Competitive Equilibria in a Market for
Indivisible Commodities with Middlemen

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In this paper, we consider a market of indivisible goods with middlemen as an assignment market. Initially, we show that the set of all imputations given by competitive equilibria of an assignment market with middlemen who trade a single unit of indivisible goods coincides with the core of the game derived from the market. Next, by using the first result, we show that the set of all imputations given by competitive equilibria of an assignment market with middlemen who trade multiple units of indivisible goods coincides with the core of the game of another three-sided assignment market generated from this market. Finally, by using the second result, we give an equivalent condition for the existence of a competitive equilibrium of an assignment market in which each middleman trades multiple units of goods. This equivalent condition can be characterized by the existence of an integral solution of a linear program.

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Keywords: Assignment games; Middlemen; Competitive equilibrium; Core; Partitioning linear program

JEL classification: C62; C71; C78; D50

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1 Introduction

Shapley and Shubik (1972) formulated a market game in which the economic agents consist of sellers and buyers, and indivisible goods are exchanged for money. This market game is called the assignment game. The assignment game deals with a market in which each seller owns one unit of indivisible goods initially and wants to sell it, and each buyer wants to purchase at most one unit of the indivisible goods. Shapley and Shubik (1972) showed that the core of the assignment game coincides with the set of optimal dual solutions to a certain linear programming problem called the assignment linear program. Since there always exist integral solutions of the assignment linear program, the core of this game is never empty. Furthermore, they showed that the core of this game coincides with the set of imputations given by competitive equilibria of the market. Typical markets to which the assignment game is applicable are housing markets, e.g., Shapley and Shubik (1972) and Kaneko (1976, 1982). Also the assignment market game is applicable to two-sided labor markets in which there are workers and enterprises, e.g., Crawford and Knoer (1981), Kelso and Crawford (1982) and Sotomayor (2002).

However, the two-sided assignment game may be inapplicable to a three-sided housing market in which there are sellers, buyers and middlemen. This is because the situation underlying the two-sided assignment market does not assume a middlemen’s role such as a matchmaker, who buys houses from sellers and sells them to buyers. There can be found the literature describing various situations that middlemen play the role of matchmakers, e.g., Rubinstein and Wolinsky (1987), Yavaş (1994) and Wimmer et al. (2000). One of these situations is that private search of sellers or buyers is very costly and middlemen reduce the costs of private search. In this situation, the utility of a buyer may depend not only on the kind of goods she bought but also on a middleman from whom she bought it. Therefore it is appropriate to formulate this type of three-sided markets as an assignment game with middlemen, namely one case of m-sided assignment games defined by Quint (1991a). The three-sided assignment game is also applicable to intermediate labor markets.

In this paper, we shall take a side of middlemen into two-sided assignment markets. Then we investigate the relationship between the competitive equilibrium of three-sided assignment markets and the core of the three-sided assignment games derived from the markets. In the analysis of assignment markets, it has been a major point to consider cores of assignment games and
competitive equilibria of the assignment markets since Shapley and Shubik (1972), e.g., Kaneko (1976, 1982) and Quinzii (1984). In the literature of three-sided assignment markets, cores of the three-sided assignment games or a generalization of this game have been mainly investigated, e.g., Quint (1991a, 1991b). In fact, Quint (1991b) gives an equivalent condition for balancedness of partitioning games defined by Kaneko and Wooders (1982), the class of which includes three-sided assignment games. This equivalent condition for balancedness is that there exist integral solutions in a certain linear programming problem called the partitioning linear program. This equivalent condition implies that the core of three-sided assignment games may be empty. Although the core of the three-sided assignment games has been investigated, few studies have been caught a light on the relationship between the core and the competitive equilibrium of the three-sided assignment markets. Our study casts a light mainly on this point.

Our paper presents a simple model of assignment markets with middlemen and its extension. In the followings, we will give a brief explanation on the simple market model. Each seller and each buyer are under the same setting as Shapley and Shubik (1972). Each middleman owns no unit of goods initially and she can trade a single unit of goods between sellers and buyers. Each middleman wants to sell to a buyer at most one unit of goods which the middleman purchases from a seller. Moreover we assume that each middleman consumes no unit of goods which she purchases from the seller. These assumptions are for simplicity of our analysis. We will extend the simple market model to the model of a complex market, namely a three-sided assignment market in which each middleman can trade multiple units of indivisible goods.

Initially, in the simple market model, we consider the relationship between the competitive equilibrium of an assignment market with middlemen and the core of the game derived from this market. We show that the set of all imputations given by competitive equilibria of the assignment market in the first stage coincides with the core of the game derived from the market. Next, in the extension of the simple market model, we consider the relationship between the competitive equilibrium of an assignment market with middlemen and the core of the game derived from the market in the second stage. By using the first result, we show that the set of all imputations given by competitive equilibria of the assignment market in the second stage coincides with the core of the game of another three-sided assignment market generated from the market. Finally, by the second result in our study and
the result of Quint (1991b), we give an equivalent condition for the existence of a competitive equilibrium of the assignment market with middlemen who trade multiple units of indivisible goods. This equivalent condition for the existence of a competitive equilibrium can be characterized by the existence of an integral solution of the partitioning linear program.

The remainder of the paper is organized as follows: in Section 2, we will give the model of an assignment market with middlemen who trade a single unit of indivisible goods. Section 3 introduces three-sided assignment games and we will give the first main result of the equivalence of the core and the competitive equilibria of the market explained in Section 2. Section 4 deals with the model of an assignment market with middlemen who trade multiple units of indivisible goods. We will give the second main result of the equivalence of the core and the competitive equilibria of the market explained in Section 4. This leads to the last main result: an equivalent condition for existence of a competitive equilibrium of the market in Section 4. The paper closes in Section 5 with short concluding remarks on our study.

2 Simple Three-sided Assignment Markets

In this section, we will introduce simple three-sided assignment markets. By simple three-sided assignment markets, we mean assignment markets with middlemen who trade a single unit of indivisible goods.

Let \((N_1, N_2, N_3)\) be a three-sided assignment market, where \(N_1 = \{i_1, i_2, \ldots, i_{n_1}\}\) \((n_1 \in \mathbb{N})\) is the set of sellers, \(N_2 = \{j_1, j_2, \ldots, j_{n_2}\}\) \((n_2 \in \mathbb{N})\) is the set of middlemen and \(N_3 = \{k_1, k_2, \ldots, k_{n_3}\}\) \((n_3 \in \mathbb{N})\) is the set of buyers. Let \(N = N_1 \cup N_2 \cup N_3\) be the set of all agents. In the market there are \(n_1\) kinds of indivisible goods and they are exchanged for money. Each seller \(i \in N_1\) owns only one unit of indivisible goods initially, namely \(\omega_i = 1\). Each middleman and each buyer own no unit of goods initially.

Let us denote the demand and supply of this market as follows:

**Sellers’ side:** Each seller \(i\) sells one unit of her goods to a middleman or consumes it by herself. Let \(x_i\) be the consumption of seller \(i\), namely \(x_i \in \{0, 1\}\).

**Middlemen’s side:** Each middleman \(j\) wants to sell at most one unit of goods to only a buyer. For this purpose, each middleman purchases at most
one unit of goods from only a seller. Let $\bar{x}_{ij}$ be the supply of seller $i$ to middleman $j$, namely $\bar{x}_{ij} \in \{0, 1\}$. Note that we assume that each middleman consumes no unit of goods which she purchases from the seller\(^1\).

**Buyers’ side:** Each buyer $k$ purchases at most one unit of goods from only a middleman in order to consume it by herself. Let $x_{ijk}$ be the consumption of buyer $k$, namely $x_{ijk} \in \{0, 1\}$. This means, if $x_{ijk} = 1$, that buyer $k$ demands one unit of goods which middleman $j$ purchases from seller $i$. Moreover, let $\bar{x}_{ijk}$ be the supply of middleman $j$, namely $\bar{x}_{ijk} \in \{0, 1\}$. This means, if $\bar{x}_{ijk} = 1$, that middleman $j$ supplies to buyer $k$ one unit of goods which she purchases from seller $i$.

From the above explanation, we give assumptions **A1**, **A2** and **A3** on constraints of goods in each agent’s side.

**A1**: For all $i \in N_1$, $X_i \equiv \{(x_i, (\bar{x}_{ij})_{j \in N_2}) \in \mathbb{Z}^{1+n_2}_+ : x_i + \sum_{j \in N_2} \bar{x}_{ij} = \omega_i = 1\}$.

**A2**: For all $j \in N_2$, $X_j \equiv \{(\bar{x}_{ijk})_{i \in N_1, k \in N_3} \in \mathbb{Z}^{n_3 \times n_2}_+ : \sum_{i \in N_1} \sum_{k \in N_3} \bar{x}_{ijk} \leq 1\}$.

**A3**: For all $k \in N_3$, $X_k \equiv \{(x_{ijk})_{i \in N_1, j \in N_2} \in \mathbb{Z}^{n_2 \times n_3}_+ : \sum_{i \in N_1} \sum_{j \in N_2} x_{ijk} \leq 1\}$.

Thus, we formalize a simple three-sided assignment market as $(N_1, N_2, N_3)$ under **A1**, **A2** and **A3**. Figure 1 is an illustration of the flow of seller $i$’s initial endowment from seller $i$ to buyer $k$ via middleman $j$ in a simple three-sided assignment market.

Each seller and each buyer have a utility function on consumption respectively. The utility functions on consumption are measured in terms of money and given by $U_i : \mathbb{Z}_+ \to \mathbb{R}$ for each $i \in N_1$ and $U_k : \mathbb{Z}^{n_3 \times n_2}_+ \to \mathbb{R}$ for each $k \in N_3$ respectively. We assume that for each $l \in N_1 \cup N_3$, $U_l(\cdot)$ is non-decreasing and $U_l(0) = 0$. Each middleman $j$ purchases at most one unit of goods at the price $p_i \in \mathbb{R}_+$ from seller $i$. Also, middleman $j$ sells at most one unit of $i$’s initial goods at the price $q_{ij} \in \mathbb{R}_+$ to buyer $k$. Let two distinct price lists be given by $p = (p_i)_{i \in N_1} \in \mathbb{R}^{n_1}_+$ and $q = (q_{ij})_{i \in N_1, j \in N_2} \in \mathbb{R}^{n_2 \times n_3}_+$ respectively.

Then each utility function is given by the followings.

\[ \cdot \text{For all } i \in N_1, U_i(x_i) + p_i(\sum_{j \in N_2} \bar{x}_{ij}). \]

\(^1\)For example, in practice, each middleman in three-sided housing markets does not consume the house which she purchases from a seller. Our assumption reflects this fact.
For all \( j \in N_2, \) \( - \sum_{i \in N_1} p_i (\sum_{k \in N_3} \tilde{x}_{ijk}) + \sum_{i \in N_1} q_{ij} (\sum_{k \in N_3} \tilde{x}_{ijk}). \)

For all \( k \in N_3, \) \( U_k ((x_{ijk})_{i \in N_1, j \in N_2}) - \sum_{i \in N_1} \sum_{j \in N_2} q_{ij} x_{ijk}. \)

We will give an interpretation only of the utility function of each middleman. This utility function consists of two components. The first is the total cost of purchasing goods from a seller. The second is revenue produced by her selling goods to a buyer.

Since in each market demand equals supply at any equilibrium, we give assumptions \( A4 \) and \( A5 \) on the market-clearing conditions.

\( A4 \) : For all \((i, j) \in N_1 \times N_2, \) \( \sum_{k \in N_3} \tilde{x}_{ijk} = \tilde{x}_{ij}. \)

\( A5 \) : For all \((i, j, k) \in N_1 \times N_2 \times N_3, \) \( x_{ijk} = \tilde{x}_{ijk}. \)

Next, let us introduce a competitive equilibrium (price) and a competitive outcome in the simple three-sided assignment market respectively.

A tuple \((\hat{p}, \hat{q}, \hat{x}) = ((\hat{p}_i)_{i \in N_1}, (\hat{q}_{ij})_{i \in N_1, j \in N_2}, ((\hat{x}_i)_{i \in N_1}, (\hat{x}_{ijk})_{i \in N_1, j \in N_2, k \in N_3})) \in \mathbb{R}_{++}^{n_1} \times \mathbb{R}_{++}^{n_1n_2} \times \mathbb{Z}_{++}^{n_1n_2n_3} \) is called a competitive equilibrium if \((\hat{p}, \hat{q}, \hat{x}) \) satisfies
(I): for all \( i \in N_1 \), \( U_i(\hat{x}_i) + \hat{p}_i(\omega_i - \hat{x}_i) \)
\[= \max_{x_i, (\hat{x}_{ijk})_{j \in N_2}} \left[U_i(x_i) + \hat{p}_i(\sum_{j \in N_2} \hat{x}_{ijk}) \right], \]

(II): for all \( j \in N_2 \), \(- \sum_{i \in N_1} \hat{p}_i(\sum_{k \in N_3} \hat{x}_{ijk}) + \sum_{i \in N_1} \hat{q}_j(\sum_{k \in N_3} \hat{x}_{ijk}) \)
\[= \max_{(\hat{x}_{ijk})_{i \in N_1, k \in N_3 \in X_j}} \left[- \sum_{i \in N_1} \hat{p}_i(\sum_{k \in N_3} \hat{x}_{ijk}) + \sum_{i \in N_1} \hat{q}_j(\sum_{k \in N_3} \hat{x}_{ijk}) \right], \]

(III): for all \( k \in N_3 \), \( U_k((\hat{x}_{ijk})_{i \in N_1, j \in N_2}) - \sum_{i \in N_1} \sum_{j \in N_2} \hat{q}_j \hat{x}_{ijk} \)
\[= \max_{(\hat{x}_{ijk})_{i \in N_1, j \in N_2 \in X_k}} \left[U_k((\hat{x}_{ijk})_{i \in N_1, j \in N_2}) - \sum_{i \in N_1} \sum_{j \in N_2} \hat{q}_j \hat{x}_{ijk} \right] \text{ and} \]

(IV): for all \( i \in N_1 \), \( \hat{x}_i + \sum_{j \in N_2} \sum_{k \in N_3} \hat{x}_{ijk} = \omega_i (= 1) \).

Note that (IV) is equivalent to A1 through A5. This means the equilibrium conditions for all goods.

We call \((\hat{p}, \hat{q})\) a competitive equilibrium price if there exists a competitive equilibrium \((\hat{p}, \hat{q}, \hat{x})\).

If there exists a competitive equilibrium \((\hat{p}, \hat{q}, \hat{x})\), then a competitive outcome \((\hat{u}, \hat{v}, \hat{w}) \in \mathbb{R}^{n_1+n_2+n_3}\) is given by the followings.

\[
\begin{align*}
\cdot & \text{ For all } i \in N_1, \ \hat{u}_i = U_i(\hat{x}_i) + \hat{p}_i(\sum_{j \in N_2} \sum_{k \in N_3} \hat{x}_{ijk}). \\
\cdot & \text{ For all } j \in N_2, \ \hat{v}_j = - \sum_{i \in N_1} \hat{p}_i(\sum_{k \in N_3} \hat{x}_{ijk}) + \sum_{i \in N_1} \hat{q}_j(\sum_{k \in N_3} \hat{x}_{ijk}). \\
\cdot & \text{ For all } k \in N_3, \ \hat{w}_k = U_k((\hat{x}_{ijk})_{i \in N_1, j \in N_2}) - \sum_{i \in N_1} \sum_{j \in N_2} \hat{q}_j \hat{x}_{ijk}. 
\end{align*}
\]

### 3 The Three-sided Assignment Game

Let \(|N| = n\) and let \(\pi\) be a subset of \(2^N\) given by \(\pi \equiv \left\{ \{i\} | i \in N \right\} \cup \left\{ \{i,j,k\} | i \in N_1, j \in N_2, k \in N_3 \right\}\). Let \(a = (a_T)_{T \in \pi} \in \mathbb{R}^{n_1+n_2+n_3}\) be a vector satisfying as follows: \(a_{\{i\}} = U_i(\omega_i)\) for all \(i \in N_1\) and \(a_{\{j\}} = a_{\{k\}} = 0\) for all \(j \in N_2\) and \(k \in N_3\). Moreover, \(a_{\{i,j,k\}} = U_k(e^{ij})\) where \(e^{ij}\) is the \(n_1n_2\)-dimensional vector such that \(e^{ij}_{i'j'} = 1\) and \(e^{ij}_{i'j'} = 0\) for all \((i', j') \neq (i, j)\). We define a \(\pi\)-partition of \(S\), \(\rho_S\), as any partition of \(S\) into \(\pi\). Let \(P_S\) be the class of all \(\pi\)-partitions of \(S\).

By Kaneko and Wooders (1982), we can give the three-sided assignment game as a partitioning game \((N, V)\), where
\[
V(S) \equiv \max_{\rho_S \in P_S} \sum_{T \in \rho_S} a_T \quad \text{for nonempty } S \subseteq N \quad \text{with } V(\emptyset) = 0.
\]
Owing to Kaneko and Wooders (1982), the core of \((N, V)\) is given by the set of utility vectors \((\hat{u}, \hat{v}, \hat{w}) \in \mathbb{R}^n\) satisfying (1) \(\sum_{i \in N_1} \hat{u}_i + \sum_{j \in N_2} \hat{v}_j + \sum_{k \in N_3} \hat{w}_k = V(N)\) and (2) for all \(T \in \pi\), \(\sum_{i \in N_1 \cap T} \hat{u}_i + \sum_{j \in N_2 \cap T} \hat{v}_j + \sum_{k \in N_3 \cap T} \hat{w}_k \geq V(T)\). The first is the efficiency condition and the second is the stability conditions.

**Lemma 1** Each competitive outcome belongs to the core of \((N, V)\).

**Proof.** Let \((\hat{u}, \hat{v}, \hat{w})\) be a competitive outcome. Then, there exists a competitive equilibrium \((\hat{p}, \hat{q}, \hat{x})\) which attains \((\hat{u}, \hat{v}, \hat{w})\).

Let \(\rho^*_S\) be a \(\pi\)-partition which attains \(V(S)\). Let \(z = ((z_i)_{i \in N_1}, (z_{ijk})_{i \in N_1, j \in N_2, k \in N_3})\) be a vector in \(\mathbb{Z}^{n_1+n_2+n_3}\) satisfying

(i) \(z_i = 1\) if \(\{i\} \in \rho^*_S\) and \(i \in N_1\),

(ii) \(z_i = 0\) if \(\{i\} \notin \rho^*_S\) and \(i \in N_1 \cap S\),

(iii) \(z_i = 1\) if \(i \in N_1 \setminus S\),

(iv) \(z_{ijk} = 1\) if \(\{i, j, k\} \in \rho^*_S\) and \(\{i, j, k\} \in N_1 \times N_2 \times N_3\) and

(v) \(z_{ijk} = 0\) otherwise.

**Step 1:** Let \(z_i = x_i\), \(z_{ijk} = x_{ijk} = \tilde{x}_{ijk}\) and \(\sum_{k \in N_3} z_{ijk} = \tilde{x}_{ij}\). Then we will show that \(z\) satisfies A1 through A3, namely the followings.

\[
\begin{align*}
\text{(1)} & \quad z_i + \sum_{j \in N_2} \sum_{k \in N_3} z_{ijk} = 1 \quad \text{for all } i \in N_1 \cap S. \\
\text{(2)} & \quad \sum_{i \in N_1} \sum_{k \in N_3} z_{ijk} \leq 1 \quad \text{for all } j \in N_2 \cap S. \\
\text{(3)} & \quad \sum_{i \in N_1} \sum_{j \in N_2} z_{ijk} \leq 1 \quad \text{for all } k \in N_3 \cap S.
\end{align*}
\]

First, we will show (1).

**Case 1:** Suppose \(\{i\} \in \rho^*_S\) and \(i \in N_1\). By (i), \(z_i = 1\). Also, by (v), we have \(z_{ijk} = 0\) for all \((j, k) \in N_2 \times N_3\). Then \(z_i + \sum_{j \in N_2} \sum_{k \in N_3} z_{ijk} = 1\).

**Case 2:** Suppose \(\{i\} \notin \rho^*_S\) and \(i \in N_1 \cap S\). By (ii), \(z_i = 0\). This supposition implies that there exists \(\{j, k\} \in N_2 \times N_3\) such that \(\{\tilde{i}, \tilde{j}, \tilde{k}\} \in \rho^*_S\). By (iv), \(z_{ijk} = 1\). Also, by (v), we have \(z_{ijk} = 0\) for all \((j, k) \in N_2 \times N_3\) such that \(j \neq \tilde{j}\) and \(k \neq \tilde{k}\). Then \(z_i + \sum_{j \in N_2} \sum_{k \in N_3} z_{ijk} = 1\). This completes the proof of (1).

Next, we will show (2).
Case 1: Suppose \( \{j\} \in \rho^*_S \) and \( \hat{j} \in N_2 \). By (v), \( z_{ijk} = 0 \) for all \((i, k) \in N_1 \times N_3 \). Then \( \sum_{j \in N_1} \sum_{k \in N_3} z_{ijk} = 0 \).

Case 2: Suppose \( \{i', j', k'\} \in \rho^*_S \) and \( \{i', j', k'\} \in N_1 \times N_2 \times N_3 \). By (iv), \( z_{i'j'k'} = 1 \). Also, by (v), we have \( z_{ij'k} = 0 \) for all \((i, k) \in N_1 \times N_3 \) such that \( i \neq i' \) and \( k \neq k' \). Then \( \sum_{j \in N_2} \sum_{k \in N_3} z_{ij'k} = 1 \). This completes the proof of (2).

The proof of (3) is the same as that of (2). Therefore we will omit the proof of (3).

Step 2: We will prove

\[
\sum_{i \in S \cap N_1} \hat{u}_i + \sum_{j \in S \cap N_2} \hat{v}_j + \sum_{k \in S \cap N_3} \hat{w}_k \geq V(S) \quad \text{for all } S \subseteq N.
\]

We have

\[
\sum_{i \in S \cap N_1} \hat{u}_i + \sum_{j \in S \cap N_2} \hat{v}_j + \sum_{k \in S \cap N_3} \hat{w}_k
= \sum_{i \in S \cap N_1} \left[ U_i(\hat{x}_i) + \hat{p}_i(\sum_{j \in N_2} \sum_{k \in N_3} \hat{x}_{ijk}) \right]
+ \sum_{j \in S \cap N_2} \left[ -\sum_{i \in N_1} \hat{p}_i(\sum_{k \in N_3} \hat{x}_{ijk}) + \sum_{i \in N_1} \hat{q}_{ij}(\sum_{k \in N_3} \hat{x}_{ijk}) \right]
+ \sum_{k \in S \cap N_3} \left[ U_k((\hat{x}_{ijk})_{i \in N_1, j \in N_2}) - \sum_{i \in N_1} \sum_{j \in N_2} \hat{q}_{ij}\hat{x}_{ijk} \right]
\geq \sum_{i \in S \cap N_1} \left[ U_i(\hat{z}_i) + \hat{p}_i(\sum_{j \in N_2} \sum_{k \in N_3} \hat{z}_{ijk}) \right]
+ \sum_{j \in S \cap N_2} \left[ -\sum_{i \in N_1} \hat{p}_i(\sum_{k \in N_3} \hat{z}_{ijk}) + \sum_{i \in N_1} \hat{q}_{ij}(\sum_{k \in N_3} \hat{z}_{ijk}) \right]
+ \sum_{k \in S \cap N_3} \left[ U_k((\hat{z}_{ijk})_{i \in N_1, j \in N_2}) - \sum_{i \in N_1} \sum_{j \in N_2} \hat{q}_{ij}\hat{z}_{ijk} \right]
\]

9
\[
\begin{align*}
&= \sum_{i \in S \cap N_1} \left[ U_i(z_i) + \hat{p}_i \left( \sum_{j \in S \cap N_2} \sum_{k \in S \cap N_3} z_{ijk} \right) \right] \\
&\quad + \sum_{j \in S \cap N_2} \left[ - \sum_{i \in S \cap N_1} \hat{p}_i \left( \sum_{k \in S \cap N_3} z_{ijk} \right) + \sum_{i \in S \cap N_1} \hat{q}_{ij} \left( \sum_{k \in S \cap N_3} z_{ijk} \right) \right] \\
&\quad + \sum_{k \in S \cap N_3} \left[ U_k((z_{ijk})_{i \in N_1, j \in N_2}) - \sum_{i \in S \cap N_1} \sum_{j \in S \cap N_2} \hat{q}_{ij} z_{ijk} \right] \\
&= \sum_{i \in S \cap N_1} [U_i(z_i)] + \sum_{k \in S \cap N_3} [U_k((z_{ijk})_{i \in N_1, j \in N_2})] \\
&= \sum_{i \in \rho_S^*} a_{\{i\}} + \sum_{\{i,j,k\} \in \rho_S^*} a_{\{i,j,k\}} = V(S).
\end{align*}
\]

Note that the first inequality holds since Step 1 and the definition of the competitive equilibrium and the second equality holds since the definition of \( z \).

**Step 3:** We have \( \sum_{i \in N_1} \hat{u}_i + \sum_{j \in N_2} \hat{v}_j + \sum_{k \in N_3} \hat{w}_k = V(N) \) since \((\hat{u}, \hat{v}, \hat{w})\) is the competitive outcome. This completes the proof of Lemma 1.

Next we will consider a linear programming. Let \( y = (y_T)_{T \in \pi} \in \mathbb{R}^{n+\alpha^*n_2n_3} \) be a vector of control variables. Let \( b = (b_i)_{i \in N} \in \mathbb{R}^n \) be a vector of control variables of a dual problem. We define a primal problem \((P)\) for a partitioning linear programming as the follows:

\[
(P) \quad \max_{(y_T)_{T \in \pi}} \sum_{T \in \pi} a_T y_T \\
\text{s.t.} \quad \sum_{T \in \pi, T \ni i} y_T = 1 \quad \text{for all } i \in N \\
\quad \quad \quad \quad y_T \geq 0 \quad \text{for all } T \in \pi.
\]

We define \((D)\) as a dual problem of \((P)\) as follows:

\[
(D) \quad \min_{(b_i)_{i \in N}} \sum_{i \in N} b_i \\
\text{s.t.} \quad \sum_{i \in T} b_i \geq a_T \quad \text{for all } T \in \pi.
\]

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We will consider the relationship among $V(\cdot)$, $(P)$ and $(D)$ stated above. Owing to the theorem in Quint (1991b), the following two statements hold.

(i) If $(P)$ solves integrally (i.e. with all 0’s and 1’s), the core of our game $(N, V)$ is nonempty and coincides with the set of optimal solutions to $(D)$.

(ii) If $(P)$ does not solve integrally, the core is empty.

If there exists at least one integral solution $y^*$ for $(P)$, let $\hat{y}$ be a vector in $\mathbb{Z}_{+}^{n_1+n_1n_2n_3}$ such that $\hat{y}_i \equiv y^*_{(i)}$ for all $i \in N_1$ and $\hat{y}_{ijk} \equiv y^*_{(i,j,k)}$ for all $\{i,j,k\} \in \pi$. Let $S_P \subseteq \mathbb{Z}_{+}^{n_1+n_1n_2n_3}$ be the set of all $\hat{y}$. Let $S_D$ be the set of solutions of $(D)$.

Let $C^*$ be given by the set of utility vectors $(\bar{u}, \bar{v}, \bar{w}) \in \mathbb{R}^n$ satisfying

(1): the $\pi$-partition efficiency conditions: for some $\hat{y} \in \{0, 1\}^{n_1+n_1n_2n_3}$,

- $\bar{u}_i + \bar{v}_j + \bar{w}_k = a_{(i,j,k)}$ if $\hat{y}_{ijk} = 1$;
- $\bar{u}_i = a_i$ if $\hat{y}_i = 1$;

(2): the stability conditions:

- $\bar{u}_i + \bar{v}_j + \bar{w}_k \geq a_{(i,j,k)}$ if $\{i,j,k\} \in \pi$;
- $\bar{u}_i \geq a_{(i)}$ if $i \in N_1$;
- $\bar{v}_j \geq a_{(j)} = 0$ if $j \in N_2$;
- $\bar{w}_k \geq a_{(k)} = 0$ if $k \in N_3$.

The following lemma shows that the core of $(N, V)$ coincides with $C^*$, hence, the $\pi$-partition efficiency conditions can be regarded as the efficiency conditions of the core.

**Lemma 2** The core of the game $(N, V)$ coincides with $C^*$.

**Proof.** Step 1: We will show that the core is a subset of $C^*$. Suppose that the core is nonempty. Since $S_P$ is nonempty, $S_D$ is also nonempty. There exists a $\hat{y}$ in $S_P$, which is derived from an integral solution $y^*$ for $(P)$. Moreover, $S_D$ coincides with the core of the game $(N, V)$. Choose any $(u^*, v^*, w^*) \in S_D$ and fix it. It suffices to prove that $(u^*, v^*, w^*)$ satisfies (i) $u'_i + v'_j + w'_k = a_{(i,j,k)}$ if $\hat{y}_{ijk} = 1$ and (ii) $u'_i = a_i$ if $\hat{y}_i = 1$. By the complementary slackness condition$^2$, $u'_i + v'_j + w'_k = a_{(i,j,k)}$ if $\hat{y}_{ijk} = 1$ and $u'_i = a_{(i)}$ if $\hat{y}_i = 1$, which completes the proof of Step 1.

$^2\hat{y}$ is a solution of $(P)$ and $b$ is a solution of $(D)$ if and only if $\sum_{T \in \pi} y_T (\sum_{t \in T} b_t - a_T) = 0$ (see Dantzig(1963) pp.135-136).
Lemma 3

The core of \((N, V)\) belongs to the set of competitive outcomes.

Proof. Suppose the core of \((N, V)\) is nonempty. Since this supposition, \(S_P\) is nonempty. Let \(\hat{y} \in \{0, 1\}^{n_1+n_1 n_2 n_3}\) be a vector in \(S_P\). Choose any \((\bar{u}, \bar{v}, \bar{w})\) in the core of \((N, V)\). We set a price list \((\hat{p}, \hat{q})\) satisfying \(\hat{p}_i = \bar{u}_i\) for all \(i \in N_1\) and \(\hat{q}_{ij} = \bar{u}_i + \bar{v}_j\) for all \(i \in N_1\) and all \(j \in N_2\).

Our target is to show that \((\hat{p}, \hat{q}, \hat{y})\) is a competitive equilibrium, which yields a competitive outcome \((\bar{u}, \bar{v}, \bar{w})\). Let \(\hat{y}_i = x^*_i\), \(\hat{y}_{ijk} = x^*_{ijk} = \hat{x}_{ijk}\) and \(\sum_{k \in N_3} \hat{y}_{ijk} = \hat{x}_{ijk}\).

Step 1: \(((x^*_i))_{i \in N_1}, (\hat{x}^*_{ijk}))_{i \in N_1, j \in N_2, k \in N_3} \in \{0, 1\}^{n_1+n_2 n_3}\) satisfies A1, \(((\hat{x}^*_{ijk}))_{i \in N_1, j \in N_2, k \in N_3} \in \{0, 1\}^{n_1 n_2 n_3}\) satisfies A2 and \(((\hat{x}^*_{ijk}))_{i \in N_1, j \in N_2, k \in N_3} \in \{0, 1\}^{n_2 n_3}\) satisfies A3. The proof of this statement will be omitted since it is a matter of calculation under \(\hat{y} \in S_P\).

Step 2: Let \(\hat{y} \in S_P\). Then

\[
\bar{v}_j = 0 \quad \text{if there exists } j \in N_2 \text{ such that } \hat{y}_{ij k} = 0 \text{ for all } (i', k') \in N_1 \times N_3; \\
\bar{w}_k = 0 \quad \text{if there exists } k \in N_3 \text{ such that } \hat{y}_{i j k} = 0 \text{ for all } (i', j') \in N_1 \times N_2.
\]

For notational convenience, let \(J \equiv \{j \in N_2 : \hat{y}_{ij k} = 0 \text{ for all } i \in N_1 \text{ and all } k \in N_3\}, \ K \equiv \{k \in N_3 : \hat{y}_{ij k} = 0 \text{ for all } i \in N_1 \text{ and all } j \in N_2\}\). Since \((\bar{u}, \bar{v}, \bar{w})\) in the core, \(\bar{v}_j \geq 0\) for all \(j \in J\) and \(\bar{w}_k \geq 0\) for all \(k \in K\).

By lemma 2, we have

\[
\sum_{j \in J} \bar{v}_j + \sum_{k \in K} \bar{w}_k = 0,
\]

which implies \(\bar{v}_j = 0\) for all \(j \in J\) and \(\bar{w}_k = 0\) for all \(k \in K\).
Step 3: We will show \((x_i^*,(\tilde{x}_{ij}^*)_{j \in N_2})\) is a maximal solution for all \(i \in N_1\), \((\tilde{x}_{ijk}^*)_{i \in N_1,k \in N_3}\) is a maximal solution for all \(j \in N_2\), and \((x_{ijk}^*)_{i \in N_1,j \in N_2}\) is a maximal solution for all \(k \in N_3\).

Substep 3-1: Consider a seller \(i \in N_1\). The seller \(i\) can gain \(a_{\{i\}}\) by consuming her own goods or gain \(\hat{p}_i\) by selling her own goods to a middleman. By A1, there are two cases.

Case 1: \(x_i^* = 1\) and \(\tilde{x}_{ij}^* = 0\) for all \(j \in N_2\).

In this case, \(a_{\{i\}} \geq \hat{p}_i\) must be satisfied. In fact, this holds since

\[
 a_{\{i\}} = \bar{u}_i \\
= \hat{p}_i.
\]

Note that the first equality holds since Lemma 2 and \(x_i^* = \hat{y}_i\).

Case 2: \(x_i^* = 0\) and there is \(j \in N_2\) such that \(\tilde{x}_{ij}^* = 1\) and \(\tilde{x}_{ij}^* = 0\) for all \(j' \in N_2 \setminus \{j\}\).

In this case, \(\hat{p}_i \geq a_{\{i\}}\) must be satisfied. In fact, this holds since

\[
 \hat{p}_i = \bar{u}_i \\
\geq a_{\{i\}}.
\]

Note that the inequality holds since Lemma 2.

Substep 3-2: Next, consider a middleman \(j \in N_2\). The middleman \(j\) can gain \(\hat{q}_{ij} - \hat{p}_i\) by buying one unit of goods from a seller \(i\) and selling it to a buyer, or gain 0 by doing nothing. By A2, there are two cases.

Case 1: \(\tilde{x}_{ijk}^* = 0\) for all \(i \in N_1\) and all \(k \in N_3\).

In this case, \(0 \geq \hat{q}_{ij} - \hat{p}_i\) must be satisfied. In fact, this holds since

\[
0 = \bar{v}_j \\
= (\bar{u}_i + \bar{v}_j) - \bar{u}_i \\
= \hat{q}_{ij} - \hat{p}_i.
\]

Note that the first equality holds since Step 2 and \(\tilde{x}_{ijk}^* = \hat{y}_{ijk}\) for all \(i \in N_1\) and \(k \in N_3\).
Case 2: There is \((i, k) \in N_1 \times N_3\) such that \(\hat{x}_{ijk}^* = 1\) and \(\hat{x}_{i'k'}^* = 0\) for all \((i', k') \in N_1 \times N_3\) such that \((i', k') \neq (i, k)\).

In this case, \(\hat{q}_{ij} - \hat{p}_i \geq \hat{q}_{i'j} - \hat{p}_{i'}\) for all \(i' \in N_1 \setminus \{i\}\) and \(\hat{q}_{ij} - \hat{p}_i \geq 0\) must be satisfied. In fact, these statements hold since

\[
\hat{q}_{ij} - \hat{p}_i = (\bar{u}_i + \bar{v}_j) - \bar{u}_i = \bar{v}_j = (\bar{u}_{i'} + \bar{v}_j) - \bar{u}_{i'} = \hat{q}_{i'j} - \hat{p}_{i'}
\]

and \(\bar{v}_j \geq 0\).

Substep 3.3: Finally, consider a buyer \(k \in N_3\). The buyer \(k\) can gain \(\hat{a}_{i,j;k} - \hat{q}_{ij}\) by buying one unit of goods of a seller \(i\) from a middleman \(j\), or gain 0 by consuming nothing. By A3, there are two cases.

Case 1: \(x_{ijk}^* = 0\) for all \(i \in N_1\) and all \(j \in N_2\).

In this case, \(0 \geq a_{\{i,j,k\}} - \hat{q}_{ij}\) must be satisfied. In fact, this holds since

\[
0 = \bar{w}_k \\
\geq a_{\{i,j,k\}} - (\bar{u}_i + \bar{v}_j) \\
= a_{\{i,j,k\}} - \hat{q}_{ij}.
\]

Note that the first equality holds since Step 2 and \(x_{ijk}^* = \hat{y}_{ijk}\) for all \(i \in N_1\) and \(j \in N_2\). Moreover, the inequality holds since Lemma 2.

Case 2: There is \((i, j) \in N_1 \times N_2\) such that \(x_{ijk}^* = 1\) and \(x_{i'j'k}^* = 0\) for all \((i', j') \in N_1 \times N_2\) such that \((i', j') \neq (i, j)\).

In this case, \(a_{\{i,j,k\}} - \hat{q}_{ij} \geq a_{\{i',j',k\}} - \hat{q}_{i'j'}\) for all \((i', j') \in N_1 \times N_2\) such that \((i', j') \neq (i, j)\) and \(a_{\{i,j,k\}} - \hat{q}_{ij} \geq 0\) must be satisfied. In fact, these statements hold since

\[
a_{\{i,j,k\}} - \hat{q}_{ij} = (\bar{u}_i + \bar{v}_j + \bar{w}_k) - (\bar{u}_i + \bar{v}_j) = \bar{w}_k \\
\geq a_{\{i',j',k\}} - (\bar{u}_{i'} + \bar{v}_{j'}) \\
= a_{\{i',j',k\}} - \hat{q}_{i'j'}
\]

and \(\bar{w}_k \geq 0\). Note that the first equality holds since Lemma 2 and \(x_{ijk}^* = \hat{y}_{ijk}\). Moreover, the inequality holds since Lemma 2.
Step 4: \((\hat{\rho}, \hat{q}, \hat{y})\) is a competitive equilibrium. It is clear by Step 1 and Step 3. Therefore a competitive outcome at \((\hat{\rho}, \hat{q}, \hat{y})\) is \((\hat{u}, \hat{v}, \hat{w})\).

Thus, we can show the first main result in the present study as follows:

**Theorem 1** The set of competitive outcomes of the simple three-sided assignment market \((N_1, N_2, N_3)\) coincides with the core of the game \((N, V)\).

**Proof.** Immediate from Lemma 1 and Lemma 3.

The following example shows that the competitive outcome of each middleman is not necessarily zero.

**Example 1** Let \(N_1 = \{i_1, i_2\}, N_2 = \{j_1, j_2\}\) and \(N_3 = \{k_1, k_2\}\). Let the utility functions of all agents be given by the followings:

\[
\begin{align*}
(i) & \quad U_i(x_i) = 0 \quad \text{for all } i \in N_1; \\
(ii) & \quad U_k(e^{ij}) = 4 \quad \text{if } (i, j, k) = (i_1, j_1, k_1); \\
(iii) & \quad U_k(e^{ij}) = 8 \quad \text{if } (i, j, k) = (i_2, j_2, k_2); \\
(iv) & \quad U_k(e^{ij}) = 1 \quad \text{otherwise}.
\end{align*}
\]

There exists only one integral solution for \((P)\), then there exists \(\hat{y} \in S_P \subseteq \mathbb{Z}^{10}_+\) such that \(\hat{y}_{i_1j_1k_1} = \hat{y}_{i_2j_2k_2} = 1\) and all other components of \(\hat{y}\) are 0. Therefore

\[
\begin{align*}
(i) & \quad V(N) = 12 \\
(ii) & \quad V(S) = 4 \quad \text{if } S \supseteq \{i_1, j_1, k_1\} \text{ and } S \neq N; \\
(iii) & \quad V(S) = 8 \quad \text{if } S \supseteq \{i_2, j_2, k_2\} \text{ and } S \neq N; \\
(iv) & \quad V(S) = 0 \quad \text{if there exists } h \in \{1, 2, 3\} \text{ such that } S \cap N_h = \emptyset; \\
(v) & \quad V(S) = 1 \quad \text{otherwise}.
\end{align*}
\]

The core of \((N, V)\) is the set of \((\hat{u}, \hat{v}, \hat{w}) = ((\hat{u}_i)_{i \in N_1}, (\hat{v}_j)_{j \in N_2}, (\hat{w}_k)_{k \in N_3})\) such that \((i)\) \((\hat{\bar{u}}, \hat{\bar{v}}, \hat{\bar{w}}) \geq 0\), \((ii)\) \(\hat{u}_i + \hat{v}_j + \hat{w}_k \geq 1\) for \((i, j, k) \in N_1 \times N_2 \times N_3\), \((iii)\) \(\hat{u}_{i_1} + \hat{v}_{j_1} + \hat{w}_{k_1} = 4\), and \((iv)\) \(\hat{u}_{i_2} + \hat{v}_{j_2} + \hat{w}_{k_2} = 8\).

Let \((\hat{u}, \hat{v}, \hat{w})\) be an imputation of the core of \((N, V)\) such that \(\hat{u}_{i_1} = \hat{u}_{i_2} = 1, \hat{v}_{j_1} = 1, \hat{v}_{j_2} = 2, \hat{w}_{k_1} = 2\) and \(\hat{w}_{k_2} = 5\). By Theorem 1, \((\hat{u}, \hat{v}, \hat{w})\) is a competitive outcome of the market \((N_1, N_2, N_3)\) under the competitive equilibrium price \((\hat{\rho}, \hat{q})\) such that \((i)\) \(\hat{\rho}_i = 1\) for each \(i \in N_1\), \((ii)\) \(\hat{q}_{ij_1} = 2\) for each \(i \in N_1\), \((iii)\) \(\hat{q}_{ij_2} = 3\) for each \(i \in N_1\).

On the other hand, the following example shows that a competitive equilibrium does not necessarily exist.
Example 2 Let $N_1 = \{i_1, i_2\}$, $N_2 = \{j_1, j_2\}$ and $N_3 = \{k_1, k_2\}$. Let the utility functions be given by

(i) $U_{i_1}(x_{i_1}) = U_{i_2}(x_{i_2}) = 0$;
(ii) $U_{k_1}(e^{ij_2}) = U_{k_2}(e^{ij_2}) = U_{k_3}(e^{ij_2}) = 1$;
(iii) $U_{k_1}(e^{ij_1}) = U_{k_1}(e^{ij_1}) = U_{k_2}(e^{ij_1}) = 0$.

Since the core of $(N, V)$ is empty by Quint (1991a), no competitive equilibrium exists by Theorem 1.

4 Extension to a Multiple Trading Case

We will extend the simple three-sided assignment market to a market in which each middleman trades more than one unit of goods. In the extension of assignment market model, keep assumptions A1, A3. However relax A2 as follows:

$A2'$: for all $j \in N_2$, $X'_j \equiv \{(x_{ijk})_{i \in N_1, k \in N_3} \in \mathbb{Z}^{n_1n_3}_+ : \sum_{i \in N_1} \sum_{k \in N_3} x_{ijk} \leq s_j\}$.

In $A2'$, $s_j \in \mathbb{N}$ ($s_j \geq 1$) is the number of units of goods which middleman $j$ can trade. In this section, we call $(N_1, N_2, N_3)$ under $A1$, $A2'$ and $A3$ the three-sided assignment market.

Let us introduce an agent normal form of the three-sided assignment market. Let $M_j$ be the set of artificial middlemen of type $j \in N_2$ satisfying $|M_j| = s_j$. This means that each $m \in M_j$ trades at most one unit of goods, thus each of them satisfies A2. Let the set of all artificial middlemen be given by $\bar{N}_2 \equiv \{m \in M \text{ for each } j \in N_2\}$, where $|\bar{N}_2| = \bar{n}_2$. We can regard the market $(N_1, \bar{N}_2, N_3)$ under $A1$, $A2$ and $A3$ as an agent normal form of the three-sided assignment market, namely an agent normal form market$^3$. Let $\bar{N} \equiv N_1 \cup \bar{N}_2 \cup N_3$, where $|\bar{N}| = \bar{n}$.

We define utility functions on consumption in the agent normal form market $(N_1, \bar{N}_2, N_3)$ as follows: $\bar{U}_{i_1} : \mathbb{Z}_+ \rightarrow \mathbb{R}$ for all $i \in N_1$ and $\bar{U}_{k_1} : \mathbb{Z}^{n_1n_2}_+ \rightarrow \mathbb{R}$ for all $k \in N_3$. We put natural assumptions as (B1) $\bar{U}_{i_1}(\cdot) = U_{i_1}(\cdot)$ for each $i \in N_1$ and (B2) for each $(i, j, k) \in N_1 \times N_2 \times N_3$, $\bar{U}_{k}(e^{im}) = U_{k}(e^{ij})$ for $m = 0$.

$^3$The basic idea underlying our agent normal form is an analogy with the agent normal form of Selten (1975).
all $m \in M_j$, where $e^{im}$ is the $n_1n_2$-dimensional vector such that $e^{im}_{im'} = 1$ and $e^{im}_{im'} = 0$ for all $(i', m') \neq (i, m)$. The assumption B2 means that each unit of goods traded by artificial middlemen of the same type $j$ yields the same utility to buyer $k$.

We define the vector $\tilde{a} \in \mathbb{R}^{n_1n_2n_3}$ satisfying (1) $\tilde{a}_{ij} = \tilde{U}_i(\omega_i)$ for all $i \in N_1$, (2) $\tilde{a}_{im} = \tilde{a}_{ik} = 0$ for all $m \in \tilde{N}_2$ and all $k \in N_3$, and (3) $\tilde{a}_{ijm} = \tilde{U}_k(\omega^m)$ for all $i \in N_1$, $m \in \tilde{N}_2$ and all $k \in N_3$. Since B2, we have

\[
\tilde{a}_{i,m,k} = \tilde{a}_{i,m',k} = 0 \quad \text{whenever } m, m' \in M_j.
\]

Let $\tilde{\pi} = \{(i) | i \in N_1 \} \cup \{ (i, j, k) | i \in N_1, j \in \tilde{N}_2, k \in N_3 \}$. Given $\tilde{\pi}$, $\tilde{p}_S$ is a $\tilde{\pi}$-partition of $S$ and $\tilde{P}_S$ is the class of all $\tilde{\pi}$-partition of $S$. We can define the assignment game of the agent normal form market $(N_1, \tilde{N}_2, N_3)$ as $(\tilde{N}, \tilde{V})$, where

\[
\tilde{V}(S) \equiv \max_{\tilde{p}_S \in \tilde{P}_S} \sum_{T \in \tilde{P}_S} \tilde{a}_T \quad \text{for nonempty } S \subseteq \tilde{N} \quad \text{with } \tilde{V}(\emptyset) = 0.
\]

Next we will consider the partitioning program derived from the game $(\tilde{N}, \tilde{V})$. Let $\tilde{S}_p$ be the set in the same manner as $S_P$. The core of the game $(\tilde{N}, \tilde{V})$ coincides with the set defined in the same manner as Lemma 2.

Owing to B2, we get the following lemma.

**Lemma 4** Let $(\tilde{u}, \tilde{v}, \tilde{w})$ be an imputation in the core of $(\tilde{N}, \tilde{V})$. Fix any $j \in N_2$. If $m, m' \in M_j$, then $\tilde{v}_m = \tilde{v}_{m'}$.

**Proof.** See Appendix. ■

In the agent normal form market $(N_1, \tilde{N}_2, N_3)$, we will introduce the set of price vectors $(\tilde{p}, \tilde{q}) \in \mathbb{R}^{n_1n_2}$ given by

\[
\Omega \equiv \left\{ (\tilde{p}, \tilde{q}) \in \mathbb{R}^{n_1n_2} : \begin{array}{c}
\text{for all } i \in N_1, j \in N_2 \\
\tilde{q}_{im} = \tilde{q}_{im'} \text{ if } m, m' \in M_j
\end{array} \right\}.
\]

This means that each artificial middleman of type $j \in M_2$ is subject to the same prices. Also, in the three-sided market $(N_1, N_2, N_3)$, we define the price vector $(p, q) \in \mathbb{R}^{n_1n_2}$ satisfying $p_i = \tilde{p}_i$ for all $i \in N_1$, and $q_{ij} = \tilde{q}_{im}$ for all $i \in N_1$, all $m \in M_j$, and all $j \in N_2$.

To compare the two markets stated above, for each $j \in N_2$ we define

\[
\tilde{x}_{ijk} = \sum_{m \in M_j} \tilde{x}_{imk}, \quad x_{ijk} = \sum_{m \in M_j} x_{imk} \quad \text{and} \quad \tilde{x}_{ij} = \sum_{m \in M_j} \tilde{x}_{im}.
\]

We can obtain the following lemma.
Lemma 5 Let \( \hat{x}_{ijk} = \sum_{m \in M_j} \hat{x}_{imk} \), \( p_i = \tilde{p}_i \) and \( q_{ij} = \tilde{q}_{im} \) for all \( i \in N_1 \), all \( k \in N_3 \), all \( m \in M_j \), and all \( j \in N_2 \). There exists a competitive equilibrium \((\tilde{p}, \tilde{q}, ((\hat{x}_{i})_{i \in N_1}, (\hat{x}_{imk})_{i \in N_1, m \in N_2, k \in N_3}))\) of the agent normal form market \((N_1, \tilde{N}_2, N_3)\) if and only if there exists a competitive equilibrium \((p, q, ((\hat{x}_{i})_{i \in N_1}, (\hat{x}_{ijk})_{i \in N_1, j \in N_2, k \in N_3}))\) of the three-sided assignment market \((N_1, N_2, N_3)\).

Proof. See Appendix. ■

Moreover, we get the following corollary by the proof of Lemma 5.

Corollary 1 Let \( E^* \subseteq \mathbb{R}^n \) be the set of competitive outcomes of the three-sided market \((N_1, N_2, N_3)\) and let \( E^{**} \) be the set of vectors in \( \mathbb{R}^n \) given by competitive outcomes of the agent normal form market \((N_1, \tilde{N}_2, N_3)\) where competitive equilibrium prices are in \( \Omega \). Then \( E^* = E^{**} \).

Note that for each vector in \( E^{**} \) an outcome of a middleman \( j \) is given by the sum of outcomes of artificial middlemen of type \( j \). Lemma 5 implies that a competitive outcome of the three-sided assignment market \((N_1, N_2, N_3)\) is given by a competitive outcome derived from the competitive equilibrium of the agent normal form market \((N_1, \tilde{N}_2, N_3)\), and conversely.

The following theorem is the second result in the present study as follows:

Theorem 2 The set of competitive outcomes of the three-sided assignment market \((N_1, N_2, N_3)\) coincides with the set of imputations of the three-sided assignment market given by the core of \((\tilde{N}, \tilde{V})\).

Proof. The proof will be completed by the following two steps.

Step 1: By Lemma 1, Lemma 3 and Lemma 4, the core of \((\tilde{N}, \tilde{V})\) coincides with the set of competitive outcomes of the agent normal form market \((N_1, \tilde{N}_2, N_3)\) supported by a price vector in \( \Omega \).

Step 2: By Corollary 1, the set of imputations of the three-sided market \((N_1, N_2, N_3)\) given by competitive outcomes of the agent normal form market \((N_1, \tilde{N}_2, N_3)\) supported by a price vector in \( \Omega \) coincides with the set of competitive outcomes of the three-sided market. ■

By Quint (1991b) and Theorem 2, we have immediately the following theorem.
Theorem 3 There exists a competitive equilibrium of the three-sided assignment market \((N_1, N_2, N_3)\) if and only if the partitioning linear program derived from \((\tilde{N}, \tilde{V})\) solves integrally.

Proof. The proof will be completed by considering the distinct two cases as follows:

Case 1: The partitioning linear program derived from \((\tilde{N}, \tilde{V})\) solves integrally.

By Quint (1991b), the core of the game \((\tilde{N}, \tilde{V})\) is nonempty. By Lemma 3, there exists a competitive equilibrium \((\tilde{p}, \tilde{q}, ((\tilde{x}_{imk})_{i \in N_1, m \in \tilde{N}_2, k \in N_3})\) of the agent normal form market \((N_1, \tilde{N}_2, N_3)\). By Lemma 4 and the proof of Lemma 3, \((\tilde{p}, \tilde{q}) \in \Omega\). By Lemma 5, there exists a competitive equilibrium of the three-sided assignment market \((N_1, N_2, N_3)\).

Case 2: The partitioning linear program derived from \((\tilde{N}, \tilde{V})\) does not solve integrally.

By Quint (1991b), the core of the game \((\tilde{N}, \tilde{V})\) is empty. By Theorem 1, the set of competitive outcomes of the agent normal form market \((N_1, \tilde{N}_2, N_3)\) is empty. This implies that there exists no competitive equilibrium of \((N_1, \tilde{N}_2, N_3)\). Then, by Lemma 5, there exists no competitive equilibrium of the three-sided assignment market \((N_1, N_2, N_3)\), which completes the proof of Theorem 3. ■

5 Concluding Remarks

We demonstrated that in assignment markets with middlemen there may not exist a competitive equilibrium. Furthermore, we presented that the payoff of each middleman may be positive if the competitive equilibrium exists. These findings may lead to interesting remarks as the followings.

Initially, although in the two-sided assignment market there always exists a competitive equilibrium, the three-sided assignment market does not necessarily have this property. Our study gives an equivalent condition for the existence of a competitive equilibrium of the three-sided assignment market. Since this condition is based on the partitioning linear program, the existence problem of a competitive equilibrium of the three-sided assignment market may have much to do with computation theory or discrete convex analysis, e.g., Murota (2003).
Next, we may connect the two-sided assignment market with the three-sided assignment market. Recall that, in the assignment market with a sole middleman, the payoff of the middleman is positive if the competitive equilibrium exists. Then an increase of the number of middlemen who are the same type as the initial middleman can yield the assignment market with middlemen whose payoffs are zero. This implies that increasing the number of homogeneous middlemen may cause a change from the three-sided assignment market to the two-sided assignment market. Therefore, the relationship between the two-sided assignment market and the three-sided assignment market may deserve investigation, which we leave to the future research.

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Appendix

Proof of Lemma 4. We will complete the proof by showing some claims in two distinct cases.

Case 1: There exists \( \hat{y} \in \tilde{S}_P \) such that for each \( m \in M_j \), \( \hat{y}_{imk} = 1 \) for some \( (i, k) \in N_1 \times N_3 \).

Claim 1 Let \((\bar{u}, \bar{v}, \bar{w})\) be an imputation in the core of \((\tilde{N}, \tilde{V})\). Suppose that there exists \( \hat{y} \in \tilde{S}_P \) such that for each \( m \in M_j \), \( \hat{y}_{imk} = 1 \) for some \( (i, k) \in N_1 \times N_3 \). If there exists \( m' \in M_j \), then \( \bar{v}_m \leq \bar{v}_{m'} \).

Proof of Claim 1. If there exists \( \hat{y} \in \tilde{S}_P \) such that \( \hat{y}_{imk} = 1 \), then \( \bar{u}_i + \bar{v}_m + \bar{w}_k = \tilde{a}_{\{i, m, k\}} \) by Lemma 2. By the definition of the core, \( \bar{u}_i + \bar{v}_{m'} + \bar{w}_k \geq \tilde{a}_{\{i, m', k\}} \), thus \( \bar{u}_i + \bar{v}_{m'} + \bar{w}_k \geq \tilde{a}_{\{i, m', k\}} = \tilde{a}_{\{i, m, k\}} = \bar{u}_i + \bar{v}_m + \bar{w}_k \), which implies \( \bar{v}_{m'} \geq \bar{v}_m \).
Claim 2. Let \((\bar{u}, \bar{v}, \bar{w})\) be an imputation in the core of \((\bar{N}, \bar{V})\). Suppose that there exists \(\hat{y} \in \hat{I}_P\) such that for each \(m \in M_j\), \(\hat{y}_{imk} = 1\) for some \((i, k) \in N_1 \times N_3\), and there exists \(\hat{y}' \in \hat{I}_P\) such that for each \(m' \in M_j\), \(\hat{y}'_{im'k'} = 1\) for some \((i', k') \in N_1 \times N_3\). Then \(\bar{v}_m = \bar{v}_{m'}\).

**Proof of Claim 2.** Immediate from Claim 1.

Case 2: For all \(\hat{y} \in \hat{I}_P\), there exists \(m \in M_j\) such that \(\hat{y}_{imk} = 0\) for all \((i, k) \in N_1 \times N_3\).

Claim 3. If there exists \(\hat{y} \in \hat{I}_P\) such that for some \(m' \in M_j\), \(\hat{y}_{im'k} = 0\) for all \((i, k) \in N_1 \times N_3\), then \(\bar{v}_{m'} = 0\).

**Proof of Claim 3.** The proof is the same as that of Step 2 of Lemma 3. Therefore we will omit the proof of Claim 3.

Claim 4. Suppose that there exists \(\hat{y} \in \hat{I}_P\) such that \(\hat{y}_{imk} = 1\) for some \((i, m, k) \in N_1 \times N_2 \times N_3\), and there exists \(m' \in M_j\) such that \(\hat{y}_{im'k} = 0\) for all \((i, k) \in N_1 \times N_3\). Then there exists \(\hat{y}' \in \hat{I}_P\) such that \(\hat{y}'_{im'k'} = 1\) for some \((i', k') \in N_1 \times N_3\).

**Proof of Claim 4.** We define \(\hat{y}'\) as follows: \(\hat{y}'_i = \hat{y}_i\) for all \(i \in N_1\), \(\hat{y}'_{im'k} = \hat{y}_{imk}\) and \(\hat{y}'_{im'k} = \hat{y}_{imk}\) for all \((i, m, k) \in N_1 \times N_2 \times N_3\) such that \((i, m, k) \neq (i, m, k)\) and \((i, m, k) \neq (i, m', k)\). Let \(y^* = (y^*_T)_{T \in \bar{T}}\) be a vector in \(\mathbb{Z}^n_+\) such that \(y^*_i = y'_i\) for all \(i \in N_1\), \(y^*_m = 1\) for all \(m \in \bar{N}_2\), \(y^*_k = 1\) for all \(k \in N_3\), and \(y^*_m = \hat{y}_{imk}\) for all \((i, m, k) \in \bar{P}\). Then, since B2, the value of \((P)\) at the integral solution \(y^*\) is equal to \(V(\bar{N})\). Therefore \(\hat{y}' \in \hat{I}_P\).

By Claim 2, Claim 3 and Claim 4, \(\bar{v}_m = 0\) for all \(m \in M_j\). Therefore Lemma 4 holds in Case 2, which completes the proof of Lemma 4.

**Proof of Lemma 5.** Before proving the lemma, it is useful to introduce the distinct two mappings. By these mappings, we can link the constraints of goods in the market \((N_1, N_2, N_3)\) and those in the market \((N_1, \bar{N}_2, N_3)\).
Let $M_j$ be given by $M_j = \{m_1(j), m_2(j), \ldots, m_{s_j}(j)\}$, where $|M_j| = s_j$. Let the mapping $f_j : [0, s_j] \cap \mathbb{Z}_+ \to \{0, 1\}^{s_j}$ be $f_j(\alpha) = (\beta_m)_{m \in M_j}$, where

$$
\beta_m = \begin{cases} 
1 & \text{if } m = m_1(j), m_2(j), \ldots, m_{s_j}(j) \\
0 & \text{otherwise}.
\end{cases}
$$

Thus we define $(x_{imk})_{m \in M_j} = f_j(x_{ijk})$, $(\bar{x}_{imk})_{m \in M_j} = f_j(\bar{x}_{ij})$, and $(\bar{x}_{im})_{m \in M_j} = f_j(\bar{x}_{ij})$. From the definition of $f_j$, $x_{ijk} = \sum_{m \in M_j} x_{imk}$, $\bar{x}_{ijk} = \sum_{m \in M_j} x_{imk}$ and $\bar{x}_{ij} = \sum_{m \in M_j} \bar{x}_{im}$.

Next, define the mapping $g_j : \{0, 1\}^{s_j} \to [0, s_j] \cap \mathbb{Z}_+$ be $g_j((\gamma_m)_{m \in M_j}) = \delta$, where

$$
\delta = \sum_{m \in M_j} \gamma_m.
$$

Thus we define $x_{ijk} = g_j((x_{imk})_{m \in M_j})$, $\bar{x}_{ijk} = g_j((\bar{x}_{imk})_{m \in M_j})$ and $\bar{x}_{ij} = g_j((\bar{x}_{im})_{m \in M_j})$.

The proof consists of five steps. Step 4 and 5 are main parts of the proof and the other steps are the preliminaries to the last two steps.

**Step 1:** We will show that under $\bar{x}_{ijk} = \sum_{m \in M_j} \bar{x}_{imk}$, $x_{ijk} = \sum_{m \in M_j} x_{imk}$ and $\bar{x}_{ij} = \sum_{m \in M_j} \bar{x}_{im}$, the following statements hold. The proof of Step 1 will be omitted since it is clear by $f_j$ and $g_j$.

(i) For any fixed $i \in \mathbb{N}_1$, there exists $(x_i, (\bar{x}_{ij})_{j \in \mathbb{N}_2}) \in \mathbb{Z}_+^{1+n_2}$ which satisfies $x_i + \sum_{j \in \mathbb{N}_2} \bar{x}_{ij} = 1$ if and only if there exists $(x_i, (\bar{x}_{im})_{m \in \mathbb{N}_2}) \in \mathbb{Z}_+^{1+n_2}$ which satisfies $x_i + \sum_{m \in \mathbb{N}_2} \bar{x}_{im} = 1$.

(ii) For any fixed $j \in \mathbb{N}_2$, there exists $(\bar{x}_{ijk})_{i \in \mathbb{N}_1, k \in \mathbb{N}_3} \in \mathbb{Z}_+^{n_1n_3}$ which satisfies $\sum_{i \in \mathbb{N}_1} \sum_{k \in \mathbb{N}_3} \bar{x}_{ijk} = s_j$ if and only if there exists $(\bar{x}_{imk})_{i \in \mathbb{N}_1, m \in \mathbb{N}_j, k \in \mathbb{N}_3} \in \mathbb{Z}_+^{n_1n_3}$ which satisfies $\sum_{i \in \mathbb{N}_1} \sum_{k \in \mathbb{N}_3} \bar{x}_{imk} = 1$ for all $m \in \mathbb{N}_j$.

(iii) For any fixed $k \in \mathbb{N}_3$, there exists $(\bar{x}_{ijk})_{i \in \mathbb{N}_1, j \in \mathbb{N}_2} \in \mathbb{Z}_+^{n_1n_2}$ which satisfies $\sum_{i \in \mathbb{N}_1} \sum_{j \in \mathbb{N}_2} \bar{x}_{ijk} = s_j$ if and only if there exists $(\bar{x}_{imk})_{i \in \mathbb{N}_1, m \in \mathbb{N}_2} \in \mathbb{Z}_+^{n_1n_2}$ which satisfies $\sum_{i \in \mathbb{N}_1} \sum_{m \in \mathbb{N}_2} \bar{x}_{imk} = 1$.

**Step 2:** Assume that $\bar{x}_{ijk} = \sum_{m \in M_j} \bar{x}_{imk}$, $x_{ijk} = \sum_{m \in M_j} x_{imk}$ and $\bar{x}_{ij} = \sum_{m \in M_j} \bar{x}_{im}$. Then we will show that the objective function which each agent in the three-sided market maximizes can be expressed by that each agent in the agent normal form market maximizes, and conversely.
First, for all $i \in N_1$,
\[
U_i(x_i) + p_i\left(\sum_{j \in N_2} \tilde{x}_{ij}\right) = U_i(x_i) + p_i\left(\sum_{j \in N_2} \sum_{m \in M_j} \tilde{x}_{imk}\right)
= U_i(x_i) + \tilde{p}_i\left(\sum_{m \in N_2} \tilde{x}_{imk}\right).
\]

Secondly, for all $j \in N_2$,
\[
\begin{align*}
- \sum_{i \in N_1} p_i\left(\sum_{k \in N_3} \tilde{x}_{ijk}\right) + \sum_{i \in N_1} q_{ij}\left(\sum_{k \in N_3} \tilde{x}_{ijk}\right) \\
&= \sum_{i \in N_1} p_i\left(\sum_{k \in N_3} \sum_{m \in M_j} \tilde{x}_{imk}\right) + \sum_{i \in N_1} q_{ij}\left(\sum_{k \in N_3} \sum_{m \in M_j} \tilde{x}_{imk}\right) \\
&= \sum_{m \in M_j} \left[-\sum_{i \in N_1} \hat{p}_i\left(\sum_{k \in N_3} \tilde{x}_{imk}\right) + \sum_{i \in N_1} \hat{q}_{im}\left(\sum_{k \in N_3} \tilde{x}_{imk}\right)\right]
\end{align*}
\]

Thus objective function of middleman $m \in \tilde{N}_2$ is
\[
- \sum_{i \in N_1} \hat{p}_i\left(\sum_{k \in N_3} \tilde{x}_{imk}\right) + \sum_{i \in N_1} \hat{q}_{im}\left(\sum_{k \in N_3} \tilde{x}_{imk}\right).
\]

Finally, for all $k \in N_3$,
\[
\begin{align*}
U_k\left(\left(\forall i \in N_1, j \in N_2\right) x_{ijk}\right) - \sum_{i \in N_1} \sum_{j \in N_2} q_{ij}x_{ijk} \\
&= \sum_{i \in N_1, j \in N_2} a_{\{i,j,k\}} x_{ijk} - \sum_{i \in N_1} \sum_{j \in N_2} q_{ij}x_{ijk} \\
&= \sum_{i \in N_1, j \in N_2} a_{\{i,j,k\}}\left(\sum_{m \in M_j} x_{imk}\right) - \sum_{i \in N_1} \sum_{j \in N_2} q_{ij}\left(\sum_{m \in M_j} x_{imk}\right) \\
&= \sum_{i \in N_1, m \in \tilde{N}_2} \hat{a}_{\{i,m,k\}} x_{imk} - \sum_{i \in N_1} \sum_{m \in \tilde{N}_2} \hat{q}_{im} x_{imk} \\
&= \hat{U}_k\left(\left(\forall i \in N_1, m \in \tilde{N}_2\right) x_{imk}\right) - \sum_{i \in N_1} \sum_{m \in \tilde{N}_2} \hat{q}_{im} x_{imk},
\end{align*}
\]

which completes the proof of Step 2.
**Step 3:** We will show that under $\tilde{x}_{ijk}^* = \sum_{m \in M_j} \tilde{x}_{imk}^*$ the following holds for each $j \in N_2$.

$$\max_{(\tilde{x}_{ijk}) \in \tilde{X}_j} \left[ -\sum_{i \in N_1} \sum_{k \in N_3} p_i(\sum_{i \in N_1} \tilde{x}_{ijk}) + \sum_{i \in N_1} q_{ij}(\sum_{k \in N_3} \tilde{x}_{ijk}) \right]$$

$$= -\sum_{i \in N_1} \sum_{k \in N_3} \tilde{p}_i(\sum_{i \in N_1} \tilde{x}_{imk}) + \sum_{i \in N_1} \tilde{q}_{im}(\sum_{k \in N_3} \tilde{x}_{imk}).$$

$\iff$ For all $m \in M_j$,

$$\max_{(\tilde{x}_{imk}) \in \tilde{X}'_m} \left[ -\sum_{i \in N_1} \sum_{k \in N_3} \tilde{p}_i(\sum_{i \in N_1} \tilde{x}_{imk}) + \sum_{i \in N_1} \tilde{q}_{im}(\sum_{k \in N_3} \tilde{x}_{imk}) \right]$$

$$= -\sum_{i \in N_1} \sum_{k \in N_3} \tilde{p}_i(\sum_{i \in N_1} \tilde{x}_{imk}) + \sum_{i \in N_1} \tilde{q}_{im}(\sum_{k \in N_3} \tilde{x}_{imk}).$$

It is clear that “if part” of Step 3 holds since Step 1 and Step 2. We will show “only if part” of Step 3. We consider the following two cases.

**Case 1:** $0 \leq \sum_{i \in N_1} \sum_{k \in N_3} \tilde{x}_{ijk}^* < s_j$.

Suppose that there exists $i' \in N_1$ such that $q_{ij} - p_i' > 0$. We can define $(\tilde{x}_{ijk}')_{i \in N_1, k \in N_3}$ which satisfies that $\tilde{x}_{i'jk'}^* = \tilde{x}_{ij'}^* + 1$ for a buyer $k' \in N_3$ and $\tilde{x}_{ijk}' = \tilde{x}_{ijk}^*$ for all $(i, k) \in N_1 \times N_3$ such that $(i, k) \neq (i', k')$. Since $\sum_{i \in N_1} \sum_{k \in N_3} \tilde{x}_{ijk}^* < s_j$, $(\tilde{x}_{ijk}')_{i \in N_1, k \in N_3}$ satisfies $A2'$. Moreover,

$$\max_{(\tilde{x}_{ijk}) \in \tilde{X}_j} \left[ -\sum_{i \in N_1} \sum_{k \in N_3} p_i(\sum_{i \in N_1} \tilde{x}_{ijk}) + \sum_{i \in N_1} q_{ij}(\sum_{k \in N_3} \tilde{x}_{ijk}) \right]$$

$$= -\sum_{i \in N_1} \sum_{k \in N_3} p_i(\sum_{i \in N_1} \tilde{x}_{ijk}) + \sum_{i \in N_1} q_{ij}(\sum_{k \in N_3} \tilde{x}_{ijk})$$

$$< -\sum_{i \in N_1} \sum_{k \in N_3} p_i(\sum_{i \in N_1} \tilde{x}_{ijk}) + \sum_{i \in N_1} q_{ij}(\sum_{k \in N_3} \tilde{x}_{ijk}) + q_{ij} - p_i'$$

$$= -\sum_{i \in N_1} \sum_{k \in N_3} p_i(\sum_{i \in N_1} \tilde{x}_{ijk}') + \sum_{i \in N_1} q_{ij}(\sum_{k \in N_3} \tilde{x}_{ijk}'),$$

which is a contradiction. Thus $\tilde{q}_{im} - \tilde{p}_i = q_{ij} - p_i \leq 0$ for all $i \in N_1$ and for all $m \in M_j$. If $q_{ij} - p_i < 0$, then $\tilde{x}_{ijk}^* = 0$, thus $\tilde{x}_{imk}^* = 0$ for all $m \in M_j$. 

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Therefore
\[
\max_{(\bar{x}_{imk})_{i \in N_1, k \in N_3} \in X_m} \left[ -\sum_{i \in N_1} \bar{p}_i \left( \sum_{k \in N_3} \bar{x}_{imk} \right) + \sum_{i \in N_1} \bar{q}_{im} \sum_{k \in N_3} \bar{x}_{imk} \right]
\]
\[
= 0
\]
\[
= -\sum_{i \in N_1} \bar{p}_i \left( \sum_{k \in N_3} \bar{x}_{imk}^* \right) + \sum_{i \in N_1} \bar{q}_{im} \left( \sum_{k \in N_3} \bar{x}_{imk}^* \right).
\]

**Case 2:** \( \sum_{i \in N_1} \sum_{k \in N_3} \bar{x}_{imk}^* = s_j. \)

It is clear that \( q_{ij} - p_i \geq 0 \) for some \((i, j) \in N_1 \times N_2. \) Let \( i^* \) be one seller \( i \in N_1 \) which attains the maximal value of \( \bar{q}_{im} - \bar{p}_i. \) Suppose that there exist \( i' \in N_1 \) such that \( \sum_{k \in N_3} \bar{x}_{imk}^* = 1 \) and \( \bar{q}_{im} - \bar{p}_i < \max_i [\bar{q}_{im} - \bar{p}_i]. \)

Then there exists \( k' \in N_3 \) such that \( \bar{x}_{imk'}^* = 1. \) We define \( (\bar{x}_{imk})_{i \in N_1, k \in N_3} \) as follows:
\( \bar{x}_{imk'}^* = \bar{x}_{imk'}^* - 1, \bar{x}_{ik'mk'}^* = \bar{x}_{ik'mk'}^* + 1 \) and \( \bar{x}_{imk} = \bar{x}_{imk}^* \) for all \((i, k) \in N_1 \times N_3 \) such that \((i, k) \neq (i', k'), (i^*, k'). \) Since \( \sum_{i \in N_1} \sum_{k \in N_3} \bar{x}_{imk}^* = \sum_{i \in N_1} \sum_{k \in N_3} \bar{x}_{imk} \) satisfies A2. Moreover,
\[
- \sum_{i \in N_1} \bar{p}_i \left( \sum_{k \in N_3} \bar{x}_{imk}^* \right) + \sum_{i \in N_1} \bar{q}_{im} \left( \sum_{k \in N_3} \bar{x}_{imk}^* \right)
\]
\[
= -\bar{p}_{i'} + \bar{q}_{im}
\]
\[
< -\bar{p}_{i^*} + \bar{q}_{im}
\]
\[
= -\sum_{i \in N_1} \bar{p}_i \left( \sum_{k \in N_3} \bar{x}_{imk}^* \right) + \sum_{i \in N_1} \bar{q}_{im} \left( \sum_{k \in N_3} \bar{x}_{imk}^* \right),
\]

which is a contradiction. Therefore \( \sum_{k \in N_3} \bar{x}_{imk}^* = 0 \) for all \( i \in N_1 \) such that \( \bar{q}_{im} - \bar{p}_i < \max_i [\bar{q}_{im} - \bar{p}_i]. \) Since \( \sum_{i \in N_1} \sum_{k \in N_3} \bar{x}_{imk}^* = 1 \) for all \( m \in M_j \) by
A2 and $|M_j| = s_j$,

$$
\max_{(\tilde{x}_{imk})\in N_1, k\in N_3 \in X_m} \left[ -\sum_{i\in N_1} \tilde{p}_i (\sum_{k\in N_3} \tilde{x}_{imk}) + \sum_{i\in N_1} \tilde{q}_{im} (\sum_{k\in N_3} \tilde{x}_{imk}) \right]
$$

$$
= \max_i [\tilde{q}_{im} - \tilde{p}_i]
$$

$$
= \max_i [\tilde{q}_{im} - \tilde{p}_i] (\sum_{k\in N_3} \tilde{x}_{imk})
$$

$$
= \sum_{i\in N_1} \max_i [\tilde{q}_{im} - \tilde{p}_i] (\tilde{x}_{imk})
$$

$$
= \sum_{i\in N_1} (\tilde{q}_{im} - \tilde{p}_i) (\sum_{k\in N_3} \tilde{x}_{imk})
$$

$$
= -\sum_{i\in N_1} \tilde{p}_i (\sum_{k\in N_3} \tilde{x}_{imk}) + \sum_{i\in N_1} \tilde{q}_{im} (\sum_{k\in N_3} \tilde{x}_{imk}),
$$

which completes the proof of Step 3.

**Step 4:** We will show “if part” of the statement in Lemma 5.

By the definition of the competitive equilibrium, there exists a competitive outcome $(\hat{u}^*, \hat{v}^*, \hat{w}^*) \in \mathbb{R}^{n_1+n_2+n_3}$ of the three-sided market $(N_1, N_2, N_3)$ such that

- for all $i \in N_1$: $\hat{u}_i^* = U_i(\hat{x}_i) + p_i(\sum_{j\in N_2} \sum_{k\in N_3} \hat{x}_{ijk})$,
- for all $j \in N_2$: $\hat{v}_j^* = -\sum_{i\in N_1} p_i (\sum_{k\in N_3} \hat{x}_{ijk}) + \sum_{i\in N_1} q_{ij} (\sum_{k\in N_3} \hat{x}_{ijk})$ and
- for all $k \in N_3$: $\hat{w}_k^* = U_k((\hat{x}_{ijk})_{i\in N_1, j\in N_2}) - \sum_{i\in N_1} \sum_{j\in N_2} q_{ij} \hat{x}_{ijk}$.

Define $x_i^* = \hat{x}_i$, $x_{ij}^* = \hat{x}_{ijk}$, $\tilde{x}_{ij}^* = \hat{x}_{ijk}$ and $\tilde{x}_{ij} = \sum_{k\in N_3} \tilde{x}_{ijk}$. It is clear that

- for all $i \in N_1$: $U_i(x_i^*) + p_i(\sum_{j\in N_2} \tilde{x}_{ij}^*) = U_i(\hat{x}_i) + p_i(\sum_{j\in N_2} \sum_{k\in N_3} \hat{x}_{ijk})$,
- for all $j \in N_2$: $-\sum_{i\in N_1} p_i (\sum_{k\in N_3} \tilde{x}_{ijk}) + \sum_{i\in N_1} q_{ij} (\sum_{k\in N_3} \tilde{x}_{ijk}) = -\sum_{i\in N_1} p_i (\sum_{k\in N_3} \hat{x}_{ijk}) + \sum_{i\in N_1} q_{ij} (\sum_{k\in N_3} \hat{x}_{ijk})$,
- for all $k \in N_3$: $U_k((x_{ijk}^*)_{i\in N_1, j\in N_2}) - \sum_{i\in N_1} \sum_{j\in N_2} q_{ij} x_{ij}^* = U_k((\hat{x}_{ijk})_{i\in N_1, j\in N_2}) - \sum_{i\in N_1} \sum_{j\in N_2} q_{ij} \hat{x}_{ijk}$.

and each of them satisfies A1, A2 and A3 respectively.
Thus \((x^*_i)_{i \in N_1}, (\tilde{x}^*_{ij})_{i \in N_1, j \in N_2}\) is a maximal solution for each \(i \in N_1\), \((\tilde{x}^*_{ijk})_{i \in N_1, j \in N_2, k \in N_3}\) is a maximal solution for each \(j \in N_2\) and \((x^*_{ijk})_{i \in N_1, j \in N_2, k \in N_3}\) is a maximal solution for each \(k \in N_3\).

Define \((x^*_{imk})_{m \in M_j} = f_j(x^*_{ij}), (\tilde{x}^*_{imk})_{m \in M_j} = f_j(\tilde{x}^*_{ijk})\) and \((\tilde{x}^*_{im})_{m \in M_j} = f_j(\tilde{x}^*_{ij})\). By Step 1, each of them satisfies A1, A2 and A3 respectively.

Define \(\hat{x}_{imk} = x^*_{imk}\). Then \(\hat{x}_{imk} = x^*_{imk} = \hat{x}_{imk}, \hat{x}_{ijk} = \sum_{m \in M_j} \hat{x}_{imk}\) and \(\tilde{x}^*_{im} = \sum_{k \in N_3} \hat{x}_{imk}\). By Step 2, we get

\[
\begin{align*}
\text{for all } i \in N_1 : \quad & \hat{u}^*_i = \bar{U}_i(\hat{x}_i) + \bar{p}_i(\sum_{m \in \bar{N}_2} \sum_{k \in N_3} \hat{x}^*_{imk}), \\
\text{for all } j \in N_2 : \quad & \hat{v}^*_j = \sum_{m \in M_j} \left[ - \sum_{i \in N_1} \bar{p}_i(\sum_{k \in N_3} \hat{x}^*_{imk}) + \sum_{i \in N_1} \bar{q}_i(\sum_{k \in N_3} \hat{x}^*_{imk}) \right] \\
\text{and} \quad & \text{for all } k \in N_3 : \quad \hat{w}^*_k = \bar{U}_k((\hat{x}^*_{imk})_{i \in N_1, m \in \bar{N}_2}) - \sum_{i \in N_1} \sum_{m \in \bar{N}_2} \bar{q}_m \hat{x}^*_{imk}.
\end{align*}
\]

It is clear that \(((x^*_i)_{i \in N_1}, (\tilde{x}^*_{im})_{i \in N_1, m \in \bar{N}_2})\) is a maximal solution for each \(i \in N_1\) and \((x^*_{im})_{i \in N_1, m \in \bar{N}_2, k \in N_3}\) is a maximal solution for each \(k \in N_3\). By Step 3, \((\tilde{x}^*_{imk})_{i \in N_1, m \in \bar{N}_2, k \in N_3}\) is a maximal solution for each \(m \in \bar{N}_2\). Moreover, \(\omega_i - \hat{x}_i = \sum_{j \in N_2} \sum_{k \in N_3} \hat{x}_{ijk} = \sum_{m \in \bar{N}_2} \sum_{k \in N_3} \hat{x}^*_{imk}\). Thus \(((\hat{x}_i)_{i \in N_1}, (\tilde{x}^*_m)_{i \in N_1, m \in \bar{N}_2}, (\hat{x}^*_{imk})_{i \in N_1, m \in \bar{N}_2, k \in N_3})\) satisfies (I), (II), (III) and (IV) in the definition of the competitive equilibrium. Therefore there exists the competitive equilibrium \((\bar{p}, \bar{q}, (\hat{x}_i)_{i \in N_1}, (\tilde{x}^*_m)_{i \in N_1, m \in \bar{N}_2}, (\hat{x}^*_{imk})_{i \in N_1, m \in \bar{N}_2, k \in N_3})\) of the agent normal form market \((N_1, \bar{N}_2, N_3)\).

**Step 5:** The remainder is to show “only if part” of the statement in Lemma 5. However, since the proof of Step 5 is the same as Step 4, the proof will be omitted.
References


