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KUMQRP DISCUSSION PAPER SERIES

DP2007-028

Game Theoretic Analyses of a Knockout of Bidding Rings

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abstract
In practice, bidding rings at an English auction frequently distribute collusive gains among the ring members via a nested knockout. This paper presents a generic nested knockout and discusses the relationship between each distributive outcome of this generic nested knockout and each solution of bidding ring games. This paper shows that the outcome yielded by each nested knockout belongs to the core. In particular, a .nest nested knockout can yield the Shapley value and a nested knockout in which the surplus of each inner ring is minimal can yield the nucleolus.

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Game Theoretic Analyses of a Knockout of Bidding Rings

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February 25, 2008

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Keywords: Bidding rings; Nested knockout; Shapley value; Nucleolus

JEL classification: C71

1 Introduction

Bidding rings have been operated throughout the world. For example, Cassady (1967) reported realistic bidding rings in many commodity fields, such as the antique trade, the fish trade, and timber rights. A bidding ring is to eliminate buyer competition and thus gain an advantage over sellers' side. Therefore bidding rings are price influencing behavior in auctions.

Cassady (1967) explained that in a realistic English auction rings allocate the object won the main English auction via a secondary English auction. This secondary auction after the main auction is called a *knockout.* When bidders have different evaluations for the object being auctioned, it is observed that knockouts are repeated until the final owner of the object is determined. This sequence of knockouts is called a *nested knockout.*

By following observations in Cassady (1967), Graham et al. (1990) formalized a single nested knockout mathematically and considered the relationship between the distributive outcome of the knockout and the Shapley value of a TU

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game derived from collusive behavior, namely the bidding ring game. However, in fact, there may exist many types of the nested knockout. This is because, in general, size of an inner ring at each knockout nesting is variable and Graham et al. (1990) focused on only one case with respect to size of an inner ring at each knockout nesting.

This paper investigates the relationship between the distributive outcome yielded by each nested knockout and each of TU game solutions such as the core, the Shapley value and the nucleolus of the bidding ring game. My research is an extension of Graham et al. (1990). In fact, I formalize a generic nested knockout mathematically. The generic nested knockout allows not only variability of size of an inner ring at each knockout nesting but also variability of the number of participants of each knockout. Therefore, the generic nested knockout can generate various nested knockouts including one considered in Graham et al. (1990).

My approach to study bidding rings, which is inspired by Graham et al. (1990), uses cooperative game theory. Most of literature on bidding rings since Graham and Marshall's (1987) seminal work concerns with mechanism design theory. The literature assumes that bidders act noncooperatively under incomplete information on auctions and it demonstrates how to operate bidding rings. However, this assumption underlying the mechanism design approach may not be appropriate under some situation that there is strong possibility of cooperation among bidders. An example is the situation that the number of all bidders at a main auction is relatively small and each bidders at the auction has approximately complete information on the auction. This example implies that some bidding rings may be operated by cooperative behavior among bidders, not by noncooperative behavior among bidders. Therefore cooperative game approach is suitable to deal with the situations mentioned above.

Initially, I show that the distributive outcome yielded by each nested knockout belongs to the core of the bidding ring game. Therefore all nested knockouts yield core imputations. Next, I show that some distributive outcomes can yield one-point TU game solutions such as the Shapley value and the nucleolus. That is to say, a nested knockout with the finest nested structure can yield the Shapley value and a nested knockout in which the surplus of each inner ring is minimal can yield the nucleolus. The nested knockout with respect to the Shapley value may be interpreted as one considered in Graham et al. (1990). The nested knockout with respect to the nucleolus may be interpreted as one in which each inner ring can guarantee its advantage for itself. Therefore, the present study demonstrates that a generic distribution rule generated from practical observations of nested knockouts can yield core imputations including the Shapley value and the nucleolus.

The remainder of the paper is organized as follows: in Section 2, I will give the model of a generic nested knockout and I will characterize some nested knockouts based on the generic one. Section 3 introduces bidding ring games and demonstrates the relationship between each distributive outcome and each solution of the bidding ring game by using formalizations in Section 2. The paper closes in Section 4 with concluding remarks on two possible applications derived from the present study.

2 The model of nested knockouts

2.1 The generic nested knockout

In practice, bidding rings at a single-object English auction distribute collusive gains among the ring members via a knockout. The knockout is a secondary English auction only among ring members after the main auction.

When ring members have different evaluations for the object, it is common to observe that knockouts are repeated until the final owner of the object is determined. Then the knockouts facilitate nested ring structure. This structure means that there is a ring formed at the present knockout within a ring formed at its previous knockout. The sequence of knockouts is called a nested knockout. A knockout is conducted at each level of knockout nesting. The procedure of a nested knockout will be explained in this subsection.

Suppose that there are n buyers in the single-object English auction. Suppose further that v_i is the evaluation of each buyer $i = 1, \cdots, n$ for the item being auctioned, and v_0 is the reservation price of the only seller for his item. Let $v_1 > v_2 > \cdots > v_n > v_0 \geq 0$.

There is possibility of a bidding rings at the main auction. Suppose that n bidders form an initial ring $R_0 = N$, where $N = \{1, \cdots, n\}$. At the main auction, buyer 1 remains active in the bidding up to v_1 and each buyer except buyer 1 remains active in the bidding up to v_0 or does not participate. Since buyer 1 can win the main auction and pays v_0 to the seller, R_0 can get the net gains $v_1 - v_0$.

After the main auction, the members in R_0 make possible a sequence of knockouts in which English auctions are used in order to determine distributions of collusive gains among all members in R_0. This nested knockout has an inductive distribution rule. The details of this rule will be also described as follows:

Suppose that in the j_{th} knockout m_j ($m_j < m_{j-1}$ and $m_0 \equiv n$) buyers form an inner ring $R_j = \{1, 2, \cdots, m_j\}$, which is contained in an inner ring of the previous knockout $R_{j-1} = \{1, 2, \cdots, m_{j-1}\}$. In the j_{th} knockout, the members of R_j appoint buyer 1 to remain active in the bidding up to v_1 and each buyer except buyer 1 to remain active in the bidding up to v_{m_j+1} or not to participate. The members of $R_{j-1} \backslash R_j$ bid competitively. Since buyer 1 can win the j_{th} knockout and pays v_{m_j+1} for the object, R_j can win the ownership of the object and can get the net gains $v_1 - v_{m_j+1}$. The difference between *the knockout gains of* R_{j-1} (to be defined inductively) and *the net gains of* R_j, namely $v_1 - v_{m_j+1}$ is then *equally* divided among all participants of the j_{th} knockout. As the result of the j_{th} knockout, each member of $R_{j-1} \backslash R_j$ can get this equal distribution as ultimate gains for colluding at the main auction. On the other hand, R_j can get $v_1 - v_0$ minus the total amount of the distribution to the members of $R_0 \backslash R_j$. I call this gains especially *the knockout gains of* R_j. It is possible for the knockouts to continue in this manner for $j = 1, 2, \cdots, t$ such that $R_t = \{1\}$. In the final knockout, buyer 1 is determined as the final owner of the object and ultimate gains of each member of R_0 for colluding at the main auction are determined.

I call ultimate gains of each member of R_0 *the distributive outcome* yielded by the generic nested knockout. From the above explanation, the distributive outcome yielded by the generic nested knockout is defined immediately as follow:

Definition 1 *Let $(z_i)_{i\in N}$ be a distributive outcome yielded by the generic nested knockout. Let $R_j = \{1, 2, \cdots, m_j\}$ where $m_j < m_{j-1}$ and let the number of participants of the j_{th} knockout be given by k_j where $m_{j-1} - m_j + 1 \leq k_j \leq m_{j-1}$. Given $N = R_0 \supset R_1 \supset \cdots \supset R_t = \{1\}$ and $(k_j)_{j=1}^t$ where $k_t = m_{t-1}$, z_i is defined inductively by $z_1 = v_1 - v_2 + z_t$ and*

$$z_i = \frac{v_{m_j+1} - v_0 - \sum_{l=m_{j-1}+1}^{n} z_l}{k_j} \quad \text{for} \;\; m_j + 1 \leq i \leq m_{j-1},$$

beginning with $m_0 \equiv n$, $\sum_{l=m_0+1}^{n} z_l \equiv 0$ and continuing for $j = 1, \cdots, t$.

2.2 Specification of the nested knockout

There may be many cases of the nested knockout. This is because size of an inner ring at each knockout nesting and the number of participants of each knockout are variable. I will characterize especially two distinct cases of the nested knockout.

2.2.1 The finest nested knockout

A simple nested knockout is the finest nested knockout in the sense that the number of knockouts is largest. I will formulate the finest nested knockout as follow:

Definition 2 *The finest nested knockout is a nested knockout in which at each j_{th} knockout, the size of an inner ring is $|R_j| = n - j$ and the number of participants is $|R_{j-1}|$.*

Then the distributive outcome of the finest nested knockout is given by the following proposition. The proof of this proposition will be omitted since it is a matter of calculation derived from Definition 1 and Definition 2.

Proposition 1 *Let $(\hat{x}_i)_{i\in N}$ be a distributive outcome yielded by the finest nested knockout. Then $\hat{x}_i$ is given by*

$$\begin{aligned} \hat{x}_i &= \sum_{j=i}^{n-1} \frac{v_j - v_{j+1}}{j} + \frac{v_n - v_0}{n} \quad \text{for } i \leq n-1 \\ \hat{x}_n &= \frac{v_n - v_0}{n}. \end{aligned}$$

2.2.2 The minimal surplus nested knockout

Let us consider a nested knockout in which the difference between the knockout gains of R_j and the net gains of R_j is minimal in each j_{th} knockout. I call this difference *the surplus of an inner ring* R_j. The net gains of R_j is the worth that R_j can get by itself. The knockout gains of R_j is the worth that R_j can get via a nested knockout distribution rule mentioned in subsection 2.1. Therefore the surplus of R_j is regarded as its advantages in forming R_j in the nested knockout.

Definition 3 *The minimal surplus nested knockout is a nested knockout in which the surplus of an inner ring at each knockout nesting is minimal.*

The above nested knockout is a nested knockout in which each inner ring can guarantee its advantage for itself. The distributive outcome of this nested knockout can be given as the following.

Proposition 2 *Let* $(x_i^*)_{i\in N}$ *be a distributive outcome yielded by the minimal surplus nested knockout. Let* $M_i = v_i - v_0$ *for each* $i \in N$*. Then* x_i^* *is given by* $x_i^* = \Lambda_t$, $x_1^* = v_1 - v_2 + \Lambda_{t'}$ *for* $m_{t-1}^* - k_t^* + 2 \le i \le m_{t-1}^*$ *and* $1 \le t \le t'$ *such that* $m_{t'}^* = 1$*, where* Λ_t, k_t^* *and* m_t^* *are defined inductively by*

$$\Lambda_t = \min_{k=2,\cdots,m_{t-1}^*} \left\{ \frac{M_{m_{t-1}^*-k+2} - \sum_{i=0}^{t-1}(k_i^*-1)\Lambda_i}{k} \right\},$$

k_t^* *denotes the largest value of* k *for which the above expression attains its minimum and* $m_t^* = m_{t-1}^* - k_t^* + 1$ *(beginning with* $m_0^* \equiv n$, $\Lambda_0 \equiv 0$ *and continuing for* $t = 1, \cdots, t'$ *such that* $m_{t'}^* = 1$*).*

Proof. The proof will be completed by the following two steps.

Step 1 : In the 1st knockout, for any fixed $k \in R_0\backslash\{1\}$, let $|R_1| = n-k+1$. The net gains of R_0 is M_1 and the net gains of R_1 is $v_1 - v_{n-k+2}$. If the number of participants of R_1 at the 1st knockout is f, then the surplus of R_1 is given by

$$M_{n-k+2} - \frac{M_{n-k+2}}{(k-1)+f}(k-1) = \frac{1}{\frac{k-1}{f}+1} \cdot M_{n-k+2}.$$

If the surplus of R_1 is minimal for any fixed k, then $f = 1$. Therefore the minimal surplus of R_1 is given by

$$\Lambda_1 = \min_{k=2,\cdots,n} \left\{ \frac{M_{n-k+2}}{k} \right\}.$$

Each member of $R_0\backslash R_1$ can gain Λ_1 as his distributive outcome. Take k_1 as a k giving $\Lambda_{(1)}$. Let $R_1 = \{1, \cdots, m_1\}$. Then $m_1 = n - k_1 + 1$.

In the 2nd knockout, for any fixed $k \in R_1\backslash\{1\}$, let $|R_2| = m_1 - k + 1$. The nested knockout gains of R_1 is $M_1 - (k_1 - 1)\Lambda_1$ and the net gains of R_2 is

$v_1 - v_{m_1-k+2}$. If the number of participants of the 2nd knockout is g, then the surplus of R_2 is given by

$$\frac{1}{\frac{k-1}{g}+1} \cdot \left(M_{m_1-k+2} - (k_1-1)\Lambda_1\right).$$

If the surplus of R_2 is minimal for any fixed k, then $g = 1$. Therefore the minimal surplus of R_2 is given by

$$\Lambda_2 = \min_{k=2,\cdots,m_1} \left\{ \frac{M_{m_1-k+2} - (k_1-1)\Lambda_1}{k} \right\}.$$

Each member of $R_1 \backslash R_2$ can gain Λ_2 as his distributive outcome. Take k_2 as a k giving Λ_2. Let $R_2 = \{1, \cdots, m_2\}$. Then $m_2 = m_1 - k_2 + 1$. Similarly, the same procedure can be repeated until the distributive outcome of each members in R_0 is determined.

Step 2 : Next, I will show that in each i_{th} nested knockout we can take $k_i = k_i^*$ such that k_i^* is the largest k giving Λ_i in order to decrease the number of the knockouts. (That is to say, the distributive outcome yielded by the minimal surplus nested knockout is independent on the number of the knockouts.)

Let k_p and k_q ($k_q < k_p$) be every k giving Λ_1, namely $\Lambda_1 = \frac{M_{n-k_p+2}}{k_p} = \frac{M_{n-k_p+2}}{k_p}$. Take k_q as a k giving Λ_1 in the 1st knockout. Let $s_1 = \Lambda_1(k_q-1)$, $m_1 = n - k_q + 1$ and $\Lambda_2 = \min_{k=2,\cdots,m_1}\left\{\frac{M_{m_1-k+2}-s_1}{k}\right\}$. I will prove (i) $\frac{M_{n-k_p+2}}{k_p} = \frac{M_{m_1-k^*+2}-s_1}{k^*}$, where $k^* = k_p - k_q + 1$ and (ii) for any $k = 2, \cdots, m_1$, $\frac{M_{n-k'+2}}{k'} \leq \frac{M_{m_1-k+2}-s_1}{k}$, where $k' = k + k_q - 1$. To show (i) and (ii) suffices for the purpose of Step 2 since the same procedure can be repeated until the t'_{th} knockout.

Proof of (i):

$$\begin{aligned}
\frac{M_{m_1-k^*+2} - s_1}{k^*} &= \frac{M_{n-k_q+1-(k_p-k_q+1)+2} - s_1}{k_p - k_q + 1} \\
&= \frac{M_{n-k_p+2} - \frac{M_{n-k_p+2}}{k_p}(k_q-1)}{k_p - k_q + 1} \\
&= \frac{M_{n-k_p+2}\left(1 - \frac{k_q-1}{k_p}\right)}{k_p - k_q + 1} = \frac{k_p - k_q + 1}{k_p} \cdot \frac{M_{n-k_p+2}}{k_p - k_q + 1} \\
&= \frac{M_{n-k_p+2}}{k_p},
\end{aligned}$$

which completes the proof of (i).

Proof of (ii): For any $k = 2, \cdots, m_1$, it sufficies to prove $k' \left(M_{m_1-k+2} - s_1\right) - k \cdot M_{n-k'+2} \geq 0$. For any $k = 2, \cdots, m_1$,

$$\begin{aligned}
& k' \left(M_{m_1-k+2} - s_1\right) - k \cdot M_{n-k'+2} \\
= \ & (k + k_q - 1) \left(M_{n-k_q+1-k+2} - \Lambda_1 (k_q - 1)\right) - k \cdot M_{n-(k+k_q-1)+2} \\
= \ & (k_q - 1) M_{n-k_q+k+3} - (k + k_q - 1)\Lambda_1 (k_q - 1) \\
= \ & (k_q - 1)(k + k_q - 1) \left[\frac{M_{n-k_q-k+3}}{k + k_q - 1} - \Lambda_1\right] \\
= \ & (k_q - 1)(k + k_q - 1) \left[\frac{M_{n-k'+2}}{k'} - \Lambda_1\right] \\
\geq \ & 0,
\end{aligned}$$

since $\frac{M_{n-k'+2}}{k'} \geq \Lambda_1$ from the definition of Λ. This completes the proof of (ii).

Therefore, in each i_{th} nested knockout we can take k_i^* as the largest value of k for which the formula of Λ_t attains its minimum and we have $m_i^* = m_{i-1}^* - k_i^* + 1$, which completes the proof. ■

Remark 1 *The distributive outcome yielded by the minimal surplus nested knockout x^* satisfies $x_1^* \geq x_2^* \geq \cdots \geq x_n^*$ owing to Step 2 in the proof of Proposition 2. In other words, the minimal surplus nested knockout has order-preserving in the sense that $x_1^* \geq x_2^* \geq \cdots \geq x_n^*$ if $v_1 > v_2 > \cdots > v_n$. Also, obviously, the distributive outcome yielded by the finest nested knockout has order-preserving.*

The following example gives the distributive outcome yielded by the minimal surplus nested knockout of a 7 buyers case.

Example (A 7-buyers case)

$$v_1 = 30, v_2 = 25, v_3 = 21, v_4 = 15, v_5 = 14, v_6 = 11, v_7 = 10, v_0 = 5.$$

The 1st knockout:

$$\Lambda_1 = \min\left\{\frac{M_7}{2}, \frac{M_6}{3}, \cdots, \frac{M_2}{7}\right\} = \min\left\{\frac{5}{2}, \frac{6}{3}, \frac{9}{4}, \frac{10}{5}, \frac{16}{6}, \frac{20}{7}\right\} = 2$$

$$k_1^* = 5, \quad x_i^* = \Lambda_1 = 2 \quad (i = 4, 5, 6, 7)$$

$$m_1^* = n - k_1^* + 1 = 3$$

The 2nd knockout:

$$\Lambda_2 = \min\left\{\frac{M_3 - (k_1^* - 1)\Lambda_1}{2}, \frac{M_2 - (k_1^* - 1)\Lambda_1}{3}\right\} = \min\left\{\frac{8}{2}, \frac{12}{3}\right\} = 4$$

$$k_2^* = 3, \quad x_i^* = \Lambda_2 = 4 \quad (i = 2, 3)$$

$$m_2^* = m_1^* - k_2^* + 1 = 1 = m_{t'}^*$$

$$x_1^* = v_1 - v_2 + \Lambda_{t'} = v_1 - v_2 + \Lambda_2 = 9$$

The distributive outcome yielded by the minimal surplus nested knockout is given by $x^* = (9, 4, 4, 2, 2, 2, 2)$. ■

3 Solutions of bidding ring games

3.1 Bidding ring games

A bidding ring game at an English auction under complete information is the TU game introduced by Graham et al. (1990). Let $N = \{1, \cdots, n\}$ be the finite set of buyers, and $S \subseteq N$ be a coalition. Let v_i be the evaluation of each buyer i for the only item being auctioned, and v_0 be the reservation price of the only seller for his item. Suppose that $v_1 > v_2 > \cdots > v_n > v_0 \geq 0$. Let us consider an English auction with a possibility of bidding ring among buyers of the auction. This situation can be described as the TU game (N, v) satisfying

$$v(S) = \begin{cases} v_1 - \max_{j \notin S} v_j & \text{if} \quad 1 \in S \\ 0 & \text{if} \quad 1 \notin S, \end{cases}$$

where $\max_{j \notin N} v_j \equiv v_0$. We call this game the bidding ring game.

The characteristic function of this game denotes the net gains that S can guarantee for itself. This function is based on the followings. First, under the English auction rule, it is a dominant strategy for each bidder to remain active until bidding reaches his evaluation. Second, in any S including buyer 1, S makes buyer 1 the sole essential bidder in the coalition. Then, if each buyer in S except buyer 1 has higher evaluation than $\max_{j \notin S} v_j$, he remains active until bidding reaches $\max_{j \notin S} v_j$. If each buyer in S except buyer 1 has lower evaluation than $\max_{j \notin S} v_j$, then he remains active until bidding reaches his evaluation. Since each bidder in $N \backslash S$ remains active until bidding reaches his evaluation, by the auction rule, the coalition S can win the auction with the net gains $v_1 - \max_{j \notin S} v_j$. Lastly, in any S not including buyer 1, this coalition cannot win the auction, hence the net gains is 0.

Remark 2 *The bidding ring game is convex (See Graham et al. (1990)). Therefore the core of this game is nonempty. Furthermore, in this game, Shapley value and the nucleolus belong to the core.*

The following subsections 3.2 and 3.3 discuss the relationship between each distributive outcome of the nested knockout and each solution of bidding ring games at an English auction. I will state here that well-known one point solutions such as the Shapley value and the nucleolus can be characterized by the nested knockouts respectively.

3.2 The core and the Shapley value

An n-dimensional vector x of the bidding ring game is a payoff vector if it satisfies that $\sum_{i \in N} x_i = v(N)$. Then the core of the present game is a set of payoff vectors x satisfying $\sum_{i \in S} x_i \geq v(S)$ for all $S \subseteq N$.

The Shapley value $\phi(v)$ of this game is a payoff vector given by the formula

$$\phi_i(v) = \sum_{S \subseteq N,\ i \notin S} \frac{|S|!(n - |S| - 1)!}{n!} \left(v(S \cup i) - v(S) \right), \quad \forall i \in N.$$

Proposition 3 *Each distributive outcome yielded by the generic nested knockout belongs to the core.*

Proof. Let $z \in \mathbb{R}^n$ be the distributive outcome yielded by the generic nested knockout. From Definition 1, z is a payoff vector. If $1 \notin S$, then $\sum_{i \in S} z_i > 0 = v(S)$. If $1 \in S$, then it suffices to show $\sum_{i \in S} z_i \geq v(S)$ such that $S = R_j (= \{1, \cdots, m_j\})$ since $z_i > 0$. Recall that R_j gets $v_1 - v_0$ minus the total amount of the distribution to the members of $N \backslash R_j$.

Then we have

$$\begin{aligned} \sum_{i \in S} z_i - v(S) &= v_1 - v_0 - \sum_{i \in N \backslash R_j} z_i - \left(v_1 - v_{m_j+1}\right) \\ &= \left(1 - \frac{m_{j-1} - m_j}{k_j}\right) \left(v_{m_j+1} - v_0 - \sum_{i \in N \backslash R_j} z_i\right) \\ &> 0, \end{aligned}$$

since $m_{j-1} - m_j < k_j$ and $v_{m_j+1} - v_0 - \sum_{i \in N \backslash R_j} z_i > 0$, which are derived from Definition 1. This completes the proof. ■

Proposition 4 *The distributive outcome yielded by the finest nested knockout is the Shapley value.*

Proof. It is obvious since Theorem 2 in Graham et al. (1990) and Proposition 1 in the present paper. ■

3.3 The nucleolus

Let X be the set of all imputations of the bidding ring game, namely $X = \{x \in \mathbb{R}^n | \sum_{i \in N} x_i = v(N),\ x_i \geq v(\{i\})\ for\ all\ i \in N\}$. Obviously, $X \neq \emptyset$. Given an imputation $x \in X$ for this game, *the excess* of a coalition S with respect to x is defined as the number $v(S) - \sum_{i \in S} x_i$ and let $e(x)$ denote the $(2^n - 2)$ dimensional vector, the components of which are all non-trivial excesses, namely the excesses of every coalition $S \neq N, \emptyset$ with respect to x, arranged in the non-increasing order. *The nucleolus* of v is defined as the set of imputations such that the vector $e(x)$ is *lexicographically minimal* over X. It is well known that the nucleolus is not empty and *a singleton* (Schmeidler (1969)).

Let v' be *the 0-normalization* of v, that is, $v'(S) = v(S) - \sum_{i \in S} v(\{i\})$ for every coalition S. It is well known that an imputation z is the nucleolus of the game v *if and only if* an imputation z' satisfying $z'_i = z_i - v(\{i\})$ for each i in N is the nucleolus of the game v'.

Proposition 5 *The distributive outcome yielded by the minimal surplus nested knockout is the nucleolus.*

Proof. Let v be the 0-normalization of a bidding ring game and let $M_i = v_i - v_0$ *for each* $i \in N$. Then,

$$v(S) = \begin{cases} M_2 & \text{if} \quad S = N \\ M_2 - \max_{i \notin S} M_i & \text{if} \quad 1 \in S \subsetneq N \\ 0 & \text{if} \quad 1 \notin S. \end{cases}$$

In the followings, I will focus on the game v stated above.

Let us consider how to divide the net gains of R_0 minus $v_1 - v_2$ among all members of R_0. By Proposition 2, we can give a modified distributive outcome as follow:

(P): *Let* $(x_i^*)_{i \in N}$ *be a modified distributive outcome yielded by the minimal surplus nested knockout. Let* $M_i = v_i - v_0$ *for each* $i \in N$. *Then* x_i^* *is given by* $x_i^* = -\lambda_t$, $x_1^* = -\lambda_{t'}$ *for* $m_{t-1}^* - k_t^* + 2 \leq i \leq m_{t-1}^*$ *and* $1 \leq t \leq t'$ *such that* $m_{t'}^* = 1$, *where* λ_t, k_t^* *and* m_t^* *are defined inductively by*

$$\lambda_t = \max_{k=2,\cdots,m_{t-1}^*} \left\{ -\frac{M_{m_{t-1}^*-k+2} + \sum_{i=0}^{t-1}(k_i^* - 1)\lambda_i}{k} \right\},$$

k_t^* *denotes the largest value of* k *for which the above expression attains its maximum and* $m_t^* = m_{t-1}^* - k_t^* + 1$ (*beginning with* $m_0^* \equiv n$, $\lambda_0 \equiv 0$ *and continuing for* $t = 1, \cdots, t'$ *such that* $m_{t'}^* = 1$).

It suffices to show the distributive outcome stated in (P) is the nucleolus of v. The proof is based on Kopelowit$'$z algorithm. Initially consider the optimal solution to the following problem (I) as

$$\begin{aligned} & \min \quad \lambda \\ s.t. \quad & \sum_{i \in S} x_i \geq -\lambda \quad \forall S \subsetneq N : 1 \notin S \\ & \sum_{i \in S} x_i \geq M_2 - \max_{i \notin S} M_i - \lambda \quad \forall S \subsetneq N : 1 \in S \\ & \sum_{i \in N} x_i = M_2 \\ & x_i \geq 0 \quad \forall i \in N. \end{aligned}$$

λ is bounded below and for $\lambda = 0$, x satisfying $x_1 = M_2$ and $x_i = 0$ for $i \neq 1$ is feasible. Hence, the optimal solution to problem (I) exists and $\lambda \leq 0$. If $1 \notin S$, then the corresponding constraint is dominated by the constraints

$$x_i \geq -\lambda \qquad i = j(S), j(S)+1, \cdots, n$$

for the coalitions $\{j(S)\}$,$\{j(S)+1\}$,$\cdots$,$\{n\}$ where $j(S) = \min\{j | j \in S, 1 \notin S\}$ because $S \subset \{j(S), j(S)+1, \cdots, n\}$ and $x_i \geq 0$ for $i = 1, \cdots, n$. If $1 \in S$, then the corresponding constraint is dominated by

$$\sum_{i=1}^{k} x_i \geq M_2 - M_{k+1} - \lambda \qquad k = 1, \cdots, \tilde{j}(S) - 1$$

for the coalition $\{1, 2, \cdots, \tilde{j}(S) - 1\}$ where $\tilde{j}(S) = \min\{j | j \notin S, 1 \in S\}$ because $x_i \geq 0$ for $i = 1, \cdots, n$. Problem (I) may, therefore, be simplified to the following problem (I$'$) as

$$\begin{aligned} & \min \quad \lambda \\ s.t. \quad & x_i \geq -\lambda \quad \forall i \in N \\ & \sum_{i=1}^{k} x_i \geq M_2 - M_{k+1} - \lambda \quad k = 2, \cdots, n-1 \\ & \sum_{i \in N} x_i = M_2 \\ & x_i \geq 0 \quad \forall i \in N. \end{aligned}$$

I now claim that the optimal value to problem (I$'$) is

$$\lambda_1 = \max\left\{-\frac{M_n}{2}, -\frac{M_{n-1}}{3}, \cdots, -\frac{M_2}{n}\right\},$$

which is as defined in (P). Since it is trivial that $\lambda \geq \max\left\{-\frac{M_n}{2}, \cdots, -\frac{M_2}{n}\right\}$, it suffices to show that $\lambda = \max\left\{-\frac{M_n}{2}, \cdots, -\frac{M_2}{n}\right\}$. Let x' satisfying $x'_1 = -\lambda + a_1$, $x'_2 = -\lambda + a_2$ and $x'_i = -\lambda$ for $i \neq 1, 2$ ($a_1 \geq 0, a_2 \geq 0$ and $a_1 + a_2 = M_2 + n\lambda$). Then I will show that x' is feasible for $\lambda = \max\left\{-\frac{M_n}{2}, \cdots, -\frac{M_2}{n}\right\}$. It is trivial that $x'_i \geq -\lambda$ for any i in N, $\sum_{i \in N} x'_i = M_2$, and $x'_i \geq 0$ because $\lambda \leq 0$. Moreover, for $k = 2, \cdots, n-1$, we have

$$\begin{aligned} \sum_{i=1}^{k} x'_i - (M_2 - M_{k+1} - \lambda) &= (n-k+1)\lambda + M_{k+1} \\ &\geq (n-k+1)\left(-\frac{M_{k+1}}{n-k+1}\right) + M_{k+1} = 0 \end{aligned}$$

because $\lambda \geq \max\left\{-\frac{M_n}{2}, \cdots, -\frac{M_2}{n}\right\}$. Therefore, for $\lambda = \max\left\{-\frac{M_n}{2}, \cdots, -\frac{M_2}{n}\right\}$, x' is feasible. This establishes the claim.

This claim implies that we have

$$\begin{aligned} x^*_i &= -\lambda_1 \quad i = n - k^*_1 + 2, \cdots, n \\ \sum_{i=1}^{n-k^*_1+1} x^*_i &= M_2 - M_{n-k^*_1+2} - \lambda_1 \end{aligned}$$

where λ_1 and k^*_1 are as defined in (P). If $k^*_1 \neq n-1$, variables $x^*_{m^*_1+1}, \cdots, x^*_n$ where m^*_1 is as defined in (P) may be eliminated from problem (I$'$), together with the additional equality constrains involving every coalition $S \subset \{m^*_1 +$

$1, m_1^* + 2, \cdots, n\}$. Problem (II) may be written

$$\begin{aligned} & \min \quad \lambda \\ s.t. \quad & x_i \geq -\lambda \quad i = 1, \cdots, m_1^* \\ & \sum_{i=1}^{k} x_i \geq M_2 - M_{k+1} - \lambda \quad k = 2, \cdots, m_1^* - 1 \\ & \sum_{i=1}^{m_1^*} x_i = M_2 + (k_1^* - 1)\lambda_1^* \\ & x_i \geq 0 \quad i = 1, \cdots, m_1^*. \end{aligned}$$

This is of precisely the same form as problem (I′), so the same procedure may be repeated for $t = 1, \cdots t'$ such that $m_{t'}^* = 1$, therefore, the proof is completed. ■

4 Concluding Remarks

I will give concluding remarks by mentioning two possible applications derived from the present study.

(1) Let v be *the monotonic game*, that is, $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$. For each coalition S, let the value of $v(N) - v(N \backslash S)$ be *the marginal contribution of* S to the grand coalition N. Note that the marginal contribution of each coalition to the grand coalition is non-negative because of the monotonicity of v. For every singleton $S = \{i\}$, the value of $v(N) - v(N \setminus S)$ is especially called *the marginal contribution of player* i to the grand coalition. Oishi (2006) defined a monotonic game as follow:

v is a game with collectively contributing coalitional leaders if it satisfies that

$$v(N) - v(N \backslash S) = \max_{i \in S} \left(v(N) - v(N \backslash \{i\})\right), \ \forall S \subseteq N.$$

The game v stated above describes a situation that for each coalition S, the marginal contribution of S to N is made by a player with the highest marginal contribution to N among all the members in S. Then the player whose marginal contribution to N is highest in S is *a collectively contributing coalitional leader*.

Oishi (2006) presented that the bidding ring game is a game with collectively contributing coalitional leaders. Therefore, the results in the present study can be generalized to games with collectively contributing coalitional leaders. As an application along this line, we can calculate the Shapley value and the nucleolus of the sewerage system game, which deals with the benefit allocation problem of cities sharing a sewer and a sewage plant on a river (See Oishi (2006)). This implies that the result of the preset study may apply to various benefit allocation problems defined on the line graph.

(2) Let (N, v) be a coalitional game. A game (N, v) will be abbreviated to v. Oishi and Nakayama (*JER*, forthcoming) defined the anti-dual of v to be

the dual of $(-v)$. As an economic meaning of the anti-dual, the anti-dual game may be considered as a cost game when we regard the original game as a profit game. A good example of the anti-dual is the relation between the airport game due to Littlechild (1974) with one aircraft in each type and the bidding ring game. Oishi and Nakayama (*JER*, forthcoming) showed that solutions such as the core, the Shapley value and the nucleolus of the anti-dual are obtained straightforwardly from original games.

The anti-dual of the bidding ring game is the simple airport game mentioned above. Therefore the generic distribution rule considered in the present study may lead to the core, the Shapley value and the nucleolus of the simple airport game.

References

[1] Cassady, R. Jr., Auctions and auctioneering. Berkeley, University of California Press, 1967.

[2] Graham,D.A. and R.C.Marshall, Collusive Bidder Behavior at Single-object Second-price and English Auctions, *Journal of Political Economy, 95*, 1217-1239, 1987.

[3] Graham,D.A., R.C.Marshall and J-F.Richard, Differential Payments within a Bidder Coalition and the Shapley Value, *American Economic Review, 80*, 493-510, 1990.

[4] Littlechild,S.C., A Simple Expression for the Nucleolus in a Special Case, *International Journal of Game Theory, 3*, 21-29, 1974.

[5] Oishi,T., The Nucleolus in a Game with Collectively Contributing Coalitional Leaders, DP2005-040, KUMQRP Discussion Paper Series, 2006.

[6] Oishi,T. and M.Nakayama, The Anti-Dual of Economic Coalitional TU Games, *The Japanese Economic Review*, forthcoming.

[7] Schmeidler,D., The Nucleolus of a Characteristic Function Game, *SIAM Journal of Applied Mathematics,17*, 1163-1170,1969.