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The anti-dual of economic coalitional TU games

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要旨

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The Anti-Dual of Economic Coalitional TU Games

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Abstract

Some of well known coalitional TU games applied to specific economic problems are shown to be connected through the relation defined as the *anti-dual*. Solutions such as the core, the Shapley value and the nucleolus of anti-dual games are obtained straightforwardly from original games.

Keywords: TU coalitional games, anti-dual, convex games, prenucleolus
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1 Introduction

Coalitional TU games have been widely applied to socio-economic problems ranging from practical to abstract ones. In these applications, the so-called characteristic function describes the value obtainable by each of possible coalitions under the socio-economic rules or systems being analyzed. Each of these games is thus a model specific to the problem under consideration, having little to do with each other in general.

We show, however, that there is a hidden relationship between games in a class appeared in the literature as specific applied TU coalitional games. We call the relationship the *anti-dual*, and present examples of games and their anti-duals both of which are well known applied economic games. This implies that coalitional games separately applied to specific problems can be still connected in an unexpected, simple way through the anti-dual relation. A typical example will be the airport game (Littlechild (1974)) in a simple case vs. the bidder collusion game (Graham et al. (1990)); namely, the bidder collusion game defined for analysing a collusive bidding behavior in the English auction is the anti-dual of the airport game imposing the cost of construction of a runway to a group of aircrafts which

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are going to use the runway. These two games were well analyzed separately without any interaction, and solutions such as the nucleolus and the Shapley value have been computed. But, these solutions of the anti-dual games are easily shown to be obtained by just reversing the sign of those of the original games. This fact may considerably facilitate computation of some solutions once the game under consideration is found to be the anti-dual of a game already known or studied.

In this note, we shall initially define the *anti-dual* of a TU coalitional game to be the *dual* of the game with opposite sign. We then show several examples of well known economic anti-dual games, including a new game that is obtained as the anti-dual of the big boss game (Muto et al. (1988)). Finally, we show some of easy general properties of the anti-dual game : the convexity is inherited, and the core as well as the nucleolus and the Shapley value is immediately obtained by just having the sign reversed.

2 The anti-dual games

Let $N = \{1, \ldots, n\}$ be a finite set, and let v be a real valued function on 2^N satisfying $v(\emptyset) = 0$. Then, the pair (N, v) will be called a TU *coalitional game* or simply a *game*, where N is the set of players and $v(S)$ is the *worth* that coalition S can obtain by itself. A game (N, v) will be abbreviated to v.

Given a game v, the *dual* of v is the game v^* such that $v^*(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$. In this paper, we define the *anti − dual* of v to be the dual of $-v$; that is, to be the game $(-v)^*$ given by

$$(-v)^*(S) = (-v)(N) - (-v)(N \setminus S) = -v(N) + v(N \setminus S), \quad \forall S \subseteq N.$$

Before presenting examples of anti-dual economic games, we first note a straightforward general property of anti-duals in the following lemma.

Lemma 1 *Let v be any game and let a be any additive game defined by $a(S) = \sum_{i \in S} a_i$ for all $S \subseteq N$. Then,* $\left(-((-v)^* + a)\right)^* = v - a$.

Proof. Let $v' = (-v)^* + a$. Then, for any $S \subseteq N$ we have that

$$\begin{aligned}
(-v')^*(S) &= -v'(N) + v'(N \setminus S) \\
&= -\Big((-v)^* + a\Big)(N) + \Big((-v)^* + a\Big)(N \setminus S) \\
&= -\Big(-v(N) + a(N)\Big) + \Big(-v(N) + v(N \setminus (N \setminus S)) + a(N \setminus S)\Big) \\
&= v(S) - a(S).
\end{aligned}$$

□

In the special case where $a \equiv 0$, that is, where $a_i = 0$ for all $i \in N$, we see that for any game v the anti-dual of $(-v)^*$ is v itself.

2.1 The airport game and the bidder collusion game

Let (N, v_A) be a simple version of a game called the *airport game* due to Littlechild (1974) with one aircraft in each type, that is,

$$v_A(S) = -\max_{i \in S} c_i, \quad \forall S \subseteq N,$$

where c_i is the construction cost of the runway for type i aircraft. We assume that the cost is increasing in the length of the runway, so that

$$c_1 > c_2 > \cdots > c_n > 0.$$

In this simple version, the problem is how to impute the cost share to each type of aircraft in N if the longest runway is to be constructed. Thus, v_A describes the situation in which every coalition, if it is to construct a runway, must construct the longest one of all types of aircrafts in the coalition.

Next, consider the following game (N, v_C) given by

$$v_C(S) = \begin{cases} c_1 - \max_{i \in N \setminus S} c_i & \text{if } 1 \in S, \\ 0 & \text{if } 1 \notin S, \end{cases}$$

where we put $\max_{i \in N \setminus N} c_i \equiv 0$ for convenience. This game was first defined by Graham et al. (1990) to investigate a *bidder collusion* in the *English auction* under complete information. Namely, c_i is here the bidder i 's evaluation of the item to be auctioned, so that any coalition including the bidder 1 can win the auction with net surplus $c_1 - \max_{i \in N \setminus S} c_i$ by making bidder 1 the sole bidder in the coalition. We may call this game v_C the *bidder collusion game*, or simply the *collusion game*. Note that the reservation price of the seller is assumed to be zero.

These two games, apparently having little to do with each other, are the anti-duals each other.

Proposition 1 *The anti-dual of a collusion game v_C is an airport game v_A, and conversely; that is,*

$$(-v_C)^* = v_A \quad \textit{and} \quad (-v_A)^* = v_C.$$

Proof. The anti-dual of a collusion game v_C is given by

$$\begin{aligned} (-v_C)^*(S) &= -v_C(N) + v_C(N \setminus S) \\ &= \begin{cases} -c_1 + c_1 - \max_{i \in S} c_i & = -\max_{i \in S} c_i, \quad \text{if } 1 \notin S, \\ -c_1 & = -\max_{i \in S} c_i, \quad \text{if } 1 \in S, \end{cases} \end{aligned}$$

that is,

$$(-v_C)^*(S) = -\max_{i \in S} c_i, \quad \forall S \subseteq N.$$

The converse follows from Lemma 1 by taking $a \equiv 0$.

□

2.2 The bankruptcy game

Another example is concerned with the well known *bankruptcy game* defined by

$$v_B(S) = \max\left(0,\ E - \sum_{i\in N\setminus S} d_i\right),\ \ \forall S \subseteq N.$$

Here, $E \geq 0$ is the estate to be divided among the creditors, and $d_i \geq 0$ is the debt to creditor $i \in N$ satisfying $E \leq \sum_{i\in N} d_i$. The value $v_B(S)$ expresses the amount that the creditors in S can guarantee themselves to obtain.

On the other hand, consider the following game that may be called the *public good game*:

$$v_P(S) = \max\left(0,\ \sum_{i\in S} d_i - E\right),\ \ \forall S \subseteq N.$$

The public good game v_P describes the situation where a fixed size of a public good can be provided at cost E by any coalition S whenever the net benefit $\sum_{i\in S} d_i - E$ to the coalition is nonnegative.

Proposition 2 *Let v_B and v_P be the bankruptcy game and the public good game, respectively, and let d be the additive game given by*

$$d(S) = \sum_{i\in S} d_i, \quad \forall S \subseteq N.$$

Then,

$$(-v_B)^* = v_P - d \quad \textit{and} \quad (-v_P)^* = v_B - d.$$

Proof. We first note that

$$v_B(S) = v_P(N \setminus S) - d(N \setminus S) + E,\ \ \forall S \subseteq N.$$

This follows from

$$\begin{aligned} v_B(S) &= \max(d(N \setminus S),\ E) - d(N \setminus S) \\ &= \max(0,\ d(N \setminus S) - E) + E - d(N \setminus S) \\ &= v_P(N \setminus S) + E - d(N \setminus S). \end{aligned}$$

Then, the anti-dual $(-v_B)^*$ of v_B is given by

$$\begin{aligned} (-v_B)^*(S) &= -[v_B(N) - v_B(N \setminus S)] \\ &= -[E - v_P(S) - E + d(S)] \\ &= v_P(S) - d(S) \\ &= (v_P - d)(S). \end{aligned}$$

The converse follows from Lemma 1 with $a = d$; namely, we have

$$(-v_P)^* = (-((-v_B)^* + d))^* = v_B - d.$$

□

Since a game $v - d$ is *strategically equivalent* to a game v, the anti-dual of a bankruptcy game is essentially a public good game, and conversely.

Note also that the game v_P is a bankruptcy game with the estate E being replaced by $d(N)-E$, the total deficit to be shared by the creditors (see e.g., Driessen 1993, p.163). Hence, it can be said also that the anti-dual of the bankruptcy game v_B is itself a bankruptcy game, assigning to each coalition S the total deficit to be shared in S.

2.3 The glove game

We have yet another example of well known games with anti-duals being essentially the same games. The so-called *glove game* is a game (N, v_G) with $N = L \cup R$, $L \cap R \neq \emptyset$ and $L, R \neq \emptyset$ such that

$$v_G(S) = \min\Big\{ |S \cap L|,\ |S \cap R| \Big\} \quad \forall S \subseteq N.$$

Members of L and R are, respectively, the owners of one left-hand glove, and the owners of one right-hand glove. The market price of a pair of gloves is assumed one. Then $v_G(S)$ describes the worth of pairs of gloves in S.

Proposition 3 *Let v_G be a glove game with $|L| = |R|$, and let m be the additive game given by*

$$m(S) = |S| \quad \forall S \subseteq N.$$

Then,

$$(-v_G)^* = v_G - m.$$

Proof. Assume $|L| = |R| = k$. For any $S \subseteq N$, we have that

$$\begin{aligned}
(-v_G)^*(S) + m(S) &= -v_G(N) + v_G(N \setminus S) + |S| \\
&= -k + \min\Big\{ |(N \setminus S) \cap L|,\ |(N \setminus S) \cap R| \Big\} + |S| \\
&= -k + \min\Big\{ k - |S \cap L|,\ k - |S \cap R| \Big\} + |S| \\
&= \min\Big\{ |S| - |S \cap L|,\ |S| - |S \cap R| \Big\} \\
&= \min\Big\{ |S \cap R|,\ |S \cap L| \Big\} \\
&= v_G(S).
\end{aligned}$$

□

Thus, the anti-dual of a square glove game v is essentially the square glove game v itself. In fact, since

$$m(S) = -\sum_{i \in S} (-v_G)^*(\{i\}) \quad \forall S \subseteq N,$$

the zero-normalized anti-dual $(-v_G)^* + m$ of v_G is v_G itself. In the recent literature, several games can be found to have the same property. For example, the Böhm-Bawerk horse market game with the Shapley value in its core is equivalent to a square glove market game (Núñez and Rafels, 2005); and the assignment game with a stable core (Solymosi and Raghavan, 2001) can be shown to extend the above result (Oishi, 2006b).

2.4 The big boss game

Our final example is somewhat an abstract game called the *big boss game* (Muto et al. (1988)). Let (N, v_{BB}) be a *monotonic* game satisfying

1. $v_{BB}(S) = 0 \quad \text{if} \quad 1 \notin S$
2. $v_{BB}(N) - v_{BB}\Big(N \setminus (N \setminus S)\Big) \geq \sum_{i \in N \setminus S} \Big(v_{BB}(N) - v_{BB}(N \setminus \{i\})\Big) \quad \text{if} \quad 1 \in S.$

Note that by the monotonicity, $v_{BB}(S) \geq 0$ for all $S \subseteq N$, and that the contribution $v_{BB}(N) - v_{BB}(N \setminus R)$ of coalition R to the grand coalition is nonnegative for all $R \subseteq N$.

Player 1 is the *big boss*, since any coalition without player 1 can obtain nothing. The second condition simply states that players other than the big boss may increase their contribution to the grand coalition by forming a coalition. For details on the solution structures and economic examples of the big boss game, the reader may refer to Muto et al. (1988).

We now identify the anti-dual of the big boss game.

Proposition 4 *The anti-dual $(-v_{BB})^*$ of the big boss game v_{BB} satisfies the following conditions.*

1. $(-v_{BB})^*(S) = (-v_{BB})^*(N) \quad \textit{if} \quad 1 \in S$
2. $(-v_{BB})^*(S) \leq \sum_{i \in S} (-v_{BB})^*(\{i\}) \quad \textit{if} \quad 1 \notin S$

Proof. Let $1 \in S$. Then, since $1 \notin N \setminus S$, we have

$$(-v_{BB})^*(S) = -v_{BB}(N) + v_{BB}(N \setminus S) = -v_{BB}(N) = (-v_{BB})^*(N).$$

Suppose next that $1 \notin S$. Then, $1 \in N \setminus S$ so that

$$\begin{aligned}(-v_{BB})^*(S) &= -\Big(v_{BB}(N) - v_{BB}(N \setminus S)\Big) \\ &\leq -\left(\sum_{i \in S} \Big(v_{BB}(N) - v_{BB}(N \setminus \{i\})\Big)\right) \\ &= \sum_{i \in S} \Big(-v_{BB}(N) + v_{BB}(N \setminus \{i\})\Big) = \sum_{i \in S} (-v_{BB})^*(\{i\}).\end{aligned}$$

□

Thus the anti-dual of the big boss game is a game v_D such that

$$v_D(S) \begin{cases} = & v_D(N) \quad \text{if} \quad 1 \in S \\ \leq & \sum_{i \in S} v_D(\{i\}) \quad \text{if} \quad 1 \notin S. \end{cases}$$

Note that $v_D(N) = -v_{BB}(N) \leq 0$, and that $v_D(\{i\}) = -v_{BB}(N) + v_{BB}(N \setminus \{i\}) \leq 0$ for all $i \in N \setminus \{1\}$.

This game also describes the unusual power of the big boss. Player 1 is the only player who plays a crucial role in any coalition; and, no other players wish to form coalitions without player 1. The power of player 1 may be even more explicitly seen if we normalize v_D so that $v_D(S) = r > 0$ for all $S \subseteq N$ with $1 \in S$. In this sense, the player 1 of the game v_D is in a similar position to that of the *dictator*, and the game v_D might be called the *dictator game*. In fact, if $v_D(S) = 0$ for all $S \subseteq N \setminus \{1\}$, which is the case if and only if $v_{BB}(S) = v_{BB}(N)$ for all $S \subseteq N$ with $1 \in S$, each of the games v_D and v_{BB} reduces to a game that is strategically equivalent to the *dictatorial simple game*

$$v(S) = \begin{cases} 1 & \text{if} \quad 1 \in S \\ 0 & \text{if} \quad 1 \notin S, \end{cases}$$

which, being an inessential game, describes the situation that every player other than 1 is completely powerless.

3 Convexity

It is well known that the bankruptcy game is convex. The public good game is convex too, since it is a bankruptcy game. The airport game and the collusion game are also known to be convex. We now show that the convexity of a game is inherited to its anti-dual.

A game (N, v) is said to be *convex* if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T), \quad \forall S, T \subseteq N.$$

Theorem 1 *The anti-dual $(-v)^*$ of v is convex iff v is convex.*

Proof. Let $S, T \subseteq N$ and assume that v is convex. Then,

$$\begin{aligned} (-v)^*(S) + (-v)^*(T) &= -[v(N) - v(N \setminus S)] - [v(N) - v(N \setminus T)] \\ &= v(N \setminus S) + v(N \setminus T) - 2v(N) \\ &= v(S^c) + v(T^c) - 2v(N) \\ &\leq v(S^c \cup T^c) + v(S^c \cap T^c) - 2v(N) \\ &= v(N \setminus (S \cap T)) + v(N \setminus (S \cup T)) - 2v(N) \\ &= (-v)^*(S \cup T) + (-v)^*(S \cap T) \end{aligned}$$

The converse follows from Lemma 1 by taking $a \equiv 0$.

□

4 Core, nucleolus and Shapley value

In the literarture, one-point solutions such as the nucleolus (Schmeidler, 1969) and the Shapley value have been applied to this class of games. Littlechild (1974) gave a recursive formula for the nucleolus of an airport game, and Littlechild and Owen (1973) computed the Shapley value. Graham et al.(1990) computed the Shapley value of a collusion game, and interpreted the payoff as one accumulated through nested knockout. Oishi (2006a) also proved that the nucleolus of the collusion game is given by a recursive formula. As for the bankruptcy game, O'Neill (1982) identified the Shapley value. Aumann and Maschler (1985) showed that the nucleolus is identical to the CG-consistent solution of the bankruptcy problem, which can be computed explicitly. In Muto et al. (1988), solutions including the core, the nucleolus and the Shapley value of the big boss game are studied in detail. As we see below, these solutions for the anti-dual can be obtained by just reversing the sign of the corresponding solutions for the original game.

An n-vector $x = (x_1, \ldots, x_n)$ of a game (N, v) is a *payoff vector* if it satisfies that $\sum_{i \in N} x_i = v(N)$. Then the core of a game v is a set of payoff vectors x satisfying $\sum_{i \in S} x_i \geq v(S)$ for all $S \subseteq N$.

The Shapley value $\phi(v) = (\phi_1(v), \ldots, \phi_n(v))$ of a game v is a payoff vector given by the formula

$$\phi_i(v) = \sum_{S \subseteq N,\ i \notin S} \frac{|S|!(n - |S| - 1)!}{n!}(v(S \cup \{i\}) - v(S)), \quad \forall i \in N.$$

To define the nucleolus, for each $S \subseteq N$ let $e(S, x) = v(S) - \sum_{i \in S} x_i$ be the *excess* of a coalition S against a payoff vector x. Let $\theta(x) = (\theta_1(x), \ldots, \theta_{2^n}(x))$ be the 2^n-dimensional vector the components of which are the excesses $e(S, x)$ arranged in the nonincreasing order. Then the *prenucleolus* of a game v is a payoff vector x for which the vector $\theta(x)$ is *lexicographically minimal* for all payoff vectors y; that is, a payoff vector x is a prenucleolus if there exists no payoff vector y and k with $1 \leq k \leq 2^n$ such that

$$\theta_j(y) = \theta_j(x), \ \ \forall j \leq k - 1; \ \text{ and } \ \theta_k(y) < \theta_k(x).$$

A prenucleolus is called the *nucleolus* if it is an *imputation*; that is, if the prenucleolus x satisfies the *individual rationality* that $x_i \geq v(\{i\})$ for all $i \in N$. Given a game (N, v), it is well known that there exists a unique prenucleolus; and a unique nucleolus if the game has an imputation (see e.g., Peleg and Sudhölter, 2003). It is also well known that for games with nonempty cores such as convex games, for example, the unique prenucleolus is the nucleolus that is necessarily contained in the core.

Theorem 2

1. *Payoff vector $-x$ is in the core of $(-v)^*$ iff x is in the core of v.*

2. *Let $\mu(v)$ be the prenucleolus of v. Then, $\mu((-v)^*) = -\mu(v)$.*

3. *Let $\phi(v)$ be the Shapley value of v. Then, $\phi((-v)^*) = -\phi(v)$.*

Proof. Given a payoff vector x of v, we have for each $S \subseteq N$,

$$\begin{aligned} v(S) - x(S) &= v(N) + (-v)^*(N \setminus S) - x(S) \\ &= (-v)^*(N \setminus S) - (-x(N \setminus S)). \end{aligned}$$

Since $-x$ is a payoff vector of $(-v)^*$ iff x is a payoff vector of v, 1 and 2 immediately follow.

To show 3, we first note a well known property of the Shapley value ϕ that $\phi(v) = \phi(v^*)$ where v^* is the dual of v (see, e.g., Peleg and Sudhölter, 2003). Then, since $\phi(-v) = -\phi(v)$, it follows that

$$\phi((-v)^*) = \phi(-v) = -\phi(v).$$

□

Thus, these solutions of the anti-dual can be obtained immediately from the original. For example, the nucleolus and the Shapley value of the public good game v_P can be obtained from those of the bankruptcy game v_B in the following way:

$$\begin{aligned} \mu(v_P) &= \mu(v_P - d) + \mu(d) = \mu((-v_B)^*) + \mu(d) = -\mu(v_B) + \mu(d), \\ \phi(v_P) &= \phi(v_P - d) + \phi(d) = \phi((-v_B)^*) + \phi(d) = -\phi(v_B) + \phi(d), \end{aligned}$$

where d being the additive game $d(S) = \sum_{i \in S} d_i, \forall S \subseteq N$,

$$\mu(d) = \phi(d) = (d_1, \ldots, d_n).$$

The nucleolus $\mu(v_D)$ of the dictator game v_D, which appears to be unknown, can also be obtained from the nucleolus $\mu(v_{BB})$ of the big boss game v_{BB}. That is, letting $M_i(v_{BB}) = v_{BB}(N) - v_{BB}(N \setminus \{i\})$ for all $i \in N$, we have from Theorem 4.2 in Muto et al. (1988) that

$$\begin{aligned} \mu(v_D) = -\mu(v_{BB}) &= \left(-v_{BB}(N) + \frac{1}{2} \sum_{i \in N \setminus \{1\}} M_i(v_{BB}),\ -\frac{1}{2} M_2(v_{BB}), \ldots,\ -\frac{1}{2} M_n(v_{BB}) \right) \\ &= \left(v_D(N) - \frac{1}{2} \sum_{i \in N \setminus \{1\}} v_D(\{i\}),\ \frac{1}{2} v_D(\{2\}), \ldots,\ \frac{1}{2} v_D(\{n\}) \right). \end{aligned}$$

When the big boss game is convex, so is the dictator game by Theorem 1. Then, since Theorem 4.5 in Muto et al. (1988) states in particular that $\mu(v_{BB}) = \phi(v_{BB})$, we have here that

$$\mu(v_D) = \phi(v_D).$$

Thus, the anti-dual of the big boss game may be a typical example in which theorems 1 and 2 can be of substantial use.

5 Concluding remarks

In the axiomatization literature, a solution $\sigma(N, v)$ of a game (N, v) is said to be *self dual* if it satisfies

$$\sigma(N, v) = \sigma(N, v^*),$$

where v^* is the *dual* of v. The Shapley value is self dual, whereas the prenucleolus is not. Theorem 2 on the other hand indicates that the two solutions satisfy the relation,

$$\sigma(N, (-v)^*) = -\sigma(N, v).$$

Thus, we might as well say that these solutions are 'self anti-dual.' The dual of a game is already incorporated in the axiomatization of solutions of a TU game. Therefore, whether or not the anti-dual of a game can also play a similar role may deserve investigation, which we leave to the future research.

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