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Abstract

In this paper, we discuss the formation of collaboration networks among firms that are located in a circular city and engage in price competition. We examine the pairwise stability of some networks and a dynamic network formation process. In addition, we consider the efficiency of networks.

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Abstract

In this paper, we discuss the formation of collaboration networks among firms that are located in a circular city and engage in price competition. We examine the pairwise stability of some networks and a dynamic network formation process. In addition, we consider the efficiency of networks.

Keywords : Network formation, Collaboration, Spatial Competition, Pairwise stable, Formation process

JEL classification codes : D85, L11, L13, R32

1 Introduction

In this paper, we will discuss collaboration networks among firms that are located in a circular city and engage in price competition. We consider that firms can form collaborative links with other firms. If two firms form a link between them, then the production costs of the firms are reduced. A set of all firms and the existing links between them is said to be a network. Some previous studies discuss Cournot and Bertrand models with formation of collaboration networks. See, for instance, Bloch (2002), Goyal and Joshi (2003, 2006), Kawamata (2004) and Okumura (2007a,b). Johnson and Gilles (2000) discuss the relationship between the players' locations and the formation of communication networks. They apply the connection model introduced by Jackson and Wolinsky (1996). However, there is no former paper analyzing a spatial competition model with formation of collaboration networks among firms.

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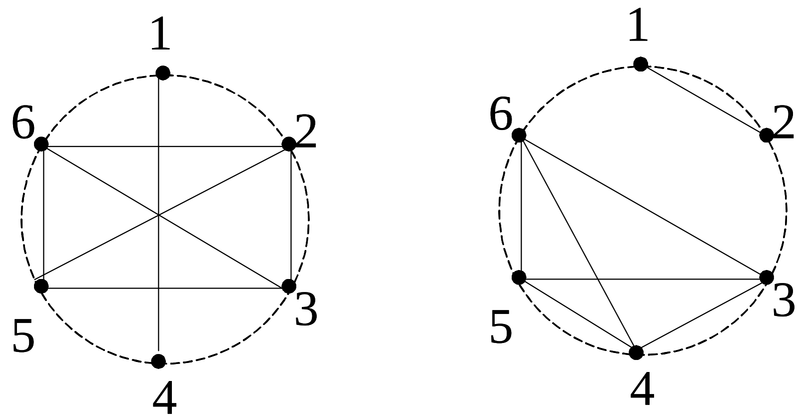

Figure 1: Two Different Networks

A network is represented by a set of nodes and edges. The nodes represent the firms and the edges represent the existing bilateral collaborative links among firms. We will analyze the stability and some market outcomes of networks. In the former models of network formation among firms, some market outcomes and the stability result of a network is same as the renamed network. For example, consider the six-firm example. Figure 1 depicts two different networks. In the network in the left of Figure 1, the link between firm 1 and firm 4 are formed, and firms 2, 3, 5 and 6 form the links with each other. On the other hand, in the network in the right of Figure 1, the link between firm 1 and firm 2 are formed, and firms 3, 4, 5 and 6 form the links with each other. In a Cournot or Bertrand oligopoly model with network formation, discussed by Goyal and Joshi (2003) for example, some market outcomes, e.g., the social surplus, of the two networks are completely same. However, this is not the case in our model. This is because, in the right network, firm 1 forms the link with an adjacent firm (firm 2), but in the left network, firm 1 forms the link with a firm (firm 4) that is not adjacent to firm 1. Thus, market outcomes, e.g., social surplus, are different between the two networks. Also, since incentives to form a link may be different, the stability result may also be different between two networks.

We show that a firm's profit function satisfies a condition: if the cost reducing effects of a link between two firms on the firms are same, then they have an incentive to form the link. This condition is also focused on by Goyal and Joshi (2003) and Okumura (2007a,b). Goyal and Joshi (2003) show that the complete network, in which all firms form the links with each other, is pairwise stable under the condition. Pairwise stability is the solution concept introduced by Jackson and Wolinsky (1996). Okumura (2007a) shows a network formation process may converge to the complete network if the number of firms is even under the condition. The formation process is introduced by Watts (2001). See, also Jackson and Watts (2002). Thus, in our model, the complete network is pairwise stable and the formation process may converge to the complete network if the number of firms is even. Okumura (2007b) also gives another model satisfying the condition. His model is a generalization of Goyal and Joshi's (2006) patent race model with network formation.

In addition, we focus on some specific networks: complete groups networks in which the firms are divided into some groups and any links between two firms in a group are formed, but any links between a firm in a group and a firm in another group are not formed. We restrict our attention to the complete groups network in which the number of groups is two. The networks in Figure 1 are such networks. We will completely characterize the pairwise stable complete groups network with two groups. Moreover, we will discuss the network formation process that can converge to some complete groups network.

In this paper, we generalize Salop's (1979) model. Salop (1979) discusses a circular city model, but he assumes that the cost function of each firm is symmetric. See also Economides (1989). No previous study focuses on the equilibrium in the case where the firms have asymmetric production cost functions. On the other hand, since our model includes the cases that the marginal costs of firms are asymmetric, we derive the solution of the model that is a generalization of Salop's (1979) model.

2 Model

We will discuss a two-stage game. At the first stage, bilateral collaborative links are formed. At the second stage, firms engage in price competition. Let the finite set of firms be $N = \{1, 2, \cdots, n\}$. We assume $n \geq 3$. Let a set of N and the collaborative links between two firms in N be a network g. We will write $ij \in g$, indicating that firms i and j are linked in g, while $ij \notin g$ indicates that i and j are not linked in g. In addition, let $g + (-)ij$ be the network obtained from g by adding (severing) a link between i and j to g. The number of links that i forms in g is given by $\eta^i(g)$.

Let $c : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ be a function satisfying $c(m) < c(m-1)$ for all

$m \in \mathbb{Z}_+$. The marginal cost of a firm is assumed to be constant and is given by $c^i = c(\eta^i(g))$ for $i \in N$. That is, firm i's marginal cost depends on the number of firms that form a link with i.

3 Spatial Competition

Here, we will consider the price competition of the firms in a given network g. That is, we will characterize the equilibrium in the second stage of the game. Each firm i chooses the price p^i to maximize their own profit. The profit of firm i is given by $(p^i - c^i)D^i$ where D^i is the demand that i faces.

There is a circular city with perimeter 1. We assume that the firms are located equidistant from one another. Let the address of firm i be $(i-1)/n$. Let the function $d : N \times N \to \mathbb{Z}_+$ be such that

$$d(i,j) = d(j,i) = \min\{j-i, n-j+i\} \text{ for } i,j \in N \text{ and } j \geq i.$$

That is, $d(i,j)$ is the distance between i and j times n. For notational simplicity, we treat $d(i,j)$ as the *distance* between i and j. In addition, let

$$\bar{d}_n = \begin{cases} n/2 \text{ if } n \text{ is even,} \\ (n-1)/2 \text{ if } n \text{ is odd.} \end{cases}$$

That is, for any given n, $\bar{d}_n = \max_{i,j\in N} d(i,j)$.

Consumers are uniformly distributed with density 1 around the city. She/He purchases at most one product from a firm. When a consumer living in $s \in [0,1]$ purchases a product from firm i, she/he incurs the transportation cost $|(i-1)/n - s|\,t$. His/Her utility is given by

$$v - |(i-1)/n - s|\,t - p^i \text{ if he/she purchases a product from } i,$$
$$0 \text{ if he/she purchases nothing.}$$

We will focus on the case where (a) all consumers purchase a product and (b) each firm sells for some consumers living in both its left and right sides. In the equilibrium, (a) is satisfied if v is large enough and (b) is satisfied if t is large enough in any network. Hence, we will assume that v and t are large enough.

A consumer who is indifferent between purchasing from i and $i+1$ lives in $s_i \in [(i-1)/n, i/n]$ satisfying

$$\begin{aligned} v - \left(s_i - \frac{i-1}{n}\right) t - p^i &= v - \left(\frac{i}{n} - s_i\right) t - p^{i+1}, \\ s_i &= \frac{p^{i+1} - p^i}{2t} + \frac{2i-1}{2n} \end{aligned}$$

Thus,

$$D^i = \frac{p^{i+1} - p^i}{2t} + \frac{p^{i-1} - p^i}{2t} + \frac{1}{n}.$$

where $p^0 = p^n$. The first order conditions are, for all $i \in \{1, 2, \cdots, n\}$,

$$p^i - \frac{p^{i-1}}{4} - \frac{p^{i+1}}{4} = z^i \equiv \frac{1}{2}\left(c^i + \frac{t}{n}\right) > 0. \tag{1}$$

A solution of this problem $p^* = (p^{1*}, p^{2*}, \cdots, p^{n*})$ will be derived as follows. For $i = 1, \cdots, n$,

$$p^{i*} = \begin{cases} \alpha_0^n z^i + \sum_{j=1}^{(n-2)/2} \alpha_j^n (z^{i+j} + z^{i-j}) + \alpha_{n/2}^{i-n/2} \text{ if } n \text{ is even}, \\ \alpha_0^n z^i + \sum_{j=1}^{(n-1)/2} \alpha_j^n (z^{i+j} + z^{i-j}) \text{ if } n \text{ is odd}, \end{cases} \tag{2}$$

where $z^{n+j} = z^j$ and $z^{-j} = z^{n-j}$ for $j = 0, 1, \cdots, n/2$. Thus, the equilibrium price of each firm is linearly dependent on z^j for all $j = 1, \cdots, n$ and the effect of z^j on the equilibrium price of i is dependent on the distance between i and j. The effect of z^j on i's price is represented by $\alpha_{d(i,j)}^n$.

Next, we will derive

$$\begin{aligned} \alpha^n &= (\alpha_0^n, \alpha_1^n, \cdots, \alpha_{d_n}^n) \\ &= \begin{cases} (\alpha_0^n, \alpha_1^n, \cdots, \alpha_{n/2}^n) \text{ if } n \text{ is even}, \\ (\alpha_0^n, \alpha_1^n, \cdots, \alpha_{(n-1)/2}^n) \text{ if } n \text{ is odd}. \end{cases} \end{aligned}$$

At first, suppose that n is even. Then, by (1) and (2), α^n satisfies

$$\alpha_0^n - \frac{1}{2}\alpha_1^n = 1, \tag{3}$$

$$-\frac{1}{4}\alpha_j^n + \alpha_{j+1}^n - \frac{1}{4}\alpha_{j+2}^n = 0 \text{ for } j = 0, \cdots, n/2 - 2, \tag{4}$$

$$\alpha_{n/2}^n - \frac{1}{2}\alpha_{n/2-1}^n = 0. \tag{5}$$

The general form of (4) is, for $k = 2, \cdots, n/2$,

$$\alpha_{n/2-k}^n = \frac{(\alpha_{n/2-1}^n - y\alpha_{n/2}^n)x^k}{x - y} - \frac{(\alpha_{n/2-1}^n - x\alpha_{n/2}^n)y^k}{x - y} \tag{6}$$

where x, y are the solutions of $-\frac{1}{4}\alpha^2 + \alpha - \frac{1}{4} = 0$.

By (5) and (6), for $k = 2, \cdots, n/2$,

$$\alpha_{n/2-k}^n = \frac{(2+\sqrt{3})^k + (2-\sqrt{3})^k}{2}\alpha_{n/2}^n. \tag{7}$$

By (3) and (7), $\alpha^n_{n/2}$ satisfies

$$\left(\frac{(2+\sqrt{3})^{n/2}+(2-\sqrt{3})^{n/2}}{2}-\frac{(2+\sqrt{3})^{n/2-1}+(2-\sqrt{3})^{n/2-1}}{4}\right)\alpha^n_{n/2}=1 \tag{8}$$

Therefore, α^n is characterized by (7) and (8) if n is even. Note that, since α^n satisfies (3), (4) and (5), we have

$$\alpha^n_0+2\sum\nolimits_{j=1}^{(n-2)/2}\alpha^n_j+\alpha^n_{n/2}=2. \tag{9}$$

Similarly, if n is odd, then

$$\begin{aligned}\alpha^n_0-\frac{1}{2}\alpha^n_1 &= 1, && (10)\\ -\frac{1}{4}\alpha^n_j+\alpha^n_{j+1}-\frac{1}{4}\alpha^n_{j+2} &= 0 \text{ for } j=0,\cdots,(n-1)/2-2, && (11)\\ \frac{3}{4}\alpha^n_{(n-1)/2-1}-\frac{1}{4}\alpha^n_{(n-1)/2} &= 0. && (12)\end{aligned}$$

By (10), (11) and (12), for $k=2,\cdots,(n-1)/2$

$$\alpha^n_{(n-1)/2-k}=A(k)\alpha^n_{(n-1)/2}, \tag{13}$$

$$\left(A((n+1)/2)-\frac{A((n+1)/2-1)}{2}\right)\alpha^n_{(n-1)/2}=1, \tag{14}$$

$$\text{where } A(k)=\frac{(\sqrt{3}+1)(2+\sqrt{3})^k+(\sqrt{3}-1)(2-\sqrt{3})^k}{2\sqrt{3}},$$

are satisfied. Note that

$$\alpha^n_0+2\sum\nolimits_{j=1}^{(n-1)/2}\alpha^n_j=2. \tag{15}$$

Thus, we have the following solution;

Proposition 1 *The equilibrium price vector p^* is, for each $n\geq 3$ and $i=1,\cdots,n$,*

$$p^{i*}=\begin{cases}\frac{t}{n}+\frac{1}{2}\left[\alpha^n_0c^i+\sum_{j=1}^{(n-2)/2}\alpha^n_j(c^{i+j}+c^{i-j})+\alpha^n_{n/2}c^{i-n/2}\right] & \textit{if } n \textit{ is even,}\\ \frac{t}{n}+\frac{1}{2}\left[\alpha^n_0c^i+\sum_{j=1}^{(n-1)/2}\alpha^n_j(c^{i+j}+c^{i-j})\right] & \textit{if } n \textit{ is odd,}\end{cases} \tag{16}$$

where $c^{n+j}=c^j$, $c^{-j}=c^{n-j}$ for $j=0,1,\cdots,n/2$, and α^n is characterized by (7), (8), (13) and (14).

The parameter α^n_j represents the effect on the equilibrium price of firm a of the marginal cost of firm b where $d(a,b)=j$. We immediately have the following result on α^n:

Corollary 1 *Parameters of the equilibrium price (16) satisfy* $2 > \alpha^n_0 > 1 > \alpha^n_1 > \cdots > \alpha^n_{\bar{d}_n} > 0$ *for all* n.

(Proof) By (7), (8), (13) and (14), we have $\alpha^n_0 > \alpha^n_1 > \cdots > \alpha^n_{\bar{d}_n} > 0$. By (3), (10) and $\alpha^n_1 > 0$, we have $\alpha^n_0 > 1$. Next, by (9), (15) and $\alpha^n_k > 0\ \forall k$, we have $2 > \alpha^n_0$. Finally, by (9) and (15), $\alpha^n_0 > 1$ and $\alpha^n_k > 0\ \forall k$, we have $1 > \alpha^n_1$. (Q.E.D.)

This corollary means that the marginal production costs of all firms affect the price of each firm. In addition, the effect on the price of i of the marginal cost of j is larger than the effect on the price of i of the marginal cost of k if and only if i is located nearer to j than to k.

4 Networks

4.1 Stability and Formation Process

We will analyze the first stage of our model. That is, we will analyze the formation of collaboration networks. Since the price of each firm is dependent on the marginal costs of all firms, the prices depend on the network g. Thus, we will write the equilibrium price vector as $p^*(g) = (p^{1*}(g), p^{2*}(g), \cdots, p^{n*}(g))$. The profit of a firm is also dependent on g. We will denote to the profit function of firm i as

$$\Pi^i(g) = D(p^{i-1*}(g), p^{i*}(g), p^{i+1*}(g))(p^{i*}(g) - c(\eta^i(g)).$$

Since (1) holds, the demand of firm i in g is

$$D(p^{i-1*}(g), p^{i*}(g), p^{i+1*}(g)) = \frac{1}{t}(p^{i*}(g) - c(\eta^i(g)).$$

Therefore,

$$\Pi^i(g) = \frac{1}{t}(p^{i*}(g) - c(\eta^i(g)))^2,$$

and

$$\begin{aligned} & \mathrm{sign}[\Pi^i(g+ij) - \Pi^i(g)] \\ = \; & \mathrm{sign}[p^{i*}(g+ij) - c(\eta^i(g)+1) - \{p^{i*}(g) - c(\eta^i(g))\}]. \qquad (17) \end{aligned}$$

By Proposition 1, we have

$$
\begin{gathered}
p^{i*}(g+ij) - c(\eta^i(g)+1) - [p^{i*}(g) - c(\eta^i(g))] = \\
\left(1 - \frac{1}{2}\alpha_0^n\right)\left(c(\eta^i(g)) - c(\eta^i(g)+1)\right) - \frac{1}{2}\alpha_{d(i,j)}^n\left(c(\eta^j(g)) - c(\eta^j(g)+1)\right) \\
\equiv f(n, \eta^i(g), \eta^j(g), d(i,j)).
\end{gathered} \tag{18}
$$

By (17), we have

$$\mathrm{sign}[f(n, \eta^i(g), \eta^j(g), d(i,j))] = \mathrm{sign}[\Pi^i(g+ij) - \Pi^i(g)].$$

That is, i wants to add the link with j if and only if $f(n, \eta^i(g), \eta^j(g), d(i,j)) > 0$. Also, i wants to sever the link with j if and only if $f(n, \eta^i(g-ij), \eta^j(g-ij), d(i,j)) < 0$.

Note the following results;

Lemma 1 *$f(n, \eta^i(g), \eta^j(g), m) > f(n, \eta^i(g), \eta^j(g), m-1)$ for all $m = 2, \cdots, n/2((n-1)/2)$.*

This Lemma is obvious from (18) and Corollary 1. Consider three firms i, l and m. Suppose that $d(i,l) < d(i,m)$ and $\eta^l(g) = \eta^m(g)$. Lemma 1 implies that, if firm i wants to add il in g, then i wants to add im. In addition, if i does not want to add im in g, then i does not want to add il in g.

Lemma 2 *Consider two firms $i, j \in N$. If $c(\eta^i(g)) - c(\eta^i(g)+1) = c(\eta^j(g)) - c(\eta^j(g)+1)$, then $f(n, \eta^i(g), \eta^j(g), d(i,j)) > 0$ and $f(n, \eta^j(g), \eta^i(g), d(i,j)) > 0$.*

(Proof) Suppose that $c(\eta^i(g)) - c(\eta^i(g)+1) = c(\eta^j(g)) - c(\eta^j(g)+1) = \Delta$. Then, by (18),

$$
\begin{aligned}
f(n, \eta^i(g), \eta^j(g), d(i,j)) &> 0 \Leftrightarrow \\
\left(1 - \frac{1}{2}\alpha_0^n - \frac{1}{2}\alpha_{d(i,j)}^n\right)\Delta &> 0.
\end{aligned}
$$

By (3), (12) and Corollary 1,

$$1 - \frac{1}{2}\alpha_0^n - \frac{1}{2}\alpha_{d(i,j)}^n > 0 \text{ for all } d(i,j) = 1, 2, \cdots, \overline{d}_n.$$

Thus, we have Lemma 2. (Q.E.D.)

Thus, in the case that the effects of the link between i and j on the marginal costs of i and j are same, then the link ij will be added if $ij \notin g$ and will not be severed if $ij \in g$.

We will discuss the stability of networks and the network formation process of networks. We will consider the following two concepts.

The network g is said to be *pairwise stable*, if the following two conditions are satisfied: (a) for all firms i, j $(i > j)$ such that $ij \in g$, $f(n, \eta^i(g - ij), \eta^j(g - ij), d(i, j)) > 0$ and $f(n, \eta^j(g - ij), \eta^i(g - ij), d(i, j)) \geq 0$ and, (b) for all firms i, j $(i > j)$ such that $ij \notin g$, if $f(n, \eta^i(g), \eta^j(g), d(i, j)) > 0$, then $f(n, \eta^j(g), \eta^i(g), d(i, j)) < 0$.

Next, we will consider the dynamic network formation process introduced by Watts (2001). This is a natural extension of pairwise stability. We consider a discrete set of points in time $\{1, 2, \cdots, t, \cdots\}$. The network formation starts from the empty network, that is, $g_1 = g^E$. At each time a pair of firms is randomly identified with positive probability. The network at the end of period t is given by g_t. Suppose that firms i and j are identified at $t + 1$ and $ij \in g_t$. If either $f(n, \eta^i(g_t - ij), \eta^j(g_t - ij), d(i, j)) < 0$ or $f(n, \eta^j(g_t - ij), \eta^i(g_t - ij), d(i, j)) < 0$, then $g_{t+1} = g_t - ij$. If otherwise, $g_{t+1} = g_t$. Suppose that firms l and m are identified at $t + 1$ and $lm \notin g_t$. If both $f(n, \eta^l(g_t), \eta^m(g_t), d(l, m)) > 0$ and $f(n, \eta^m(g_t), \eta^l(g_t), d(l, m)) > 0$, then $g_{t+1} = g_t + lm$. If otherwise, $g_{t+1} = g_t$. If in $g_T = \bar{g}$ no pair of firms adds or severs the link, that is, $\bar{g} = g_T = g_{T+1} = g_{T+2} = \cdots$, then this dynamic process converges to $\bar{g}$.

It is obvious that the formation process converges to only a pairwise stable network.[1] However, this process may converge to some pairwise stable network with probability zero. Kawamata (2004) and Okumura (2007a,b) give some examples.

4.1.1 Complete network

Here, we will focus on the complete network. By Lemma 1, we have the following result:

Proposition 2 *The complete network is pairwise stable. Moreover, if n is even, the network formation process converges to the complete network with positive probability.*

[1] The fomation process may converge to no network. In that case, the formation process will be in a closed cycle. See, Jackson and Watts (2002) on this point.

The proof of this result is direct from Lemma 1, Goyal and Joshi (2003, Theorem 3.1) and Okumura (2007a, Theorem 1). This is because they consider a class of games satisfying the condition that the links between two players that have same number of links will be added (will not be severed). Lemma 1 implies that our model also satisfies this condition. Thus, we have Proposition 2.

4.1.2 Complete groups networks

We will focus on some other networks. The stability result of some networks are dependent on the function $c(\cdot)$. But, there are some networks that are not pairwise stable for any $c(\cdot)$. That is, if there are two firms i and j such that $ij \notin g$ and $\eta^i(g) = \eta^j(g)$, then g is not pairwise stable. This is because, by Lemma 1, i and j have an incentive to add ij. We will focus on some special networks in which firms are divided into some groups and each firm in a group form links with the firms in the group but form no link with the firms in the other groups. We will denote to the networks as *the complete groups networks*. Obviously, by Lemma 2, a complete groups network is not pairwise stable if there are two groups with same size. On the other hand, if the number of firms in each group is different from each other, then the network can be pairwise stable. This is because for any two firms i and j such that $ij \notin g$ and $\eta^i(g) \neq \eta^j(g)$. Moreover, we will restrict our attention to the complete groups networks where the number of groups are two. The two groups are given by $X = \{x_1, x_2, \cdots, x_k\} \neq \emptyset$ and $Y = \{y_1, y_2, \cdots, y_{n-k}\} \neq \emptyset$, where $(x_1, x_2, \cdots, x_k, y_1, y_2, \cdots, y_{n-k})$ is a permutation of N. Let $g(n,k)$ denote to a complete groups network where the firms divided into X and Y.

Lemma 3 *Consider a complete groups network $g(n,k)$. There exists a pair x, y such that $x \in X$, $y \in Y$ and $d(x,y) = \overline{d}_n$ in $g(n,k)$ if n or k is odd.*

(Proof) If n is even, then each firm i has unique firm j such that $d(i,j) = \overline{d}_n = n/2$ in $g(n,k)$. Thus, if n is even and k is odd, then there is at least pair of firms $x \in X$ and $y \in Y$ such that $d(x,y) = n/2 = \overline{d}_n$ in $g(n,k)$.

Next, if n is odd, then each firm i has just two firms j and $j+1$ such that $d(i,j) = d(i,j+1) = (n-1)/2 = \overline{d}_n$ in $g(n,k)$. Since neither X nor Y is empty, there is $j \in X$ such that $j+1 \in Y$. Since i is in either X or Y, then there is at least one pair of firms $x \in X$ and $y \in Y$ such that $d(x,y) = (n-1)/2 = \overline{d}_n$. (Q.E.D)

This Lemma implies that, when the firms are divided into two groups, there is a pair of firms $x \in X$ and $y \in Y$ where x is the firm located at farthest point from y if n or k is odd.

Lemma 4 *Suppose that n is even. (i) Then there is $g(n,k)$ which includes no pair x,y such that $x \in X$, $y \in Y$ and $d(x,y) = \overline{d}_n$ if and only if k is even. (ii) Moreover, in the network, there is a pair x,y such that $x \in X$, $y \in Y$ and $d(x,y) = \overline{d}_n - 1$.*

(Proof) The necessity part of (i) is immediate from Lemma 3. We will show the sufficiency part. Consider a complete groups network $g(n,k)$ where a group of firms $X = (x_1, x_2, \cdots, x_k)$ is such that $d(x_i, x_{i+1}) = n/2 = \overline{d}_n$ for all $i = 1, 3, 5, \cdots, k-1$ where k is even. In $g(n,k)$, for any $x \in X$, there is no $y \in Y$ such that $d(x,y) = n/2 = \overline{d}_n$. Thus, we have the sufficiency part of (i). Next, we will show (ii). Since neither X nor Y is empty, there is $j \in X$ such that $j+1 \in Y$. Therefore, there is $i \in X$, such that $d(i, j+1) = (n-2)/2 = \overline{d}_n - 1$. (Q.E.D)

By using Lemma 3 and 4, we will completely characterize the stability condition of the complete groups network. The condition depends on the number of the firms n and the size of a group k. We have the following two results:

Proposition 3 *Consider a complete groups network $g(n,k)$. Suppose that $k \neq n-k$, and n or k is odd. Then, the network is pairwise stable if and only if*

$$\begin{aligned} f(n, k-1, n-k-1, \overline{d}_n) &< 0 \textit{ or} \\ f(n, n-k-1, k-1, \overline{d}_n) &< 0. \end{aligned}$$

Proposition 4 *Consider a complete groups network $g(n,k)$ where $k \neq n-k$, and n and k are even. (iii) If $X = (x_1, x_2, \cdots, x_k)$ is such that $d(x_i, x_{i+1}) = \overline{d}_n$ for all $i = 1, 3, 5, \cdots, k-1$, then $g(n,k)$ is pairwise stable if and only if $f(n, k-1, n-k-1, \overline{d}_n - 1) < 0$ or $f(n, n-k-1, k-1, \overline{d}_n - 1) < 0$. (iv) If otherwise, then $g(n,k)$ is pairwise stable if and only if $f(n, k-1, n-k-1, \overline{d}_n) < 0$ or $f(n, n-k-1, k-1, \overline{d}_n) < 0$.*

(Proof of Propositions 3 and 4) Consider a complete groups network $g(n,k)$ where $k \neq n-k$. We will check the firms have no incentive to sever any link in $g(n,k)$. Since $\eta^x(g(n,k)) = k-1$ and $\eta^y(g(n,k)) = n-k-1$ for any $x \in X$, $y \in Y$, every firm has no incentive to sever any existing link in $g(n,k)$ because of Lemma 2.

Thus, we will characterize the condition that any link between $x \in X$ and $y \in Y$ will not be added in $g(n,k)$. Let a pair of firms be $x' \in X$ and $y' \in Y$ where $d(x',y') := d' \geq d(x,y)$ for all pairs $x \in X$ and $y \in Y$ in $g(n,k)$. Note that $d' \leq \bar{d}_n$. By Lemma 1, the necessary and sufficient condition of pairwise stability of $g(n,k)$ is that either x' or y' has no incentive to add the link between $\bar{x}$ and $\bar{y}$: $f(n,k-1,n-k-1,\bar{d}) < 0$ or $f(n,n-k-1,k-1,\bar{d}) < 0$.

In Lemma 3 and 4, we have already identified $\bar{d}$ in $g(n,k)$ for all n and k. Thus, we have Proposition 3 and 4. (Q.E.D)

Consider two different complete groups networks where n and k are same. By Proposition 3, the sufficient and necessary conditions for pairwise stability of the two complete groups networks are same if n or k is odd. That is, the condition is that a firm x in a group has no incentive to add link with a firm y which is in the other group and is located farthest from x. However, by Proposition 4, the sufficient and necessary conditions for pairwise stability may be different between the two networks if n and k is even. We will give an example in the subsequent section.

Finally, we focus on the network formation process.

Corollary 2 *If $g(n,k)$ is pairwise stable and both n and k is even, then the network formation process converges to $g(n,k)$ with positive probability.*

This result is immediate from a result of Okumura (2007, Theorem 1). Though the result is on the complete network, it can be applied for complete groups networks. Thus, in the case where the size of each complete groups is even, the network formation can converge to a complete groups network if it is pairwise stable. The condition of pairwise stability of the networks is given in Proposition 4.

4.2 Socially Efficient Network

Next, we will analyze the socially efficient network, in which the social surplus is higher than in any other networks.

Proposition 5 *The complete network is the uniquely socially efficient network.*

(Proof) In the second stage of this model, we focus on the case where all consumers purchase a product in any network. Thus, in equilibrium, the social surplus will be

$v-$(sum of the total production costs)$-$(sum of the total transportation costs).

In the complete network, the marginal production cost of all firms are $c(n-1)$. Thus, the sum of total production costs in the complete network is smaller than that in any other network. Moreover, by Proposition 1, the prices of all firms are equal to $t/n + c(n-1)$. Thus, all consumers purchase from the nearest firm, because the prices of all firms are same. Therefore, the sum of total transportation costs in the complete network is smaller than or equal to that in any other network. Hence, the complete network is the uniquely socially stable network. (Q.E.D)

By Proposition 2 and 5, the uniquely socially efficient network is pairwise stable and the formation process converges to the efficient network with positive probability if n is even.

5 Six-firm Example

In this section, we will discuss the case of $n = 6$. We focus on a complete groups network with two groups: $g(6, 2)$. Let us derive the stability condition of each $g(6, 2)$. At first, consider g^A in Figure 2. The network g^A is pairwise stable if and only if either (a) firm 1 (or 4) has no incentive to add the link with 2 (or 3 or 4 or 5); namely $f(6, 1, 3, 2) < 0$, or (b) firm 2 (or 3 or 4 or 5) has no incentive to add the link with 1 (or 4); namely $f(6, 3, 1, 2) < 0$. Since

$$\alpha_0^6 = \frac{52}{45},\ \alpha_1^6 = \frac{14}{45},\ \alpha_2^6 = \frac{4}{45},\ \alpha_3^6 = \frac{2}{45},$$

by (7) and (8), the condition is

$$\begin{aligned} f(6,1,3,2) &= \frac{19}{45}(c(1)-c(2)) - \frac{2}{45}(c(3)-c(4)) < 0 \text{ or} \\ f(6,3,1,2) &= \frac{19}{45}(c(3)-c(4)) - \frac{2}{45}(c(1)-c(2)) < 0, \\ &\Leftrightarrow \frac{19}{2} < \frac{c(3)-c(4)}{c(1)-c(2)} \text{ or } \frac{2}{19} > \frac{c(3)-c(4)}{c(1)-c(2)}. \end{aligned} \tag{19}$$

On the other hand, the stability condition of g^B is that the link between

1 and 4 (or 3 and 6) will not be formed. That is, the condition is

$$\begin{aligned} f(6,1,3,3) &= \frac{19}{45}(c(1)-c(2)) - \frac{1}{45}(c(3)-c(4)) < 0 \text{ or} \\ f(6,3,1,3) &= \frac{19}{45}(c(3)-c(4)) - \frac{1}{45}(c(1)-c(2)) < 0, \\ &\Leftrightarrow \; 19 < \frac{c(3)-c(4)}{c(1)-c(2)} \text{ or } \frac{1}{19} > \frac{c(3)-c(4)}{c(1)-c(2)}. \end{aligned} \tag{20}$$

Similarly, the stability condition of g^C is that (20) is satisfied.

Note that, by Proposition 4, the stability condition of each $g(6,2)$ is either that (19) is satisfied or that (20) is satisfied.

Obviously, the stability condition of g^A is weaker than those of g^B and g^C.

Next, we will focus on the social surplus of each network: g^A, g^B, g^C. Let the social surplus of a network g be $W(g)$. Then, we have

$$W(g^A) > W(g^B) > W(g^C).$$

The welfare of g^A is the largest of the three networks. This is because the quantity of the firms that have $c(1)$ in g^A is smaller than those in other networks. That is, since the firms located near a firm with $c(1)$ have $c(3)$ in g^A, many consumers will purchase from the firms with $c(3)$.[2] Similarly, we have the intuition of $W(g^B) > W(g^C)$.

Note that the social surplus of each $g(6,2)$ is equal to either $W(g^A)$, $W(g^B)$ or $W(g^C)$. Thus, g^A has the weakest condition on pairwise stability and is the most efficient network among the complete groups networks such that one group consists of two firms and the other group consists of four firms.

6 Concluding Remarks

In this paper, we mainly focus on the complete network and the complete groups networks with two groups. There are some other networks that may be pairwise stable. For example, some complete groups networks with more than 3 groups can be pairwise stable. The networks in Figure 3 are complete groups networks with three groups and can be pairwise stable.

[2] Of course, there is a loss of the transportation costs of the consumers in each network. The loss of the transportation costs are largest in g^A. But, we can ignore the loss, because the loss is quite small compared with the loss from high production costs.

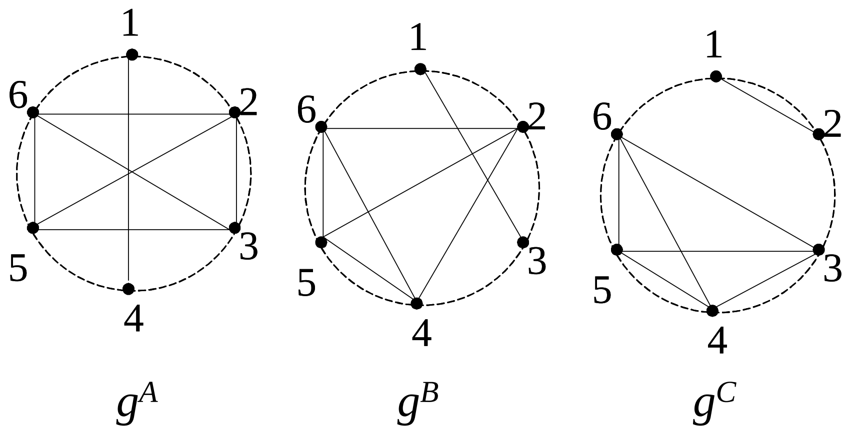

Figure 2: Three Complete Groups Networks with Two Groups

The stability conditions of the networks are different from each other. The characterization of the complete groups networks with more than 3 groups is a remaining problem.

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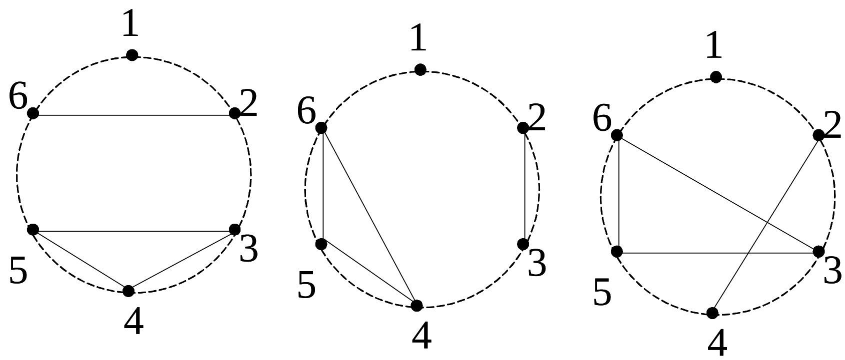

Figure 3: Three Complete Groups Networks with three Groups

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