proposition 1 *A three-sided assignment game* (N, V) *is* π^**-balanced if and only if the game is balanced.*

Proof. It is clear that if part is true. We will show that only if part is true. Let γ be a balanced family which is not a π-family. Suppose $x \in \bigcap_{T \in \gamma} V(T)$. If $S \in \gamma$ does not belong to π, then there is a π-partition P^*_S of S with $x \in \bigcap_{T \in P^*_S} V(T)$ by the definition of the game. For $T \in \pi$, let $\gamma_T = \{S : S \in \gamma,\ S \notin \pi \text{ and } T \in P^*_S\}$. We define $\hat{\gamma}$ and $\hat{\delta}$ by

$$\hat{\gamma} = \{T : T \in \gamma \text{ and } T \in \pi\} \cup \left(\bigcup_{S \in \gamma,\ S \notin \pi} P^*_S \right) \quad \text{and}$$

$$\hat{\delta}_T = \begin{cases} \delta_T + \sum_{S \in \gamma_T} \delta_S & \text{if } T \in \gamma \text{ and } T \in \pi; \\ \sum_{S \in \gamma_T} \delta_S & \text{if } T \notin \gamma \text{ and } T \in \pi; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\hat{\delta} = (\hat{\delta}_S)_{S \in 2^N \setminus \{\emptyset\}}$ is a balancing weight vector for $\hat{\gamma}$ and this $\hat{\gamma}$ is a balanced π-family. Then we have $x \in \bigcap_{T \in \hat{\gamma}} V(T)$, which implies that $x \in V(N)$. ■

4 Plane stochastic matrix representation

We will give another sufficient condition for non-empty cores of (N, V). This condition is characterized by a certain property of plane stochastic matrixes.

For any non-empty coalition S, if n by n by n zero-one (three-dimensional) matrix $A_S = (a_{S:ijk})$, all rows, columns and heights of which are indexed by members in N, satisfies

$$\begin{aligned} \sum_{i \in N} \sum_{j \in N} a_{S:ijk} &= \begin{cases} 1 & \text{if } k \in S \\ 0 & \text{if } k \notin S, \end{cases} \\ \sum_{j \in N} \sum_{k \in N} a_{S:ijk} &= \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases} \quad \text{and} \\ \sum_{k \in N} \sum_{i \in N} a_{S:ijk} &= \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{if } j \notin S, \end{cases} \end{aligned} \tag{1}$$

then we call A_S an S-*permutation matrix.*

We define $D_S(b) = (d_{S:ijk}(b))$ for each b in $\mathbb{R}^N$ and each non-empty coalition S as follows:

$$d_{S:ijk}(b) = \begin{cases} 1 & \text{if } i \in S \cap N_1,\ j \in N_2,\ k \in N_3,\ b \in V(\{i,j,k\}); \\ 1 & \text{if } i \in S \cap N_1,\ j \in N_1,\ k \in N_1,\ b \in V(\{i\}); \\ 1 & \text{if } i \in S \cap N_2,\ j \in N_2,\ k \in N_2,\ b \in V(\{j\}); \\ 1 & \text{if } i \in S \cap N_3,\ j \in N_3,\ k \in N_3,\ b \in V(\{k\}); \\ 1 & \text{if } i \in S \cap N_1,\ j \in N_3,\ k \in N_2; \\ 1 & \text{if } i \in S \cap N_2,\ j \in N_1,\ k \in N_3; \\ 1 & \text{if } i \in S \cap N_2,\ j \in N_3,\ k \in N_1; \\ 1 & \text{if } i \in S \cap N_3,\ j \in N_1,\ k \in N_2; \\ 1 & \text{if } i \in S \cap N_3,\ j \in N_2,\ k \in N_1; \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Lemma 1 $V(S) = \left\{b \in \mathbb{R}^N : D_S(b) \geq A_S \text{ for some } S\text{-permutation matrix } A_S\right\}$ *for all* $S \subseteq N$.

Proof. The proof will be completed by the following two steps.
Step 1: Suppose that there is an S-permutation matrix A_S such that $D_S(b) \geq A_S$. Let $\{(i,j,k) : a_{S:ijk} = 1,\ i \in N_1,\ j \in N_2,\ k \in N_3\} = \{(i_1, j_1, k_1), \cdots, (i_k, j_k, k_k)\}$. Note that $i_t,\ j_t,\ k_t \in S$ for all $t = 1, \cdots, k$ by (1). Let $N_1 \cap S \setminus \{i_1, \cdots, i_k\} = \{i(1), \cdots, i(p)\}$, $N_2 \cap S \setminus \{j_1, \cdots, j_k\} = \{j(1), \cdots, j(q)\}$ and $N_3 \cap S \setminus \{k_1, \cdots, k_k\} = \{k(1), \cdots, k(r)\}$. We define a π-partition $p_S = \{T_1, \cdots, T_f\}$ $(f = k + p + q + r)$ by

$$\begin{array}{ll} T_t = \{i_t, j_t, k_t\} & \text{for all } t = 1, \cdots, k, \\ T_{k+t} = \{i(t)\} & \text{for all } t = 1, \cdots, p, \\ T_{k+p+t} = \{j(t)\} & \text{for all } t = 1, \cdots, q, \\ T_{k+p+q+t} = \{k(t)\} & \text{for all } t = 1, \cdots, r. \end{array}$$

Since $a_{S:i_t j_t k_t} = d_{S:i_t j_t k_t}(b) = 1$, $i_t \in N_1 \cap S$, $j_t \in N_2 \cap S$ and $k_t \in N_3 \cap S$, we have $b \in V(\{i_t, j_t, k_t\})$ by (2). Since $a_{S:i(t)ii=1}$ for some $i \in N_1 \cap S$, $a_{S:j(t)jj=1}$ for some $j \in N_2 \cap S$ and $a_{S:k(t)kk} = 1$ for some $k \in N_3 \cap S$ by (1) and the definitions of $i(t)$, $j(t)$ and $k(t)$, we have $d_{S:i(t)ii}(b) = 1$, $d_{S:j(t)jj}(b) = 1$ and $d_{S:k(t)kk}(b) = 1$, which implies $b \in V(\{i\})$, $b \in V(\{j\})$ and $b \in V(\{k\})$ by (2). Therefore $b \in V(T_t)$ for all $t = 1, \cdots, f$, i.e., $b \in \cap_{T \in p_S} V(T)$.

Step 2: Let $b \in \cap_{T \in p_S} V(T)$ for some π-partition p_S. Let

$$\begin{array}{l} \{T \in p_S : |T| = 3\} = \{\{i_1, j_1, k_1\}, \cdots, \{i_t, j_t, k_t\}\}, \\ \{T \in p_S : |T| = 1 \text{ and } T \subseteq N_1\} = \{\{i(1)\}, \cdots, \{i(p)\}\}, \\ \{T \in p_S : |T| = 1 \text{ and } T \subseteq N_2\} = \{\{j(1)\}, \cdots, \{j(q)\}\} \text{ and} \\ \{T \in p_S : |T| = 1 \text{ and } T \subseteq N_3\} = \{\{k(1)\}, \cdots, \{k(r)\}\}. \end{array}$$

Without losing the generality, assume $p \leq q \leq r$. We define an S-permutation matrix $A_S = (a_{S:ijk})$ as follows.

For all i_t $(t = 1, \cdots, k)$:

$$a_{S:i_t jk} = \begin{cases} 1 & \text{if } j = j_t,\ k = k_t \\ 0 & \text{otherwise,} \end{cases}$$

for all $i(t)$ $(t = 1, \cdots, p)$:

$$a_{S:i(t)jk} = \begin{cases} 1 & \text{if } j = j(t),\ k = k(t) \\ 0 & \text{otherwise,} \end{cases}$$

for all $i \in N_1 \backslash S$:

$$a_{S:ijk} = 0 \text{ for all } j, k \in N,$$

for all $i \in N_2 \cup N_3$:

$$a_{S:ijk} = \begin{cases} a_{S:kij} & \text{if } i \in I,\ j \in N_3 \text{ and } k \in N_1; \\ a_{S:jki} & \text{if } i \in I,\ j \in N_1 \text{ and } k \in N_2; \\ 1 & \text{if } i \in \{j(p+1), \cdots, j(q)\} \text{ and } i = j = k; \\ 1 & \text{if } i \in \{k(p+1), \cdots, k(r)\} \text{ and } i = j = k; \\ 0 & \text{otherwise,} \end{cases}$$

where $I = \{j_1, \cdots, j_k, k_1, \cdots, k_k, j(1), \cdots, j(p), k(1), \cdots k(p)\}$.

It is easily verified that this A_S is an S-permutation matrix. We will show $D_S(b) \geq A_S$. When $a_{s:ijk} = 0$, it is always true that $d_{s:ijk}(b) \geq a_{S:ijk}$. So, suppose $a_{S:ijk} = 1$. If $(i, j, k) = (i_t, j_t, k_t)$ for some $t \leq k$, then $b \in V(\{i_t, j_t, k_t\})$ since $b \in \cap_{T \in p_S} V(T)$, which implies $d_{s:ijk}(b) = 1$. Let $(i, j, k) = (i(t), j(t), k(t))$ for some $t \leq p$. Then $b \in V(\{i(t)\}) \cap V(\{j(t)\}) \cap V(\{k(t)\}) \subseteq V(\{i(t), j(t), k(t)\})$ by $b \in \cap_{T \in p_S} V(T)$ and the superadditivity of V. This implies $d_{s:ijk}(b) = 1$. When $i = j(t),\ j = j(t)$ and $k = j(t)$ $(p+1 \leq t \leq q)$, it is true by the same reason that $b \in V(\{j(t)\})$, which implies $d_{s:ijk}(b) = 1$. When $i = k(t),\ j = k(t)$ and $k = k(t)$ $(q+1 \leq t \leq r)$, it is true by the same reason that $b \in V(\{k(t)\})$, which implies $d_{s:ijk}(b) = 1$. Therefore $D_S(b) \geq A_S$. ■

Let γ be an arbitrary balanced family of coalitions, and let $b \in \cap_{S \in \gamma} V(S)$. Let $(\delta_S)_{S \in 2^N \backslash \{\emptyset\}}$ be balancing weights for γ. Since $b \in V(S)$ for each $S \in \gamma$, there exists an A_S such that $D_S(b) \geq A_S$ because of Lemma 1. Let $B = \sum_{S \in \gamma} \delta_S A_S$. In the following lemma, we can see the crucial fact about B.

Lemma 2 *Every three-dimensional matrix B is plane stochastic, i.e., it is non-negative and has all row-, column- and height-sums equal to 1.*

Proof. This follows directly from the definition of balancing weights and

A_S; in fact, the k_{th} height sum is

$$\begin{aligned}\sum_{i\in N}\sum_{j\in N}\sum_{S\in\gamma}\delta_S a_{S:ijk} &= \sum_{S\in\gamma}\delta_S\sum_{i\in N}\sum_{j\in N}a_{S:ijk} \\ &= \sum_{S\in\gamma}\delta_S\begin{cases}1 & \text{if } k\in S\\ 0 & \text{if } k\notin S\end{cases} \\ &= \sum_{S\in\gamma:\ S\ni k}\delta_S = 1,\end{aligned}$$

and the argument for the row- and column- sums is the same. ■

Owing to Lemma 2, we will introduce a certain property of a plane stochastic matrix as follows:

Definition 1 *If it is possible to change an arbitrary plane stochastic matrix B into an N-permutation matrix A_N, i.e., to eliminate any fractional entries without changing the row-or column-or height-sums and to do so without disturbing any entries which are already 0 or 1, we say that B has the integral property.*

Proposition 2 *A three-sided assignment game is balanced if and only if every plane stochastic matrix B has the integral property.*

Proof. The proof will be completed by the following two steps.
Step 1: Suppose that every plane stochastic matrix B has the integral property. Let $b\in\cap_{S\in\gamma}V(S)$ for an arbitrary balanced family γ. We will show $b\in V(N)$. It holds that

$$D_N(b)=\sum_{S\in\gamma}\delta_S D_S(b).$$

For, if $d_{N:ijk}(b)=1$, then $d_{S:ijk}(b)=1$ if $i\in S$ and $d_{S:ijk}(b)=0$ if $i\notin S$ by (2), which implies

$$\sum_{S\in\gamma}\delta_S D_S(b)=\sum_{S\in\gamma:\ S\ni i}\delta_S=1=d_{N:ijk}(b),$$

and If $d_{N:ijk}(b)=0$, then $d_{S:ijk}(b)=0$ for all $S\in\gamma$, which implies

$$\sum_{S\in\gamma}\delta_S D_S(b)=0=d_{N:ijk}(b).$$

Since $b \in V(S)$ for each $S \in \gamma$, there exists an A_S such that $D_S(b) \geq A_S$ because of Lemma 1. So,

$$D_N(b) \geq B.$$

By the supposition, we have $D_N(b) \geq A_N$, which implies $b \in V(N)$.

Step 2: Suppose that a plane stochastic matrix B does not have the integral property. Let a balanced family γ and $b \in \cap_{S\in\gamma} V(S)$ such that $D_N(b) \geq B$. We will show $b \notin V(N)$. By the supposition, we have $D_N(b) \geq C$, where $C \neq A_N$. Of course, C is a plane stochastic matrix. It holds that both A_N and C are not comparable. For, if $C > A_N$, then at least one row- or column- or height- sum is more than 1, which is a contradiction. The argument for $C < A_N$ is the same. Therefore $b \notin V(N)$ by Lemma 1. ■

We obtain the following main result immediately from Proposition 1 and 2.

Theorem 1 *Let γ be an arbitrary balanced family of coalitions and let $B = \sum_{S\in\gamma} \delta_S A_S$. Then the following three statements are equivalent each other.*

i) A three-sided assignment game without sidepaymets (N, V) is balanced.
ii) The three-sided assignment game (N, V) is π^*-balanced.
iii) Every plane stochastic matrix B has the integral property.

5 Concluding Remarks

Oishi and Sakaue (2009) presented a simple model of three-sided assignment markets with sidepayments and its extension. Oishi and Sakaue (2009) showed that the set of all imputations given by competitive equilibria of an assignment market with middlemen who trade a single unit of indivisible goods coincides with the core of the game derived from the market. Therefore, whether or not the set of all imputations given by competitive equilibria of three-sided assignment markets without sidepayments coincides with the core of the game considered in the present study may reserve investigation, which we leave to the future research.

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