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This paper uses a variety expansion growth model to show that an economy with a time lag between innovation and widespread use of the new product can experience growth cycles. By allowing the diffusion of innovation, the economy can exhibit period-by-period indeterminacy of expectations. If agents expect that high investment and rapid growth will take place in the future, the economy actually grows faster, and if not, the economy grows slowly.

＊ Staff A 21st Century Center of Excellence Project, Keio University
Part-Time Lecturer Faculty of Economics, Yokohama National University

Graduate School of Economics and Graduate School of Business and Commerce,
Keio University
2-15-45 Mita, Minato-ku, Tokyo 108-8345, Japan

Endogenous Growth Cycles

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Abstract

This paper uses a variety expansion growth model to show that an economy with a time lag between innovation and widespread use of the new product can experience growth cycles. By allowing the diffusion of innovation, the economy can exhibit period-by-period indeterminacy of expectations. If agents expect that high investment and rapid growth will take place in the future, the economy actually grows faster, and if not, the economy grows slowly.

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1 Introduction

Are business cycles caused simply by random shocks around a deterministic trend or are they the result of more fundamental linkages between long-run growth and short-run fluctuations? The literature has traditionally stressed exogenous shocks or nonlinearites, but raising questions about the separation between growth and fluctuations is important for several reasons. First, if cycles of investment and the growth rate are endogenously determined, the mechanisms by which growing economies fluctuate should be clarified to design policies that stabilize the economy. Second, cross-country evidence implies a negative partial correlation between growth and volatility (Ramey and Ramey, 1995). Third, asymmetric behaviors of the economy during upturns and downturns seem to be determined by endogenous forces, while it is clear that exogenous shocks are sources of aggregate fluctuations. Nevertheless, macroeconomists have dealt with the mechanisms of long-run growth and sources of short-run fluctuations separately.

Recently, several authors have presented models that allow us to understand the fundamental linkage between short-run fluctuations and long-run growth. [1] The present study, in line with recent work, constructs a variety expansion model of endogenous

[*] *E-mail address:* yfulkawa@ynu.ac.jp

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[1] See Evans et al. (1998), Francois and Shi (1999), Matsuyama (1999), Francois and Lloyd-Ellis (2003), and Mukherji (2005).

growth in which innovation is set not only as the determinant of long-run growth, but also as a source of short-run fluctuations.[2] Specifically, I show that expectational indeterminacy and thereby self-fulfilling growth cycles endogenously arise in the model presented below.

The crucial assumption in this model is that I allow for technology diffusion to occur gradually: that is, there is a time lag between the invention of new products (i.e., innovation) and their widespread use. The gradual diffusion of innovation has not been set as a source of endogenous growth cycles in the literature.

This paper models the gradual diffusion of innovation by assuming that two sectors exist in final production, one high-tech and the other low-tech. Both sectors use a variety of differentiated intermediate inputs to supply intermediate goods to the final sector. To incorporate the diffusion of innovation, I assume that only the high-tech sector uses the newly invented intermediate goods/inputs, whereas the low-tech sector uses only the old intermediate inputs that were invented in the past.

This assumption, namely the gradual diffusion of innovation, provides a novel explanation of empirically observed variations in the growth rates of some variables, in contrast to the received wisdom for a normal variety expansion model. That is, there is global indeterminacy in the model, which results from the fact that multiple perfect-foresight equilibrium paths exist for any initial value in the one-dimensional dynamic system of the model, and then the law of motion is given as a correspondence in the normal backward dynamics, rather than as a function. This implies that the economy can exhibit endogenous business cycles fueled by self-fulfilling beliefs. It is also shown that such multiplicity emerges when the potential innovation capacity, which is formally defined in Section 3, is in an intermediate range, that is, when the innovation capacity is not too large and not too small.

The logic behind the result is as follows. Allowing for the gradual diffusion of innovation, an innovation is adopted by the high-tech sector temporarily: innovation increases the relative productivity of the high-tech sector compared to the low-tech sector, leading to a decline in the relative price of the high-tech sector product. As a consequence, innovation encourages demand for the high-tech sector. The increased demand for the high-tech sector, which is the only sector to adopt innovation temporarily, implies an increase in the temporary demand for innovation. This paper assumes that the monopoly power is temporary (the value of innovation depends on *temporary* monopoly rents), and hence the increased temporary demand for innovation directly stimulates an incentive to innovate. Therefore, innovation reinforces itself within a period (i.e., the self-reinforcing mechanism of innovation), and thus the expectational indeterminacy of investment in R&D. Because of this "self-reinforcing mechanism," which is the key element of the current paper, if agents expect a period of high investment and then rapid growth, the economy actually grows rapidly, and vice versa. [3]

The contribution of the present paper hence is to show, by allowing for the gradual diffusion of innovation, that innovation, the engine of growth, can also be a source of

[2]The variety expansion model is one of the more important models in the endogenous growth literature; see Romer (1990) and Grossman and Helpman (1991).

[3]Clearly, two considerations are important for the result of the current paper: the gradual diffusion of innovation and the temporary nature of the monopoly. These are all that is needed to generate expectational indeterminacy.

self-fulfilling fluctuations. This is because, as is apparent from the above, the expectational indeterminacy of this paper is generated by the self-reinforcing mechanism of innovation, which emerges owing to the gradual diffusion of innovation.

As mentioned above, the model developed here is related to the recent literature that has examined *innovation-based* models in which growing economies fluctuate endogenously. A closely related paper is the study by Evans et al. (1998), who constructed rational expectations model of growth cycles based on expectational indeterminacy, as in the present model. However, while growth cycles in both the latter and the current models are caused by expectational indeterminacy, the two models differ in several aspects. First, the Evans et al. model does not have a time lag between innovation and its widespread use, whereas the current model stresses the gradual diffusion of innovation, contrary to other standard growth models. The diffusion of innovation is the key element of this paper. Allowing the gradual diffusion of innovation, this paper clearly sets innovation not only as the engine of long-run growth, but also as the source of fluctuations. Second, Evans et al. directly assume that intermediate products are complementary, while the results of the present study do not rely on any direct complementarity. I show that, without assuming such complementarity, the economy can experience growth cycles. Third, although the Evans et al. analysis focuses on the stability of the dynamics implied by a simple learning rule, my focus is rather on a link between growth cycles and innovation, and the role of (fundamentally determined) parameters such as the potential innovation capacity in the emergence of growth cycles. This allows implications for a stabilization policy. Finally, the paper clarifies the nature of the cycles that the model exhibits (e.g., the existence of all periodic cycles and topological chaos), contrary to existing models in the growth cycle literature, including that of Evans et al. (1998).[4]

The following papers, which are related, do not shed light on the diffusion of innovation or show the existence of all periodic cycles. Francois and Lloyd-Ellis (2003), for instance, modify a quality-ladder model by allowing for the elasticity of intertemporal substitution to exceed unity and by introducing the possibility of storage. This model exhibits endogenous clustering of innovation and its implementation and associated implementation cycles. The theory of implementation cycles originated with Shleifer (1986), who, however, could not analyze the interaction between growth and cycles because innovations were assumed to occur exogenously, and hence long-run growth was also exogenous. In the study by Matsuyama (1999), the balanced growth path is unstable and the economy achieves endogenous growth cycles by moving back and forth between two phases, one with capital accumulation and no innovation, and the other with both factor accumulation and innovation. The latter model differs from Romer (1990) and Grossman and Helpman (1991, Ch. 3.2), in that the innovators of new intermediate goods enjoy a one-period monopoly power. Under the same assumption, Francois and Shi (1999) construct a quality-ladder model of implementation cycles, in which they assume that innovating the new generation of technology requires that human capital is accumulated over time. Contrary to these models, I focus on another side of innovation, namely the diffusion of innovation, to demonstrate endogenous growth

[4]The economy considered below can experience *complex* cycles of the growth rate, in the sense that the economy never converges to one cycle or another, or to any periodic orbits (i.e., topological chaos).

cycles driven by the self-fulfilling belief of the agents.

The paper is structured as follows. Section 2 outlines the basic model and characterizes the market equilibrium conditions. Section 3 provides the law of motion of the economy and characterizes the dynamic equilibria, in which the conditions required for the uniqueness and the global indeterminacy of equilibrium paths are provided. The existence of a three-period cycle is also proved. Section 4 focuses on the indeterminacy of the model, and Section 5 provides a conclusion.

2 The Basic Model

The basic model essentially differs from Grossman and Helpman (1991, Ch. 3.2) in the following respects. First, I allow for technology diffusion to occur gradually: that is, there is a time lag between innovation and the widespread use of the new products. This implies that two sectors exist in production, one high-tech and the other low-tech. Second, as in Matsuyama (1999), legal protection of intellectual property rights is imperfect, in the sense that since an innovation is imitated costlessly after one period, the innovators of new products enjoy only temporary profit power. Both parameters are necessary for the result of this paper, as shown later.

Time is discrete, extends from zero to infinity, and is indexed by t. I consider a closed economy that admits an infinitely lived representative consumer who supplies N units of labor inelastically and consumes the homogenous final good, which is taken as the numeraire. For simplicity, I assume that N is constant over time.

2.1 Households

At any period, households solve the following maximization problem:

$$\max_{\{C_t\}_{t=0}^{\infty},\{A_t\}_{t=1}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t \ln C_t \ , \tag{1}$$

$$\text{subject to} \quad A_{t+1} + C_t = (1+r_t)A_t + w_t N \ , \tag{2}$$

where $\beta \in (0,1)$ denotes the rate of time preference, A_t denotes the household's stock of assets from $t-1$ to t, which consists of shares of R&D firms, r_t denotes the interest factor from $t-1$ to t, and w_t denotes wage income. The Euler equation and the transversality condition characterize the solution to this maximization problem:

$$\frac{C_{t+1}}{C_t} = \beta(1+r_{t+1}), \forall t \geq 0 \ , \tag{3}$$

$$\lim_{T\to\infty} \beta^T \frac{A_{T+1}}{C_T} = 0 \ . \tag{4}$$

Consumption grows at a rate equal to the rate of time preference multiplied by the interest factor.

2.2 Production

Consumption arises from an output aggregate produced for two goods, Y^L and Y^H, with constant elasticity of substitution, $\varepsilon > 1$. I specify the production function for final goods as:

$$C_t = \left[\theta \left(Y_t^L\right)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\theta)\left(Y_t^H\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} , \tag{5}$$

where $\theta \in (0,1)$ is a distribution parameter that determines how important the two goods are in terms of aggregate production.

The final good is produced by competitive firms, so the product prices need to satisfy:

$$\frac{P_t^H Y_t^H}{P_t^L Y_t^L} = \left(\frac{\theta}{1-\theta}\right)^{-\varepsilon} \left(\frac{P_t^H}{P_t^L}\right)^{1-\varepsilon} . \tag{6}$$

The response of the relative price to the relative expenditure depends on the elasticity of substitution, ε. In addition, I choose the price of the final consumption good as the numeraire, so that the profit-maximizing condition, which requires that the marginal cost of producing the final good is equal to one, can be expressed as:

$$1 = \left[\theta^{\varepsilon} \left(P_t^L\right)^{1-\varepsilon} + (1-\theta)^{\varepsilon} \left(P_t^H\right)^{1-\varepsilon}\right] . \tag{7}$$

Next, I describe the two intermediate sectors, high-tech and low-tech, which are the key elements in this model.

The two intermediate goods differ only in their level of technological sophistication. The high-tech sector, Y^H, utilizes leading-edge technology (the technology invented in t), n_t, while the low-tech sector, Y^L, utilizes only old technology (invented in the previous period), n_{t-1}. Each good is the composite of perishable, differentiated inputs (or machines), which has a symmetric CES form:

$$Y_t^L = \left[\int_0^{n_{t-1}} y_t^L(j)^{\frac{\sigma-1}{\sigma}} dj\right]^{\frac{\sigma}{\sigma-1}} , \qquad Y_t^H = \left[\int_0^{n_t} y_t^H(j)^{\frac{\sigma-1}{\sigma}} dj\right]^{\frac{\sigma}{\sigma-1}} , \tag{8}$$

where $y_t^m(j),\ m = L,H$ is the number of variety j (machine j) used in sector m and the elasticity of substitution between every pair of inputs is equal to $\sigma > 1$. The asymmetry across sectors captures the gradual diffusion of technology; that is, there is a time lag between innovation and its widespread use in all sectors. For simplicity, I assume that the length of the lag is one period.

In any one period, only a subset of differentiated inputs $[0, n_t]$ is available in a marketplace, and the varieties available expand over time, with free entry to the market for differentiated machines.

Assume that intellectual property rights are not perfectly protected, in the sense that the innovator enjoys only temporary monopoly power.[5] On one hand, all the machines

[5]This assumption is adopted in many papers; see, for example, Shleifer (1986), Francois and Shi (1999), and Matsuyama (1999). However, the role for this assumption here is different from that in previous models.

in the range $[0, n_{t-1}]$ are invented prior to period t, and thus are supplied *competitively* in period t. On the other, the machines of variety $j \in [n_{t-1}, n_t]$ cannot be invented, adopted in the high-tech sector, and supplied *monopolistically* by the innovator. It is worth pointing out that because these "new" machines are in use only in high-tech production, low-tech production cannot be directly affected by the present innovation, which is captured by the number of newly introduced machines $[n_{t-1}, n_t]$, but only by past innovation, which is captured by the number of machines already introduced $[0, n_{t-1}]$.

It is well known that the demand functions of the two intermediate sectors for each differentiated machine can be represented by:

$$y_t^L(j) = P^L Y^L \frac{p(j)^{-\sigma}}{(P^L)^{1-\sigma}} , \qquad y_t^H(j) = P^H Y^H \frac{p(j)^{-\sigma}}{(P^H)^{1-\sigma}} , \tag{9}$$

where the former is the demand function of the low-tech sector for machine j, $j \in [0, n_{t-1}]$ and the latter is that of the high-tech sector for machine j, $j \in [0, n_t]$. Note that the price elasticity of demand for the machines is constant at $\sigma > 1$ in sectors L and H.

Here I assume that producing a unit of machines requires a unit of labor as input (one-for-one technology). The marginal cost of producing machines is then w_t. It follows that the old machines are sold at the marginal cost, $p_t(j) = w_t$, $\forall j \in [0, n_{t-1}]$, whereas the new machines are sold at $p_t(j) = \sigma w_t / (\sigma - 1)$, $\forall j \in [n_{t-1}, n_t]$, if they exist.

Combined with Eqs. (8) and (9), these machine prices lead to the following prices of the two intermediate goods:

$$P_t^L = \left[\int_0^{n_{t-1}} p_t(j)^{1-\sigma} dj \right]^{\frac{1}{1-\sigma}} = w_t n_{t-1}^{\frac{1}{1-\sigma}} , \tag{10}$$

$$P_t^H = \left[\int_0^{n_t} p_t(j)^{1-\sigma} dj \right]^{\frac{1}{1-\sigma}} = w_t n_{t-1}^{\frac{1}{1-\sigma}} \left(1 + \hat{\sigma} g_t\right)^{\frac{1}{1-\sigma}} , \tag{11}$$

where $\hat{\sigma} = (\sigma/(\sigma-1))^{1-\sigma}$, and $g_t = (n_t - n_{t-1})/n_{t-1}$ denotes the rate of growth of technical changes. I often refer to the rate of technical change, g_t, as the rate of economic growth.

Clearly, from Eq. (9) and the machine prices, all the machines enter the two intermediate sectors symmetrically: $y_t^L(j) = y_t^L$, $y_t^H(j) = y_t^H$, $\forall j \in [0, n_{t-1}]$ and $y_t^H(j) = \hat{y}_t^H$, $\forall j \in [n_{t-1}, n_t]$. It follows from Eqs. (9), (10), and (11) that the machine demands can be rewritten as follows:

$$y_t^L = \frac{\alpha_t C_t}{w_t n_{t-1}} , \tag{12}$$

$$y_t^H = \frac{(1-\alpha_t) C_t}{n_{t-1} w_t (1 + \hat{\sigma} g_t)} , \tag{13}$$

$$\hat{y}_t^H = \left(\frac{\sigma - 1}{\sigma} \right)^{\sigma} \frac{(1-\alpha_t) C_t}{n_{t-1} w_t (1 + \hat{\sigma} g_t)} , \tag{14}$$

where α_t denotes the factor share of the low-tech intermediate input in aggregate production; that is, $\alpha_t \equiv P_t^L Y_t^L / C_t = P_t^L Y_t^L / (P_t^L Y_t^L + P_t^H Y_t^H)$.[6] As is apparent from the definition of α, $\alpha_t C_t$ and $(1-\alpha_t)C_t$ are equal to $P^L Y^L$ and $P^H Y^H$, respectively. Now using (6), (10), and (11), I can express the factor share of the low-tech sector, α_t, as a function of the growth rate of varieties:

$$\alpha_t = \alpha(g_t) = \frac{\hat{\theta}}{\hat{\theta} + (1+\hat{\sigma} g_t)^{\lambda}} \ , \tag{15}$$

where $\hat{\theta} \equiv \left(\frac{\theta}{1-\theta}\right)^{\varepsilon}$ and $\lambda \equiv \frac{\varepsilon-1}{\sigma-1} > 0$.[7] It is easy to verify some properties of the function α: $\alpha'(g) < 0$, $\alpha(0) = \theta^{\varepsilon}/[\theta^{\varepsilon} + (1-\theta)^{\varepsilon}]$, and $\alpha(+\infty) = 0$.

Lastly, using $p_t(j) = \sigma w_t/(\sigma-1),\ j \in [n_{t-1}, n_t]$, and (14), the period-$t$ profit of technology monopolists can be expressed as:

$$\pi_t = \frac{w_t}{\sigma - 1} \hat{y}_t^H = \frac{(\sigma-1)^{\sigma-1}}{\sigma^{\sigma}} \frac{[1-\alpha(g_t)]C_t}{n_{t-1}(1+\hat{\sigma} g_t)} \ . \tag{16}$$

Inspection of (16) reveals that two forces determine the relationship between the innovator's profits and the growth rate:

1. The direct effect: the price of the high-tech intermediate input, P_t^H, decreases as a function of the rate of technical change, g_t, because of the technological sophistication; see (11). A reduction in the price of the high-tech input, P_t^H, depresses the revenue of the high-tech sector and thereby decreases its demand for machines, $y_t^H(j)$, as shown in (9). It follows that a higher rate of technical change, g_t, leads to lower high-tech sector demand for machines, $y_t^H(j)$, reducing the innovator's profits. This effect is captured by the term $1/(1+\hat{\sigma} g_t)$.

2. The indirect (relative price) effect: more innovation leads to a lower relative price of the high-tech good compared to the low-tech good, P^H/P^L. As a result, demand by the final sector for intermediate goods switches from the low-tech good, Y_t^L, to the high-tech good, Y_t^H, as shown in (6). An increase in Y_t^H increases the demand for newly introduced machines and thus the innovator's profits. This effect is captured by the term $(1-\alpha(g_t))C_t$.

These two opposing forces determine the relationship in equilibrium. If the former dominates the latter, monopolistic profits decrease with the amount of innovation and there will be decreasing aggregate returns in the R&D sector. If the relative price effect is sufficiently strong, there will be increasing aggregate returns. By differentiating (16) with respect to the growth rate for given n and C,

$$\frac{d\pi}{dg} = \frac{\hat{\sigma}^2 C (1+\hat{\sigma} g)^{\lambda-2} \left[\hat{\theta} \frac{\varepsilon-\sigma}{\sigma-1} - (1+\hat{\sigma} g)^{\lambda}\right]}{\sigma n \left[\hat{\theta} + (1+\hat{\sigma} g)^{\lambda}\right]^2} \ ,$$

[6] Note that $P_t^L Y_t^L + P_t^H Y_t^H$ is equal to C because the production function of consumption goods, (5), is linearly homogeneous in Y^L and Y^H.

[7] α_t can be rewritten as $\alpha_t = 1/[1 + (P^H Y^H)/(P^L Y^L)]$. It is straightforward to verify Eq. (15) using Eqs. (6), (10), and (11).

which implies that $\pi(g)$ has inverse U-shaped configurations if and only if $\varepsilon > \sigma + \hat{\theta}(\sigma-1)$. As depicted in Fig. 1, given any $\pi \in (\pi(0), \max_y \pi(y))$, two possible values of the growth rate exist, a higher value (g') and a lower value (g''), if the relative price effect is sufficiently large (ε is sufficiently large). This multiplicity derived from positive relationships between the amount of innovation and the profit of each innovator is of some interest. Since the current amount of innovation has a positive impact on the high-tech sector and no impact on the low-tech sector, complementarity between the high-tech sector and innovators exists in the economy. This complementary relationship plays a crucial role in generating the multiplicity associated with socially increasing returns in the R&D sector.

As discussed in the following section, this potential existence of multiplicity leads to global fluctuations driven by self-fulfilling beliefs.

2.3 Innovation

Innovating new machines in period t requires start-up costs, a^{RD}/n_{t-1} units of labor, per variety *in the previous period.* In period $t-1$, firms finance start-up costs by issuing shares, which give the holders a claim to the profits, and then exclusively supply a new variety only in period t. Owing to the one-period protection of intellectual property rights, the value of innovation in period t is equal to $V_t = \pi_t/(1+r_t)$. In equilibrium, the value, V_t, and the start-up costs, $a^{RD}w_{t-1}/n_{t-1}$, are equalized if some entry occurs. No entry occurs whenever start-up costs exceed the value of innovating new varieties. The following equation sums up these free entry conditions:

$$V_t \leq \frac{a^{RD}w_{t-1}}{n_{t-1}}, \quad (n_t - n_{t-1})\left(V_t - \frac{a^{RD}w_{t-1}}{n_{t-1}}\right) = 0. \tag{17}$$

Unlike models with perfect protection of intellectual property, the value of innovation is determined by the temporary monopoly rent, rather than by future rents, in this model. Therefore, an increase in the temporary rent in period t leads directly to an increase in the innovation value in the model, while a temporary rent has little impact on the value of innovation, which is represented as the sum of discounted rents, in models with a permanent monopoly.

2.4 Labor

In this subsection, I use labor-market clearing conditions to close this model. While N units of labor are supplied by an infinitely lived agent in any period, labor demands consist of two factors: the demands for manufacturing machines and their use as start-up operations by R&D firms. Then I obtain:

$$N = (n_{t+1} - n_t)\frac{a^{RD}}{n_t} + n_{t-1}(y_t^L + y_t^H) + (n_t - n_{t-1})\hat{y}_t^H \ ,$$

or, by substituting (12), (13), and (14) into this equation,

$$N = a^{RD} g_{t+1} + \frac{C_t}{w_t} L(g_t)\ , \tag{18}$$

$$L(g_t) \equiv \alpha(g_t) + [1-\alpha(g_t)] \frac{1 + \left(\frac{\sigma}{\sigma-1}\right)^{-\sigma} g_t}{1+\hat{\sigma} g_t}\ , \tag{19}$$

where $a^{RD} g_{t+1}$ and $L(g_t) C_t / w_t$ represent employment in R&D and manufacturing, respectively.

3 Dynamic Equilibria

Dynamic equilibria are characterized completely by (3), (4), (7), (15), (16), (17), and (18). By substituting (3) and (16) into (17), I obtain:

$$\frac{C_t}{w_t} \begin{cases} = \dfrac{a^{RD} \sigma^{\sigma} (1+\hat{\sigma} g_{t+1})}{\beta (\sigma-1)^{\sigma-1} \left[1-\alpha(g_{t+1})\right]} & \text{if } g_{t+1} > 0\ , \\ \leq \dfrac{a^{RD} \left(1+\hat{\theta}\right) \sigma^{\sigma}}{\beta (\sigma-1)^{\sigma-1}} & \text{if } g_{t+1} = 0\ . \end{cases} \tag{20}$$

From (18) and (20),

$$L(g_t) \begin{cases} = \delta(g_{t+1}) \equiv \dfrac{\eta \left(1+\hat{\theta}\right) \left(N - a^{RD} g_{t+1}\right) (1+\hat{\sigma} g_{t+1})^{\lambda-1}}{\hat{\theta} + (1+\hat{\sigma} g_{t+1})^{\lambda}} & \text{if } g_{t+1} > 0\ , \\ \geq \delta(0) = \eta N & \text{if } g_{t+1} = 0\ , \end{cases} \tag{21}$$

where $\eta \equiv \beta(\sigma-1)^{\sigma-1} / [a^{RD} \sigma^{\sigma} (1+\hat{\theta})]$. Finally, using Eqs. (7), (10), and (11), we obtain:

$$\hat{n}_{t-1} = \frac{1}{\theta^{\varepsilon} + (1-\theta)^{\varepsilon} (1+\hat{\sigma} g_t)^{\lambda}}\ , \tag{22}$$

where $\hat{n}_{t-1} \equiv \left(n_{t-1}^{1/(\sigma-1)} / w_t\right)^{\varepsilon-1}$ and the growth rate, g_t, is decreasing in $\hat{n}_{t-1}$. The growth rate thus increases with the wage rate for the given range of differentiated machines available.

For the initial range of varieties the economy inherits, n_{-1}, a perfect-foresight equilibrium path of this economy, $\{g_t, \hat{n}_{t-1}\}_{t=0}^{+\infty}$, satisfies the transversality condition, (4),[8] and the law of motion for g_t, (19) and (21), as well as one side condition (22).[9] Then the system can be expressed as the one-dimensional map of g_t.

As shown in the Appendix (note Lemmas 1 and 2), functions L and δ have the configuration depicted in Figure 2(a)–(d), where $\bar{g}$ satisfies $\delta(\hat{g}) = L(\bar{g})$, and let $\bar{\bar{g}}$ satisfy $\eta N < L(x)$, $\forall x \in [0, \bar{\bar{g}})$; then this will be the critical level of the growth rate, below

[8]The transversality condition is always satisfied in the model (see the Appendix).

[9]The value g_0 is *not* given as an initial condition, although n_{-1} is. Eq. (22) actually determines g_0 for a given range of varieties, n_{-1}. The next section discusses the determination of g_0.

which either development traps and/or multiple equilibria exist, as follows. There are now four possibilities regarding the configuration of the equilibrium law of motion in this economy, (21). The following proposition summarizes these and the main result of this paper. Its proof appears in the Appendix. Define $\kappa_1 \equiv \sigma^{\sigma}(1+\hat{\theta})/[\beta(\sigma-1)^{\sigma-1}]$ and $\kappa_2 \equiv (1+\hat{\theta})/[\hat{\sigma}(\hat{\theta}(\lambda-1)-1)]$.

Proposition 1 *Assume $0 < \kappa_2 < \kappa_1$.*[10] *The following four regimes are possible:*

1. ***Stable steady state:*** *Suppose that $N/a^{RD} > \kappa_1$. Then, for any initial value, g_0, the economy experiences unbounded growth and converges to a unique BGP, $g^* > 0$.*

2. ***Global development traps:*** *Suppose that $N/a^{RD} < \kappa_2$.*[11] *Then, for any initial value of the growth rate, the economy is always led to a trap $g^* = 0$ in finite time.*

3. ***Persistent growth cycles****: Suppose $N/a^{RD} \in (\kappa_2, \kappa_1)$ and $\hat{g}$ satisfies $\delta(\hat{g}) \geq 1$. Then, for any $g_t \in [0, \overline{\overline{g}}]$ and for any $t \geq 0$, there are three possible perfect-foresight equilibrium paths, denoted by $g^0 = 0$, $\overline{d}(g_t)$, $\overline{\overline{d}}(g_t)$, satisfying the equilibrium law of motion, (21). The economy can experience persistent growth cycles driven by self-fulfilling beliefs.*

4. ***Cycles and traps****: Suppose that $N/a^{RD} \in (\kappa_2, \kappa_1)$ and $\hat{g}$ satisfies $\delta(\hat{g}) < 1$. Then, for any $g_t \in [0, \overline{g})$ and for any $t \geq 0$, the economy is trapped in the no-growth equilibrium. For any $g_t \in [\overline{g}, \overline{\overline{g}}]$, there exist three possible perfect-foresight equilibrium paths and the economy can fluctuate.*

Proposition 1 distinguishes between four possibilities for the economy, depending on the nature of the dynamics for the growth rate g_t. Each case is characterized by the size of N/a^{RD}.

The first case is represented in Figure 3, and arises whenever ηN exceeds unity (equivalently, $N/a^{RD} > \kappa_1$). This occurs for small start-up costs, a^{RD}, and/or when a representative agent is sufficiently patient. In other words, the economy goes to case 1 when the potential innovation capacity, N/a^{RD}, is sufficiently large.[12] Let $d(g_t)$ be equilibrium paths corresponding to g_t (i.e., $g_{t+1} = d(g_t)$), which satisfies the law of motion, (21), consisting of the functions L and δ. Given $g_t \geq 0$, $d(g_t)$ is unique, and is then a well-defined function. In this case, for a parameter, $z \in [0,1]$, the equation $z = \delta(g)$ has a unique solution g, regardless of the configuration of δ, as depicted in Figure

[10] By definition, the assumption $0 < \kappa_2 < \kappa_1$ is characterized by a larger ε, a larger θ, and/or a smaller σ. A smaller σ would be justified by the following fact. Hall (1986) estimates the markup ratios of some 50 industries at the two-digit SIC code level, and shows that the markup ratio is greater than 1.5 in most industries and greater than 3 in a few. Noting that the markup is represented by $\sigma/(\sigma-1)$, Hall's estimates imply $\sigma < 3$ and $\sigma < 1.5$, respectively. Therefore, I can state that, following Hall (1986), σ should be small enough to fall below 3.

[11] Under the assumptions, the other possibility exists: there is a range of initial levels of the growth rate such that the economy experiences convergence to a BGP with $g^* > 0$. Outside of this range, the economy remains in a lower stage of development. For simplicity, I exclude this possibility because it did not arise in the numerical calculations.

[12] Here I interpret N/a^{RD} as the potential innovation capacity because N/a^{RD} is the maximum growth rate g when all of the labor force N is devoted to R&D activity.

2(a). Moreover, the growth rate that satisfies $L(g) > \eta N$ never exists. Then, there is a unique equilibrium path, and the economy does not experience any fluctuations. Clearly $\delta(x) > L(x)$ iff $x \in [0, g^*)$ holds, so that the unique BGP is globally stable in the normal backward dynamics. Hence, for any initial value of the growth rate, g_t converges to the BGP, $g^* > 0$, monotonically. These converging paths do not violate the transversality condition, and the economy thus achieves sustainable growth.

Case 2, with a high a^{RD} and/or low β, is represented in Figure 4. In other words, this case is characterized by sufficiently small innovation capacity, N/a^{RD}. For all initial levels of the growth rate, the function $d(g)$ is below the 45° line, and the growth rate converges to zero in finite time. In this case, development trap phenomena will be always observed, and once trapped, the economy never grows.

My main interest lies in cases with cyclical behavior, corresponding to cases 3 and 4, depicted in Figures 5 and 6, respectively. These cases are characterized by an intermediate range of the potential innovation capacity: $N/a^{RD} \in (\kappa_2, \kappa_1)$. In these cases, for any $g_t \in [0, \overline{\overline{g}})$, there are three possible perfect-foresight equilibrium paths in the backward dynamics, g_{t+1}, and therefore the model predicts a growth-cycle phenomenon driven by self-fulfilling beliefs. If the agents expect a stably converging path, the economy converges to the BGP stably. However, if the agents expect fluctuations, the economy fluctuates, and if the agents expect that the economy never grows, then it never does. All expectations that are consistent with the transversality condition and the law of motion can be self-fulfilling.

The economic intuition behind the existence of these self-fulfilling cycles is easy to grasp. Allowing for the gradual diffusion of innovation, new intermediate machines (or innovations) are adopted only by the high-tech sector within the period when these machines are newly introduced. It follows that the introduction of new machines increases the productivity of the high-tech sector temporarily,[13] leading to a decline in the relative price of the products of the high-tech sector. This lower cost of the high-tech products implies a greater demand by consumption good firms for the goods of the high-tech sector. Because the high-tech sector is the only buyer of *newly* introduced machines, the introduction of new machines enhances the demand for the new machines and therefore increases the temporary monopoly rent. Owing to the temporary monopoly power, the temporary rent has a much greater impact on the value of innovation, as already mentioned in Section 2.3. Thus, the greater the temporary rent, the larger is the value of innovation. Finally, innovations reinforce themselves within the period.

Such a "self-reinforcing" mechanism of innovation generates period-by-period indeterminacy of expectation endogenously. The self-reinforcing mechanism of innovation is reflected in the one-peak configuration of the profit function $\pi(g)$ in (16), especially in the upward-sloping region of g, where monopoly rent increases as a result of an increase in the number of newly introduced machines. There are multiple possible g_t for a given π_t, so that g_t that satisfies the free entry condition is not unique for a given g_{t-1}. This implies that the equilibrium paths are indeterminate.

When the self-reinforcing mechanism of innovation dominates, it makes expectation about the future state indeterminate. In this case, if agents expect that investment

[13]The productivity of the low-tech sector remains constant within this period.

in R&D will be so high that rapid growth will take place tomorrow, the economy grows faster. If agents expect that investment will hardly take place and the growth rate will be low, the economy actually grows slowly. That is, the growing economy can fluctuate, depending on the expectations, which is the main finding of this paper.

Clearly, two considerations are needed for the generation of expectational indeterminacy. One is the gradual diffusion of innovation, which implies the presence of the high-tech and low-tech sectors in this model. The demand shift from the low-tech to the high-tech sector in response to the introduction of new machines is essential for the result of this paper. The other consideration is the imperfect protection of intellectual property rights, in the sense that the market power of monopolists is temporary. As mentioned repeatedly, this increases the impact of temporary rent on the value of innovation. Such a large impact means that the demand shift to the high-tech sector directly enhances innovators.[14]

I now let g_L^* and g_H^* represent the low and the high steady states, which satisfy $g_L^* = \bar{d}(g_L^*)$ and $g_H^* = \bar{\bar{d}}(g_H^*)$. The following is easily derived from Figure 2(c) and (d).

Proposition 2 $\frac{d}{dg_H^*}\bar{\bar{d}}(g_H^*) \in (0,1)$ *holds.*

This implies that the higher steady state, g_H^*, is asymptotically stable in the normal backward dynamics. Hence, this steady state can be approached from any g_0 in case 3.

Moreover, in both cases with indeterminacy, the forward-looking analysis does not vary the backward-looking case. It also ensures that the dynamic behavior of the economy can be expectationally indeterminate and cyclical. As shown in Figure 5, if periodic cycles exist, they do not violate the transversality condition, and their existence is guaranteed by Proposition 3 stated below.

I now apply these results (Propositions 1 and 2) to discuss the policy implications of this model.

How can an economy escape from a development trap or business fluctuations? What is better for the stable development of economies? This paper suggests the following two answers. One is related to the fundamental factors of the economy. As shown in Proposition 1, case 2 is characterized by a low N/a^{RD}. This has a natural implication: an economy with small innovation capacity will be caught in a development trap and will never grow. If the innovation capacity, N/a^{RD}, is in an intermediate range, (κ_2, κ_1), the economy will exhibit expectational indeterminacy (cases 3 and 4). Finally, the economy experiences stable growth when the R&D technology is sufficiently sophisticated (larger innovation capacity) to bring it out of the regimes of global indeterminacy; $\kappa_1 < N/a^{RD}$. Therefore, the model suggests encouragement of innovation capacity, N/a^{RD}, which is associated with a reduction in the start-up cost for R&D, as a preferred policy for development and stabilization.

Another way out of the fluctuations is related to the self-fulfilling beliefs and expectations of agents. Affecting the beliefs of agents can stabilize the economy without

[14]The purpose of this paper is to show the effects of innovation diffusion on the dynamic behaviors of the economy. This might raise a question regarding the necessity for the temporary nature of monopoly power. This assumption makes the analysis drastically simple, and is thus imposed here. However, I believe that the self-reinforcing mechanism emphasized here goes far enough in a model with permanent monopoly, even if weak.

the need for transition through the regimes. In cases 3 and 4, if the higher equilibrium path, $\overline{\overline{d}}(g_t)$, is realized in all periods, Proposition 2 states that the economy will *monotonically* converge to the higher steady state, g_H^*.[15]

The focus of my remaining interest is the nature of fluctuations in this economy. The following result provides a sufficient condition that guarantees a period-three cycle in this economy (see the Appendix for a proof).[16]

Proposition 3 *In case 3, a period-three equilibrium cycle exists if $g_H^* < \overline{\overline{g}}$. In case 4, such a cycle exists if both $g_H^* < \overline{\overline{g}}$ and $\overline{g} < \underline{d}(g_H^*)$ hold.*

The former is depicted in Figure 5. In general, if a period-three cycle exists in the normal backward dynamics, such a cycle also occurs in the forward dynamics, and it is worth mentioning that Proposition 3 is established in the forward analysis.

Example 1 *Consider two economies: (A) an economy with $\beta = 0.95$, $\varepsilon = 4.5$, $\theta = 0.6$, $\sigma = 3$, $a^{RD} = 0.495$, and $N = 21.2$; and (B) an economy with $\beta = 0.95$, $\varepsilon = 4.5$, $\theta = 0.6$, $\sigma = 3$, $a^{RD} = 0.495$, and $N = 21.09$. In both economies, $N/a^{RD} \in (\kappa_2, \kappa_1)$ holds and hence both correspond to cyclical cases 3 and 4 in Proposition 1.*

Numerical calculations imply that economy A corresponds to case 3 ($\delta(\hat{g}) > 1$) and that economy B corresponds to case 4 ($\delta(\hat{g}) < 1$). It is also easy to verify that these economies satisfy the assumptions of Proposition 3: $g_H^* < \overline{\overline{g}}$ in economy A and $g_H^* < \overline{\overline{g}}$ and $\overline{g} < \underline{d}(g_H^*)$ in economy B. It follows from Proposition 3 that, in both economies, equilibrium cycles exist for all periods (see footnote 16).

Finally, using these examples, I show in the remainder of this section that the current model can exhibit topological chaos. It is well known that period-three cycles imply topological chaos (Li and Yorke 1975): there is an uncountable set of initial values on I such that the economy never converges to any steady state or to any periodic orbit.[17]

I first verify that the above examples satisfy some assumptions to be imposed in the Li and Yorke theorem. I define again a mapping $d \equiv \{\underline{d}, \overline{d}, \overline{\overline{d}}\}$ such that d maps I into itself. I is an interval. If I take $I = I_A \equiv [0, g_H^*]$, d is a continuous mapping on I_A in the case of economy A. In addition, Proposition 3 still can apply to I_A,[18] so that economy A has period-three cycles on I. Hence, economy A exhibits topological chaos, applying the Li and Yorke theorem. I define $I_B \equiv [\overline{g}, g_H^*]$, then it can be shown that economy B also exhibits period-three cycles and topological chaos on I_B.

An example of period-three and chaos can also be shown in the forward dynamics. For notational convenience, I define a function $D \equiv d^{-1}$, in which d^{-1} is an inverse image of mapping d. Set the parameters as $\beta = 0.9$, $\varepsilon = 4.5$, $\theta = 0.6$, $\sigma = 3.3$, $a^{RD} = 0.495$, and $N = 27.37$. Then $\overline{\overline{g}} = 4.24469 < g_H^* = 5.36541$ and $\delta(\hat{g}) = 0.998995 < 1$

[15]This paper does not show that these stabilization policies improve welfare.

[16]It is easy to verify the existence of cycles for all periods using a straightforward extension of the proof shown in the Appendix.

[17]There is a large body of literature on the economic applications of topological chaos; see, for example, Nishimura and Yano (1996) and Mitra (2001).

[18]As is apparent from the proof in the Appendix, if the assumption in Proposition 3 is satisfied, period-three cycles can be proven for the interval I_A in case 3.

(case 4), as described in Figure 7. In this example, period-three cycles can be found in the forward dynamics. Taking the initial value of g as $g_0 = 3.8$, $D(3.8) \simeq 1.95087$, $D^2(3.8) = 0.0544822$, and $D^3(3.8) = 3.94081$; then $D^3(g_0) > g_0 > D(g_0) > D^2(g_0)$ is satisfied. Clearly, $D : [0, g_H^*] \to [0, g_H^*]$ is continuous. Therefore, taking into account that $D^3(g_0) > g_0 > D(g_0) > D^2(g_0)$ is a sufficient condition in the Li and Yorke theorem, it can be proven that D exhibits a period-q cycle for any integer $q > 1$ and topological chaos, applying the Li and Yorke (1975) theorem.[19] Therefore, the dynamics of the growth rate, g, can be cyclical and, depending on circumstances, even chaotic in both the natural backward dynamics and the forward dynamics.

In summary, an economy with gradual diffusion of innovation exhibits expectational indeterminacy and thereby self-fulfilling growth cycles when the innovation capacity is in an intermediate range (Proposition 1, cases 3 and 4), whereas it monotonically converges to the steady state when the innovation capacity is sufficiently high (case 1).[20] The numerical calculations imply that the current model for cases 3 and 4 can exhibit all periodic cycles (Proposition 3; Example 1) and thus chaotic equilibria in the sense of Li and Yorke (1975).

As already mentioned, this paper suggests that there are two ways to stabilize an economy experiencing self-fulfilling growth cycles that may be chaotic. First, the economy converges monotonically to the unique steady state g^* when the potential innovation capacity N/a^{RD} is sufficiently large to exceed κ_1 (Proposition 1). Second, the economy converges monotonically to the higher steady state g_H^* when agents expect a high-growth path $\overline{\overline{d}}$ in every period (Proposition 2). Therefore, the current paper suggests that a cyclical (possibly chaotic) economy can be stabilized (a) by increasing the potential innovation capacity or (b) by affecting the agents' belief regarding future states.

4 Indeterminacy

Two potential sources of indeterminacy exist in the model. One, which is discussed in the previous section and represents the core feature of the analysis, arises because, for a given $g_t \in [\overline{g}, \overline{\overline{g}}]$ in Figure 5 or $[0, \overline{\overline{g}}]$ in Figure 6, there are three possible perfect-foresight equilibrium paths and the law of motion is expressed as a correspondence in the backward dynamics, rather than as a function. This "global" indeterminacy appears in cases 3 and 4, which are characterized by high a^{RD} and low β. Propositions 2 and 3 correspond to this global indeterminacy and the self-fulfilling prophecy. The former implies that such an economy can monotonically converge to $g^* > 0$ when agents expect that will monotonically converge, and the economy also fluctuates when agents expect that it will fluctuate. The latter expresses the nature of this fluctuation. These results are valid in the global dynamics.

The other source of indeterminacy involves the determinant of initial growth rates. More precisely, g_0 is not given, while n_{-1} is given. I focus here on case 1, for which

[19]Without applying the Li and Yorke theorem, it is easy to verify the existence of cycles of all periods by numerical calculation.

[20]When the potential innovation capacity is low enough, the economy converges to the no-growth steady state (case 2).

fluctuations are not observed in the global dynamics. In case 1, the one-dimensional dynamic system (21) derived from this model has the unique and stable steady state depicted in Figure 3. As discussed, the initial value of the system, g_0, is not given. Hence, for any initial value of the state variable, n_{-1}, there are many possible equilibrium paths, $(g_0, g_1, g_2, \cdots)$, satisfying both the law of motion (21) and the transversality condition (4) because of the stability of the steady state (Benhabib and Farmer, 1994; Boldrin and Rustichini, 1994).[21] If, as in "normal" models of endogenous growth, the system has a unstable steady state, both (21) and (4) hold only in the steady state and the equilibrium is determinate.

Recall that Eq. (22) holds in every period. In period $t = 0$, for a given n_{-1}, an equilibrium path $n_{-1}^{1/(\sigma-1)}/w_0$ is consistent with (4) and (21) for *any* $w_0 \geq n_{-1}^{1/(\sigma-1)} \left[\theta^{\varepsilon} + (1-\theta)^{1-\varepsilon}\right]^{1/(\varepsilon-1)}$ because w is not a state variable and jumps freely. The following statement summarizes the above discussion.

This economy can exhibit indeterminacy of the initial values of the dynamic system. It is well known that if an equilibrium is locally indeterminate, sunspot phenomena can emerge in the neighborhood of a steady state. Thus, even in case 1, this economy can experience growth cycles near a steady state.[22]

5 Concluding Remarks

This paper has established the existence of global indeterminacy of the equilibrium paths in a variety expansion model of endogenous growth that allows for the diffusion of technology. More specifically, I show that endogenous growth cycles driven by self-fulfilling beliefs can be observed, as can development trap phenomena. If agents expect that the economy will grow at a constant rate, then it does so, and if the agents expect fluctuations, then the economy fluctuates.

The transition to a case with stable growth is characterized by lower start-up costs and more patient representative agents. One way to overcome fluctuations is to alter these fundamental parameters. Another method also exists that is related to the beliefs of agents. Even in the regime of global indeterminacy, the economy can experience stable growth if agents expect that the higher equilibrium path will always be realized.

The contribution of this paper is to show that the gradual diffusion of innovation, which has not been stressed in the literature on endogenous growth cycles, generates expectational indeterminacy through the self-reinforcing mechanism of innovation discussed in the paper, leading to self-fulfilling fluctuations of growing economies. The result of this paper hence implies that innovation is both the engine of long-run growth and the source of short-run fluctuations in an economy in which innovation gradually diffuses throughout the economy. This is because a self-reinforcing mechanism of innovation emerges due to the gradual diffusion of innovation.

[21] For more recent work, see Mino (2001), Nishimura and Venditti (2004), and Nishimura et al. (2005).

[22] It is easy to show this indeterminacy without the assumption of technology diffusion. In other words, a variety expansion model with temporary monopoly power predicts growth cycle phenomena with sunspot equilibria. See Furukawa (2005) for details.

Appendix

The transversality condition.

Multiplying (18) by w_t and using $(n_{t+1}-n_t)V_{t+1} = a^{RD} w_t g_{t+1}$ and $(n_t - n_{t-1})\pi_t = \frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}\frac{g_t[1-\alpha(g_t)]C_t}{1+\hat{\sigma}g_t}$, which are derived from (16) and (17), yields the national income account:

$$(n_{t+1}-n_t)V_{t+1} + C_t = (n_t - n_{t-1})\pi_t + w_t N \ . \tag{23}$$

From this national income account and the intertemporal budget constraint (2), we can obtain the asset market clearing condition:

$$A_t = (n_t - n_{t-1})V_t = \frac{(n_t - n_{t-1})\pi_t}{1+r_t} \ , \tag{24}$$

which, together with (3), (4), and (16), can be rewritten as

$$\lim_{T\to+\infty} \beta^T \frac{A_{T+1}}{C_T} = \lim_{T\to+\infty} \beta^{T+1} \frac{g_{T+1}[1-\alpha(g_{T+1})]}{1+\hat{\sigma}g_{T+1}} \leq \lim_{T\to+\infty} \beta^{T+1} Z[1-\alpha(Z)] = 0 \ ,$$

where Z is a supremum of g. Hence, the transversality condition is satisfied whenever a feasible set of the growth rate is bounded. As shown in the following section, the supremum of g exists, and thus the transversality condition is always satisfied in this model.

The configurations of L and δ.

The following lemmas summarize the properties of the functions L and δ.

Lemma 1 *$L(g)$ satisfies the following properties. (a) $L'(g) \leq 0$, $\forall g \geq 0$ holds. (b) $L(0) = 1$. (c) $\lim_{g\to+\infty} L(g) = (\sigma-1)/\sigma \in (0,1)$.*

Lemma 2 *(a) $\delta'(g)$ satisfies the following*

$$\delta'(g) \begin{cases} < 0, \ \forall g \in (0,+\infty) & \text{iff } \frac{N}{a^{RD}} \leq \kappa_2 \ , \\ \begin{cases} > 0, \ \forall g \in (0,\bar{g}) \\ \leq 0, \ \forall g \in [\bar{g},+\infty) \end{cases} & \text{iff } \frac{N}{a^{RD}} > \kappa_2 \ , \end{cases} \tag{25}$$

where $\hat{g}$ is the threshold value below which δ is increasing in g, and then $\hat{g}$ globally maximizes the function δ. (b) $\delta(x) \leq 0$ holds iff $x \geq N/a^{RD}$. (c) $\delta(0) > L(0)$ holds iff $\eta N > 1$.

Proof of Lemma 1. Differentiating (19) with respect to g_t yields

$$L'(g_t) = \frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma(1+\hat{\sigma}g_t)^2}\left[\alpha(g_t) - 1 - \hat{\theta}^{-1}\lambda\hat{\sigma}g_t\alpha(g_t)^2(1+\hat{\sigma}g_t)^\lambda\right] \leq 0 \ ,$$

since $1-\alpha(g_t) \geq 0$. The remainder of the proof is straightforward. $\|$

Proof of Lemma 2. Differentiating (21) yields the following equation:

$$\delta'(g) = \eta(1+\hat{\theta})\delta_1(g)\delta_2(g) \ ,$$

for which the following functions are defined:

$$\delta_1(g) \equiv \frac{(1+\hat{\sigma}g)^{\lambda-2}}{\left[\hat{\theta}+(1+\hat{\sigma}g)^\lambda\right]^2} \ ,$$
$$\delta_2(g) \equiv \hat{\theta}\left[(\hat{\sigma}N(\lambda-1) - a^{RD}) - \hat{\sigma}\lambda a^{RD} g\right] - (1+\hat{\sigma}g)^\lambda(a^{RD}+N\hat{\sigma}) \ .$$

The function $\delta(g)$ is increasing in g, $\delta'(g) \geq 0$, if and only if g $\delta_2(g) \geq 0$ holds, since $\delta_1(g)$ is always positive. The necessary and sufficient condition for $\delta_2(g) \geq 0$ is

$$\hat{\theta}(N\hat{\sigma}(\lambda-1)-a^{RD}) \geq \hat{\theta}\hat{\sigma}\lambda a^{RD}g+(a^{RD}+N\hat{\sigma})(1+\sigma g)^{\lambda} \ .$$

While the RHS of this inequality is increasing in g, and equal to $a^{RD}+N\hat{\sigma}$ when $g=0$, the LHS is constant, and there exists a sufficiently large $\hat{g}$ such that the RHS exceeds the LHS if and only if $g > \hat{g}$ whenever $\hat{\theta}\left[N\hat{\sigma}(\lambda-1)-a^{RD}\right] > a^{RD}+N\hat{\sigma}$ holds. Hence, the function $\delta(g)$ has an inverse U-shaped configuration if and only if this holds. Note that $\left[N\hat{\sigma}(\lambda-1)-a^{RD}\right]$ is always positive because of the assumption $\kappa_2 > 0$. ||

These lemmas confirm the configurations depicted in Figure 2(a)–(d). ||

Proof of Proposition 1.

Using Lemmas 1 and 2, the graphical analysis in Figure 2 suffices. ||

Proof of Proposition 3.

In case 3, I define $H(g_0) \equiv \bar{d}\left\{\bar{\bar{d}}\left[\bar{\bar{d}}(g_0)\right]\right\}-g_0$, and then the function H is continuous and satisfies $H'(g) < 0$. If $H(g)=0$ has a solution, there exists a period-three cycle. As depicted in Figure 5, for any $g_0 \in [0, g_L^*]$, the assumption $g_H^* < \bar{\bar{g}}$ ensures that

$$0 < \bar{d}\left\{\bar{\bar{d}}\left[\bar{\bar{d}}(g_L^*)\right]\right\} < \bar{d}\left\{\bar{\bar{d}}\left[\bar{\bar{d}}(0)\right]\right\} < g_L^*$$

holds. This implies that $H(0) > 0$ and $H(g_L^*) < 0$ holds, and $H(g_0)=0$ thus has a solution in $(0, g_L^*)$.

In case 4, for any $g \in (\bar{g}, g_L^*)$, the assumptions $\bar{g} < \bar{d}(g_H^*)$ and $g_H^* < \bar{\bar{g}}$ ensure that $\bar{d}\left\{\bar{\bar{d}}\left[\bar{\bar{d}}(g_0)\right]\right\} > \bar{g}$ holds. The remainder of the proof is the same as for case 3. ||

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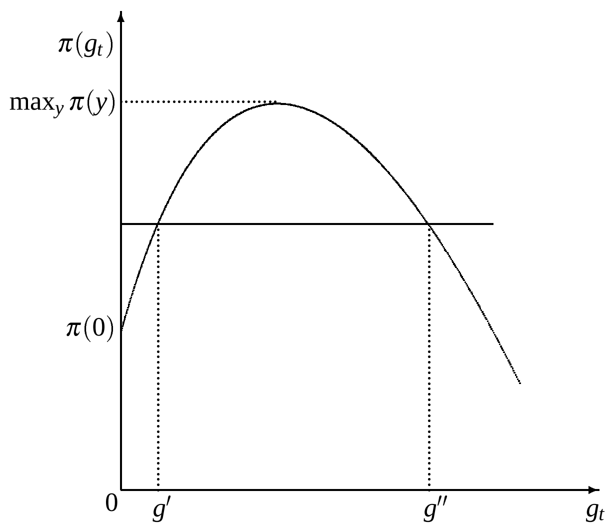

Figure 1: The profit function

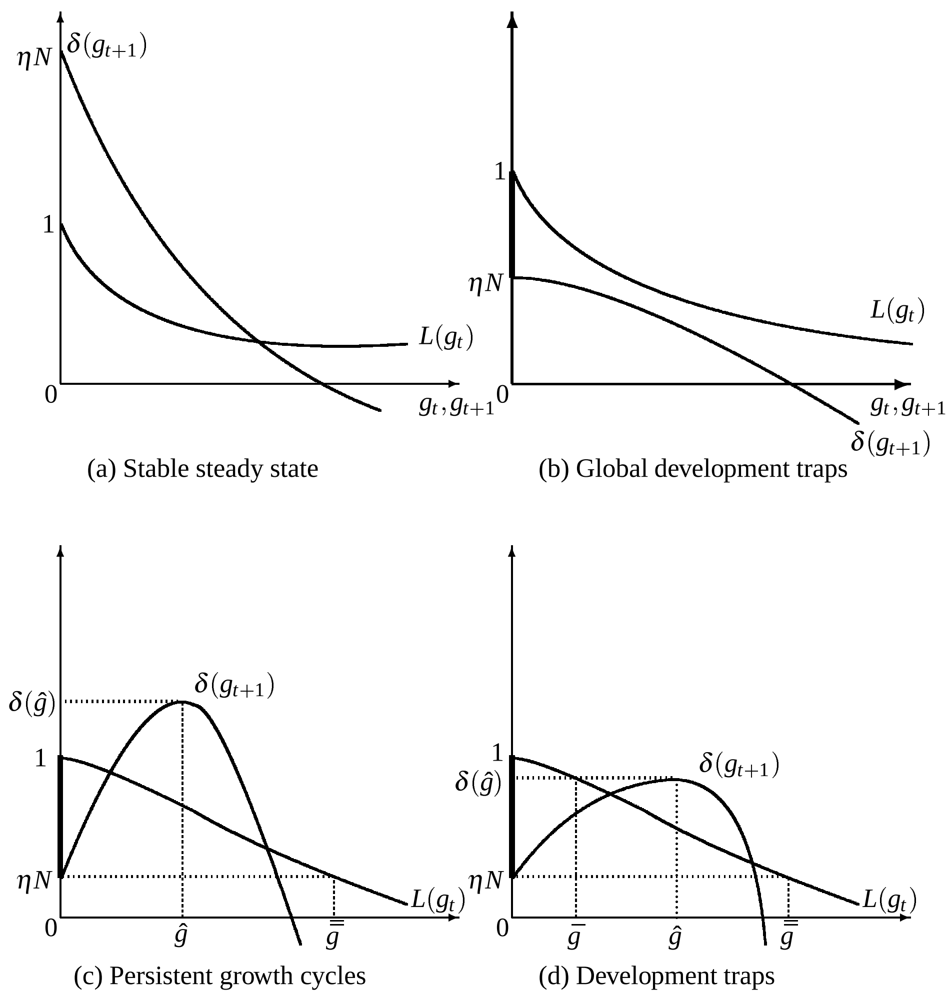

Figure 2: Functions L and δ

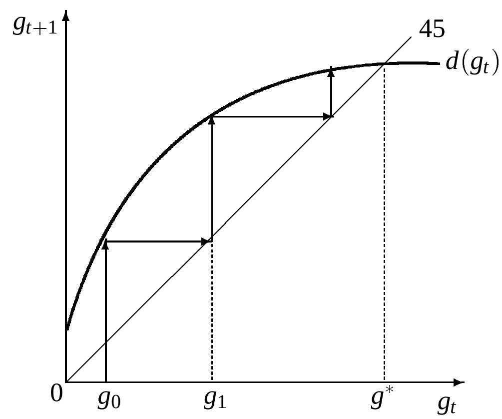

Figure 3: Stable steady state (case 1)

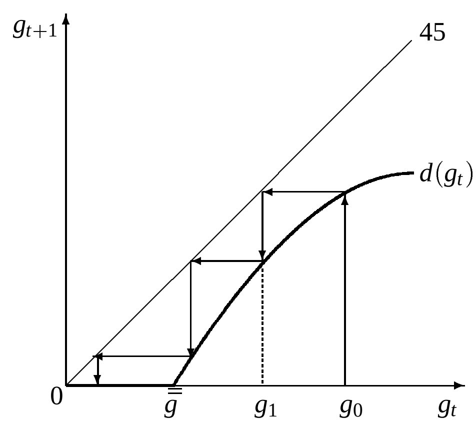

Figure 4: Global development traps (case 2)

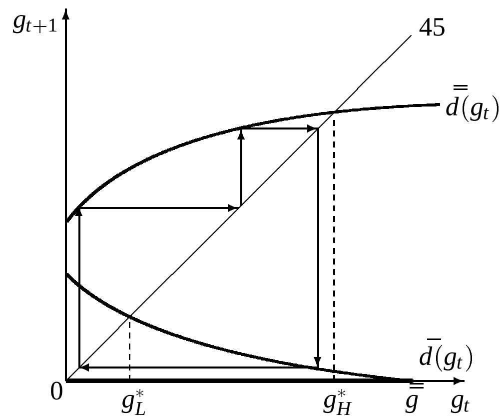

Figure 5: Persistent growth cycles (case 3)

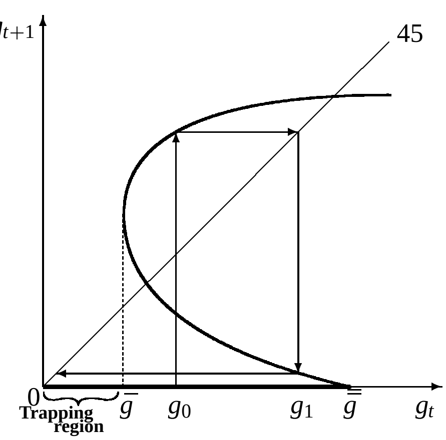

Figure 6: Cycles and traps (case 4)

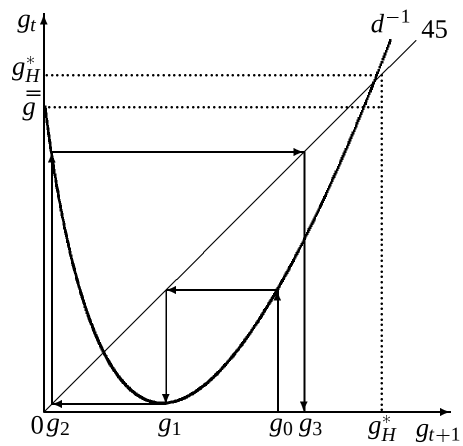

Figure 7: Period three in the forward dynamics: $\beta = 0.9$, $\varepsilon = 4.5$, $\theta = 0.6$, $\sigma = 3.3$, $a^{RD} = 0.495$, and $N = 27.37$. Then, $\overline{\overline{g}} = 4.24469 < g_H^* = 5.36541$ and $\delta(\hat{g}) = 0.998995 < 1$ (case 4).