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We provide an algorithm which finds an α-core strategy of any game in this class after O(n3・M) operations, where M is the maximum size of a strategy set of any player. The algorithm is also related to the vNM stable set: given any payoff vector not in the α-core, this algorithm returns an imputation in the α-core which dominates it. The idea of this method is based on the property of reduced games.

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Abstract

In this paper, we discuss the computational complexity of strategic cores of a class of n-person games defined by Masuzawa (2003, *Int.Jour.Game Theory 32* 479-83), which includes economic situations with monotone externality. We provide an algorithm which finds an α-core strategy of any game in this class after $O(n^3 \cdot M)$ operations, where M is the maximum size of a strategy set of any player. The algorithm is also related to the vNM stable set: given any payoff vector not in the α-core, this algorithm returns an imputation in the α-core which dominates it. The idea of this method is based on the property of reduced games.

Keywords:
NTU games, polynomial time algorithms, the α-core,
reduced games, ordinally convex game, marginal worth vectors,
monotone externality, public good provision,
international environmental pollution

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1 Introduction

Games with punishment-dominance relations are strategic form games introduced by Masuzawa (2003). In any game in this class, for any player i and any pair of his/her strategies $\{x_i, y_i\}$, the change of strategy from x_i to y_i makes the others' payoffs unanimously worse off for all the others' fixed strategies or unanimously better off for all the others' fixed strategies: strategy sets are complete with respect to *punishment-dominance relations*. In economics, the monotone externality implies this condition. The voluntary contribution game for production of a public good, the n-person prisoners' dilemma game, Cournot's oligopoly model of quantity competition, are all in this class. Masuzawa (2003) has shown that any game in this class yields a balanced and ordinally convex α-coalitional game, and that the α-core is, therefore, nonempty.

The aim of this paper is to propose an efficient algorithm to find a payoff vector in the α-core and the associated α-core strategy of a game with punishment-dominance relations. Here, we say that an algorithm for solutions on strategic form game is efficient if every computation by the algorithm terminates within a polynomial time with respect to the number of players and the sum of strategies over the players, where not only every elementary operation but also every evaluation of the payoff of a player is counted as one step.

First, I need to mention a reason why we discuss efficient computation of cores of games in this class. A solution concept indicates a set of outcomes, or a probability distribution on possible outcomes to be chosen by the players or to be recommended by the social planner. However, standard cooperative solutions are too complex to obtain efficient algorithms to find them for general classes of games, because the number of possible coalitions and that of combinations of strategies both increase exponentially as the population of players increases. Scarf (1967) has provided a constructive proof of nonemptiness of cores of NTU balanced coalitional games, which is an algorithm to find cores of NTU games given by coalitional forms. However, even in the case when the game is finite and the corner vectors characterizing the values of the coalitional game can be efficiently obtained, his algorithm is not efficient enough. It is because the algorithm checks all possible coalitions one after another at every pivoting step, the number of which is exponentially increasing with respect to the number of players.

Therefore, game theorists have focused on the computational complexity of solutions of special classes of games. On the one hand, some negative results are obtained : some solution concepts on some classes of games have been proved to be in the class of difficult problems. However, on the other hand, efficient algorithms are offered for other classes of games. They terminate within polynomial time with respect to the number of players or the length of "data" which characterize instances. In Bilbao (2000, chapter 4) a survey of computational complexity of solution concepts on classes of TU games is given.

In this paper, we show that the α-core of our class of games also overcomes such computational difficulties. We provide two efficient algorithms: one of them decides whether any given payoff vector is in the α-core or not, and the other finds a member of the α-core.

After introducing definitions and notations in the next section, we discuss, in section 3, an algorithm, called "algorithm 1", to decide whether or not any given payoff vector is in the α-core. Given a payoff vector attainable by the grand coalition, this algorithm yields a nonempty coalition (and their strategy) which improves upon the payoff vector if it is not in the α-core, otherwise the algorithm returns the empty coalition. Note that this algorithm for deciding a member of the α-core is not trivial. If a TU coalitional game is given by an oracle, it is easy to check whether a given payoff vector is in the core or not by the use of super-modular function maximization (sub-modular function minimization) algorithms due to Gröchel, Lovász, and Schrijver (1981), Iwata, Fleischer, and Fujishige (2001), Schrijver (2000), Iwata (2002). However, they are not applicable for NTU games.

In section 4, we propose an algorithm, called "algorithm 3", to find an α-core strategy efficiently by the use of algorithm 1 as a subroutine. The idea of this method will be stated in terms of strategic *reduced games*, the definition of which is given in subsection 4.1. A reduced game has smaller population players. We construct a member of the α-core of a larger population game from those of smaller population games.

Note that various notions of reduced games are proposed not only for coalitional games but also for strategic form games in order to characterize strategic solution concepts in terms of reduced games, as is in Peleg and Tijs (1996) and Takamiya (2001) . However, the definitions of them are not consistent with that of Greenberg-Peleg coalitional reduced games by Greenberg (1985) and Peleg (1985, 1986), who have extended the traditional TU coalitional reduced game by Davis and Maschler (1965) to the NTU coalitional game. On the contrary, our reduced strategic form game is consistent with Greenberg-Peleg reduced game: the α-coalitional game of our strategic reduced game is equal to the Greenberg-Peleg reduced game of the α-coalitional game of the original game. This remarkable property is important to realize our method.

In section 5, the property of the outcome obtained by algorithm 3 will be discussed in detail. If the reference point is given by a symmetric strategy profile, this algorithm terminates with a symmetric α-core strategy. On the other hand, Given a payoff vector attainable by the grand coalition as a reference point, algorithm 3 yields an imputation which dominates it if it is outside the α-core, otherwise it returns the reference point itself. Thus, our algorithm prescribes " the standard of behavior", the result of which is the vNM stable set.

In section 6, it will be shown that, with a slight modification, this algorithm can compute a payoff vector which gives every player his/her marginal contribution according to an ordering, called a *marginal worth vector*. As a direct consequence of it, we obtain that the α-cores of games with punishment-dominance relations include all marginal worth vectors. Here, note that the core of an ordinally convex game does not always include marginal worth vectors as is pointed out by Hendticx, Borm, and Timmer (2002), while a TU coalitional game is convex if and only if all marginal worth vector is in the core.

Note that our algorithm for finding a member of the α-core is also not trivial regardless of these properties of the marginal worth vectors. In other words, it is not easy, in general, to obtain a marginal worth vector of a strategic form game. The reasons for this are twofold. First, we will focus on the core of a situation without sidepayments, the structure of which is more complicated than a TU coalitional game. Second, we assume that the initial data is given not by a coalitional game but by a strategic form game and we will find not only payoffs but also the associated strategies. If a TU coalitional convex game is given by an oracle, it is easy to obtain a payoff vector in the core. In fact, we only have to compute a marginal worth vector, which can be done with in $O(n)$. However, when we confront a strategic form game or a coalitional game in a situation without sidepayments, it is not easy even in the case that we know in advance that the marginal worth vectors of the game are in the core.

In section 7, a numerical example of asymmetric 20-person game with $2^{20\times 4}$ strategy profiles will be discussed to illustrate our algorithms.

2 Preliminaries

We introduce notations and definitions. Let $N := \{1, 2, \ldots, n\}$ be a set of *players*. For all $i \in N$, let $(A^i, \geq^i)$ be a *utility space* of player i, where $\geq^i$ is a reflexive, anti-symmetry, and transitive binary relation on A^i. An element of A^i is called a *payoff*, $\geq^i$ a *preference relation*. Throughout this paper, it is assumed that A^i is a compact topological space, and that $\{a \in A^i : a \geq b\}$ and $\{a \in A^i : b \geq a\}$ are both closed for all $b \in A^i$. For all $i \in N$ and all $a^i, b^i \in A^i$, we write $a^i >^i b^i$ if $a^i \geq^i b^i$ and not $a^i \neq b^i$. Sometimes, A^i is abbreviated as A, $\geq^i$ as $\geq$, and $>^i$ as $>$.

Note that the completeness of $\geq^i$ is not assumed, which makes it possible to regard a player not only as an individual but also as a society of individuals which adopts the unanimity rule. If we consider an international environmental pollution problem such as the CO_2 problem today, this generalization is essential.

The Cartesian product of A^i in S is denoted by A^S. Typical elements of A^S are denoted by $a^S, b^S, c^S \ldots$, and called *payoff vectors* of S, the restriction into A^T of which are sometimes denoted by $a^T, b^T, c^T, \ldots$ respectively for all $T \subset S$. We write that, for all $a^S, b^S \in A^S$, $a^S \geq b^S$ if and only if, for all $i \in S$, $a^i \geq b^i$. Moreover, we write that, for all $a^S, b^S \in A^S$, $a^S \gg b^S$ if and only if, for all $i \in S$, $a^i > b^i$.

A *strategic form game* is a list $G := (N, (X^i)_{i \in N}, (u^i)_{i \in N})$ such that $N := \{1, 2, \ldots n\}$ is a nonempty finite set of players, and, for all $i \in N$, X^i is a nonempty set of *strategies* of $i \in N$, $u^i : \prod_{i \in N} X^i \to A$ is a *payoff function* of player i. A subset of N is called a *coalition*. For all $S \subset N$ $(S \neq \emptyset)$, we refer to the Cartesian product of X^i in S by X^S. Typical elements of X^S are denoted by $w^S, x^S, y^S, z^S, \ldots$. Sometimes, we refer to the restrictions of $w^S, x^S, y^S, z^S, \ldots$ to X^T by $w^T, x^T, y^T, z^T, \ldots$ respectively, where $T \subset S$. By u^S, we refer to a vector valued function defined by $u^S(x^N) := (u^i(x^N))_{i \in S}$ for all $S \subset N$. Throughout this paper, we assume:

C(compactness of strategy space and continuity of payoff functions)
For all $i \in N$, X^i is a compact topological space, and u^i is continuous.

Next, we introduce the notion of the *coalitional game*, which plays an important role to realize the properties of cooperative solutions. An *NTU coalitional game* is a correspondence

$$V : 2^N \setminus \{\emptyset\} \twoheadrightarrow A^N \text{ such that}$$

for all $a^N, b^N \in A^N$ and all $S \in 2^N \setminus \{\emptyset\}$, if $a^N \in V(S)$ and $a^S \geq b^S$, then $b^N \in V(S)$.

That is, a payoff vector a^S can be guaranteed by S itself if and only if

$$(a^S, b^{N\setminus S}) \in V(S) \text{ for all } b^{N\setminus S} \in A^{N\setminus S}.$$

Clearly, value $V(S)$ does not convey any information about any player in $N \setminus S$. Thus, sometimes, it is convenient to denote the projection of $V(S)$ onto A^S by $\hat{V}(S)$; that is,

$$\hat{V}(S) = \{a^S \in A^S : (a^S, b^{N\setminus S}) \in V(S) \text{ for all } b^{N\setminus S} \in A^{N\setminus S}\}.$$

We say that $S \subset N$ *improves upon* $a^N \in A^N$ *via* $b^S \in A^S$ if and only if

$$b^S \in \hat{V}(S), \text{ and } b^S \gg a^S.$$

A payoff vector $a^N \in A^N$ is in the *core of* V if (i) $a^N \in V(N)$ and (ii) there exists no $S \neq \emptyset$ which improves upon a^N.

The notion of the α-core can be stated in terms of the coalitional game. Let $G := (N, (X^i)_{i\in N}, (u^i)_{i\in N})$ be a strategic form game. The *α-coalitional game associated with* G is an NTU coalitional game, $V_\alpha : 2^N \setminus \{\emptyset\} \twoheadrightarrow A^N$, defined by

$$V_\alpha(S) := \begin{cases} \bigcup_{x^S \in X^S} \bigcap_{z^{N\setminus S} \in X^{N\setminus S}} \{b^N \in A^N : b^S \leq u^S(x^S, z^{N\setminus S})\} & \text{if } S \neq N, \\ \bigcup_{x^S \in X^S} \{b^N \in A^N : b^S \leq u^S(x^S)\} & \text{otherwise.} \end{cases}$$

The *α-core* of a strategic game G is defined as the core of its α-coalitional game V_α.

To discuss the α-core not only as a set of payoff vectors but also as one of strategy profiles, let us introduce additional terminologies. We say that a strategy profile $x^N \in X^N$ is an *α-core strategy* of G if and only if payoff vector $u^N(x^N) \in A^N$ is in the α-core of G. We say that S *improves upon* $a^N \in A^N$ *via* y^S if,

$$\text{for all } z^{N\setminus S} \in X^{N\setminus S}, u^S(y^S, z^{N\setminus S}) \gg a^S.$$

Note that, under **C**, $a^N \in V_\alpha(N)$ is in the α-core if and only if it can not be improved upon by any coalition via any strategy.

We introduce the punishment-dominance relation. For all $i \in N$ and all $x^i, y^i \in X^i$, we write $P(x^i, y^i)$ and say that x^i is *punishment-dominant over* y^i if,

$$\text{for all } z^{N\setminus\{i\}} \in X^{N\setminus\{i\}} \text{and all } j \in N \setminus \{i\},\ u^j(z^{N\setminus\{i\}}, x^i) \leq u^j(z^{N\setminus\{i\}}, y^i).$$

That is, $P(x^i, y^i)$ implies that a change of the strategy of a player $i \in N$ from y^i to x^i makes the other players' payoffs unanimously worse off whatever a strategy profile of the others' is. Obviously, the punishment-dominance relation is reflexive and transitive.

The following property will play an important role throughout this paper. Suppose that $x^N, y^N \in X^N$ satisfy that $P(x^i, y^i)$ for all $i \in N$. Then,

$$\text{for all } S \subset N \text{ and all } z^{N\setminus S} \in X^{N\setminus S}, u^{N\setminus S}(x^S, z^{N\setminus S}) \leq u^{N\setminus S}(y^S, z^{N\setminus S}).$$

Indeed, if $|S| = 1$, it is obvious. Suppose that it holds for all $S \subset N$ such that $|S| < k$. Then, for all $S \subset N$ such that $|S| = k$ and $j \in S$, the following holds:

$$\begin{aligned} u^{N\setminus S}(x^S, z^{N\setminus S}) &= u^{N\setminus S}(x^j, x^{S\setminus\{j\}}, z^{N\setminus S}) \\ &\leq u^{N\setminus S}(y^j, x^{S\setminus\{j\}}, z^{N\setminus S}) \\ &\leq u^{N\setminus S}(y^j, y^{S\setminus\{j\}}, z^{N\setminus S}) = u^{N\setminus S}(y^S, z^{N\setminus S}). \end{aligned}$$

In this paper we assume that:

PD(completeness with respect to the punishment-dominance relation)
For all $i \in N$ and all $x^i, y^i \in X^i$, either $P(x^i, y^i)$ or $P(y^i, x^i)$.

We call a game which satisfies **PD** a *game with punishment dominance relation* or *a game with PD* for short. In economics, this condition means monotone externality. In Masuzawa (2003), some economic instances of game with PD are provided. Without completeness of $(A_i, \geq_i)_{i\in N}$, one can easily show that:

Lemma 1 (Masuzawa (2003))
Suppose that a strategic form game, G, satisfies **PD**. Then, the α coalitional game, V_α, satisfies that,
for all $S, T \subset N$, $V_\alpha(S) \bigcap V_\alpha(T) \subset V_\alpha(S \cup T)$.

From this lemma, the α-coalitional game is ordinally convex and balanced and, thus, the α-core is nonempty if, for all $i \in N$, A^i is the real number with natural topology and $\geq^i$ means " larger that or equal to". In this paper, we will derive a non-emptiness result without the completeness of $\geq^i$.

Note that the lemma needs no convexity condition on strategy sets. Thus, the lemma 1 holds for finite games, which, hereafter, we will discuss exclusively. We assume that:

F(finiteness of strategy space)
For all $i \in N$, X^i is finite.

While economic situations are usually and conventionally represented by games with infinite strategy sets, they can also be represented as finite games by taking the minimum unit of a scale into the consideration. For example, in case of an oligopoly game of Cournot's with bounded capacity, which satisfies **PD**, the minimum unit of a scale of the product makes the game finite.

We assume that strategic form game G is given by a list, $(N, (X^i)_{i \in N})$, and an "oracle" which, for all $i \in N$ and all $x^N, y^N \in X^N$, decides whether or not $u^i(x^N) \geq^i u^i(y^N)$. That is, not only an elementary operation but also a payoff comparison is counted as one step for the evaluation of the time complexity of an algorithm. Even if the cost of the payoff comparison depends on n, the analysis of the time complexity may be useful unless the cost is not so rapidly increasing in n. This assumption is parallel to the oracle-polynomial time discussion of TU coalitional games where an evaluation of the worth of a coalition is counted as $O(1)$, as is discussed in Bilbao (2000), Faigle, Kern, and Kuipers (2001), and so on. We will discuss this assumption once again in the concluding remarks.

For all $i \in N$, let $\hat{P}^i$ be a binary on X^i such that for all $i \in N$ and all $x^i, y^i \in X^i$ (i) $\hat{P}^i(x^i, y^i)$ implies $P(x^i, y^i)$ and (ii) exactly one of three alternatives, $\hat{P}^i(x^i, y^i)$, $\hat{P}^i(y^i, x^i)$, and $x^i = y^i$, holds. We assume that, for all $i \in N$, X^i is sorted with respect to $\hat{P}^i$. That is, each of a reference to the maximum of X^i, one to the next larger element and one to the next smaller element with respect to $\hat{P}^i$ for any given $x^i \in X^i$ is counted as one step respectively. In some economic situation, this condition is plausible. For example, in a public good provision game discussed in Masuzawa (2003), for a contribution level of the private good for public good provision, the next punishment-dominant strategy is the next lower level of the contribution. One may think that the oracle for payoff comparison is sufficient and the new assumption is not necessary. However, without this condition, it may cost a considerable time to check which of $P(x^i, y^i)$ or $P(y^i, x^i)$ holds for any given pair $\{x^i, y^i\}$ of X^i, because condition PD does not require the strict inequality.

In a usual model of computation treating the integers such as *random access machine*, which is also implicitly adopted here, this sorted strategy assumption is essentially equivalent to assuming that:

SOR(sorted strategy sets with respect to the punishment-dominance relation)
For all $i \in N$, $X^i = \{0, 1, 2, \dots, m^i\}$, and $P(x^i, y^i)$ if $x^i < y^i$.

SOR comprises an assumption on the data structure of a game with **PD** and **F**. That is, under **SOR**, for variable $\boldsymbol{x}^i$ indicating i's strategy, each of "$\boldsymbol{x} := \boldsymbol{x}^i - 1$" , "$\boldsymbol{x}^i := \boldsymbol{x} + 1$" is counted as one step.

Sometimes we will use notation $0^i, 1^i, 2^i, \dots$ to distinguish strategies $0, 1, 2 \dots$ of player i's from those of the others'. For all $S \subset N$, we define $0^S \in X^S$ by $0^S := (0^i)_{i \in S}$. For all $Y \subset X^N$, we say that $z^N \in X^N$ is the *maximum* of Y if $z^N \in Y$ and, for all $y^N \in Y$, $z^N \geq y^N$, and that $z^N \in X^N$ is a *maximal* of Y if $z^N \in Y$ and there exists no $y^N \in Y$ such that $y^N \geq z^N$ and $y^N \neq z^N$.

Obviously, if **SOR** is satisfied, then, for all $i \in N$ and all $y^i \in X^i$, $P(0^i, y^i)$. Thus, a strategy profile $x^N \in X^N$ is an α-core strategy if and only if there exists no pair of $S \neq \emptyset$ and $y^N \in X^N$ such that $u^S(y^N) \gg u^S(x^N)$ and $y^{N \setminus S} = 0^{N \setminus S}$. In this paper, we assume **SOR**, which is stronger than (**PD**+**F**).

3 Finding the maximum improving coalition

In this section, we will introduce a class of combinatorial optimization problems, and an efficient algorithm to solve them. As application, we obtain an efficient algorithm to check members of the α-core in this section, and, in the next section, we will derive a method to find an element of the α-core from the algorithm.

Let N_1, N_2 be coalitions such that $\{N_1, N_2\}$ is a partition of N. Let a^N be any element of A^N. Consider the following problem, called a *Maximum Guaranteeing Strategy Problem* (MGSP for short):

> Find the maximum element of feasible solutions, $x^N \in X^N$, of the following inequality system, (1):

$$\begin{cases} \text{for all } i \in N_1, \quad x^i = 0 \ \text{ or } u^i(x^N) \geq a^i, \\ \text{for all } i \in N_2, \quad x^i = 0 \ \text{ or } u^i(x^N) > a^i. \end{cases} \tag{1}$$

Here, we say that $x^N \in X^N$ is a *feasible* solution of (1) if it satisfies (1). Clearly system (1) has at least one feasible solution, 0^N. For all $i \in N$ and all $x^N \in X^N$, we say that *the i-th condition holds* for x^N if and only if

$$i \in \{j \in N_1 \colon x^j = 0^j \text{ or } u^j(x^N) \geq a^j.\} \bigcup \{j \in N_2 \colon x^j = 0^j \text{ or } u^j(x^N) > a^j.\}.$$

In order to see the deep connection of MGSPs and the α-cores of our games, here, consider the case when

$$N_1 = \emptyset, N_2 = N, \text{ and } a^N \in V_\alpha(N).$$

Then, for any feasible solution y^N of (1), coalition

$$S := \{i \in N : u^i(y^N) > a^i\} \text{ improves upon } a^N \text{ via } y^S$$

because, for all $i \in N$, $\neg(u^i(y^N) > a^i)$ implies that $y^i = 0^i$.

Especially, the maximum tells us whether or not a^N is in the α-core. Let y_*^N be the maximum of feasible solutions, the existence of which will be proved later. Suppose that $S := \{i : u^i(y_*^N) > a^i\} \neq \emptyset$. Note that this condition is satisfied as long as $y_*^N \neq 0^N$. Then, $y_*^{N \setminus S} = 0^{N \setminus S}$. Thus, $a^N \in V_\alpha(N)$ is not in the α-core and S improves upon $a^N \in A^N$ via y_*^S.

Conversely, suppose that $S := \{i : u^i(y_*^N) > a^i\} = \emptyset$. Then, $y_*^N = 0^N$ and by the definition of the maximum, there exists no $z^N \in X^N$ such that $T := \{i : u^i(z^N) > a^i\} \neq \emptyset$ and $z^{N \setminus T} = 0^{N \setminus T}$. Indeed, if so,

$$z^N \neq y_*^N, z^i \geq y_*^i = 0^i \text{ for all } i \in N, \text{ and } z^N \text{ is a feasible solution of (1)},$$

which contradicts the definition of y_*^N. Thus, $a^N \in A^N$ is in the α-core.

Let us return to the general discussion of MGSPs. We provide a polynomial time algorithm which finds the maximum. As a corollary, we obtain the result that the maximum exists.

Let $h^N \in X^N$ be any strategy profile such that $h^N \geq z^N$ for all feasible solutions $z^N \in X^N$ of (1). Note that $(m^i)_{i \in N}$ is a candidate for such h^N. As will be shown later, this generalization contributes to the low level of the time complexity of the algorithm to find a member of the α-core.

Consider the following algorithm.

Algorithm 1

Input: a game G with **SOR**, a vector $a^N \in A^N$,

a partition $\{N_1, N_2\}$ of N,

$h^N \in X^N$ such that $h^N \geq z^N$ for all feasible $z^N \in X^N$ for (1).

Output: $\boldsymbol{y}^N \in X^N$.

Step 1 Set $\boldsymbol{y}^i := h^i$ for all $i \in N$;

Set $\boldsymbol{S} := \{i \in N : h^N$satisfies the i-th condition.$\}$;

Step 2 **while** $\boldsymbol{S} \neq N$ **begin**

Set $\boldsymbol{y}^i := \boldsymbol{y}^i - 1$ for all $i \in N \setminus \boldsymbol{S}$;

Set $\boldsymbol{S} := \{i \in N : \boldsymbol{y}^N$satisfies the i-th condition.$\}$ **end**.

Proposition 1

(i) Algorithm 1 terminates within finite steps.

Let y_*^N be an output of algorithm 1 and define $S_* \subset N$ by

$$S_* := \{i \in N_1 : u^i(y_*^N) \geq a^i\} \cup \{i \in N_2 : u^i(y_*^N) > a^i\}.$$

Then,

(ii) y_*^N is the maximum of feasible solutions for (1),

(iii) for all feasible $z^N \in X^N$ of (1),

$$S_* \supset \{i \in N_1 : u^i(z^N) \geq a^i\} \cup \{i \in N_2 : u^i(z^N) > a^i\}.$$

Proof: Let us show by recursion of steps of the computation that, every time that $\boldsymbol{S}$ is updated, $\boldsymbol{y}^N \geq z^N$ for all feasible solutions z^N.

Let $z^N \in X^N$ be a feasible solution. Suppose that, at a point of time of the computation, the value of $\boldsymbol{y^N}$ is y^N, $y^N \geq z^N$, and $\boldsymbol{S} = \{j \in N :$ the i-th condition holds for $y^N\}$. That is, we consider a point of time immediately after $\boldsymbol{S}$ is updated according to the current value of $\boldsymbol{y}$.

Define $T \subset N$ by

$$T := \{i \in N : y^i = z^i\}.$$

Then, for all $i \in T$,

$$u^i(y^N) \geq u^i(y^i, z^{N \setminus \{i\}}) = u^i(z^i, z^{N \setminus \{i\}}).$$

Thus, for all $i \in T$, the i-th condition holds not only for z^N but also for y^N. It follows that $T \subset \boldsymbol{S}$ then. Thus, $\boldsymbol{y}^N \geq z^N$ after the next update of $\boldsymbol{S}$. Recursively, we obtain the result that $\boldsymbol{y}^N \geq z^N$ in any step of the computation.

Because the feasible solution, z^N, is arbitrarily chosen, $\boldsymbol{y}^N$ is the maximum if it is feasible. Moreover, $\boldsymbol{y}^N$ is strictly decreasing as long as it is not feasible. Thus, finally, $\boldsymbol{y}^N$ becomes a feasible solution, which is the maximum, and, then, the computation terminates.

Next, consider (iii). It is clear that $y_*^N \geq z^N$ because y_*^N is the maximum of feasible solutions and z^N is a feasible solution.
Suppose that $i \notin S_*$ and $i \in N_1$, ($i \in N_2$, resp.). Then

$$0^i = y_*^i \geq z^i \geq 0^i$$

and, thus,

$$u^i(y_*^N) \geq u^i(y_*^i, z^{N\setminus\{i\}}) = u^i(z^N).$$

It follows that $\neg(u^i(z^N) \geq a^i)$, ($\neg(u^i(z^N) > a^i)$, resp.). Therefore,

$$i \notin \{j \in N_1 : u^j(z^N) \geq a^j\} \cup \{j \in N_2 : u^j(z^N) > a^j\}.\Box$$

Now let us examine the time complexity of algorithm 1. Any cycle costs $O(n)$, and, in any cycle, at most one of players decreases his strategy by 1. Thus, the time complexity of algorithm 1 is

$$O(n \cdot \sum_{i \in N} m^i).$$

Precisely, it terminates within

$$O(n \cdot \sum_{i \in N} (h^i - y_*^i)) \text{ times operations},$$

where y_*^N is the value of $\boldsymbol{y}^N$ when the algorithm terminates. One may think that this evaluation is not adequate for the evaluation of the algorithm because it depends not only on input but also on output. However, as will be shown later, this fact is important when we repeat the algorithm to obtain a member of the α-core.

To back to the problem of finding an improving coalition, we obtain the following result as a corollary of proposition 1:

Corollary 1
Let $N_1 = \emptyset$, $N_2 = N$, and let $a^N \in A^N$ be any payoff vector in $V_\alpha(N)$. Let $y_*^N \in X^N$ be the output of the computation by algorithm 1 in this case, and define S_* by $S_* := \{i \in N : u^i(y_*^N) > a^i\}$. Then, the following statements hold.

1. If $a^N \in V_\alpha(N)$ is in the α-core, then $S_* = \emptyset$ and $y_*^N = 0^N$.

2. If $a^N \in V_\alpha(N)$ is not in the α-core, then

   (a) $S_* \neq \emptyset$ and S_* improves upon a^N via $y_*^{S_*}$,

   (b) S_* is the largest improving coalition and y_*^N is the maximum improving strategy for a^N: $T \subset S_*$, $y_*^N \geq z^N$ if T improves upon a^N via z^T and $z^{N\setminus T} = 0^{N\setminus T}$.

4 Finding an α-core strategy

In the previous section, we have discussed a polynomial time algorithm to solve MGSPs. From this, we will derive an efficient method to find an α-core strategy of any game in this class.

4.1 Reduced games

The basic idea of our method can be stated in terms of reduced games defined by Greenberg (1985) and Peleg (1985, 1986). They have defined the[1] *reduced coalitional game*, $V_* : 2^S \twoheadrightarrow A^S$, of a NTU coalitional game, V, with respect to $S \subset N$ $(S \neq N)$ and $a^{N\setminus S} \in A^{N\setminus S}$ by, for all $T \subset S$ $(T \neq S)$,

$$V_*(T) := \bigcup_{R \subset N\setminus S.} \bigcup_{c^{N\setminus S} \in A^{N\setminus S}.} \{c^S : (c^S, c^{N\setminus S}) \in V(T \cup R), c^R \geq a^R\}, \text{ and}$$

$$V_*(S) := \bigcup_{c^{N\setminus S} \in A^{N\setminus S}} \{c^S : (c^S, c^{N\setminus S}) \in V(N), c^{N\setminus S} \geq a^{N\setminus S}.\}.$$

They have derived the non-emptiness of the core of an ordinally convex game recursively from the properties of the reduced games.

Here, we will define the reduced games of a strategic form game with **SOR**, the α-coalitional games of which are equal to Greenberg-Peleg reduced games of the

[1]This definition is due to Peleg (1986), which is slightly different from the others.

α-coalitional game of the original strategic game . Following this, we will extend the α-cores of the smaller population games to those of the larger population games.

First we define *subgames.*

Definition 1
For all $S \subset N$ ($\emptyset \neq S \neq N$), we define *the subgame of* G *with respect to* S, denoted by $G(S)$, as a strategic form game such that

- the set of players is S,
- the strategy set of $i \in S$ is X^i,
- the payoff function of $i \in S$ is defined by $u^i(x^S, 0^{N\setminus S})$ for all $x^S \in X^S$.

It can be easily shown that the α-coalitional game, V_α^S, of $G(S)$ satisfies that, for all $T \subset S$ ($T \neq \emptyset$)

$$V_\alpha^S(T) = \{c^S : c^T \in \hat{V}(T).\}$$

Definition 2
Let $S \subset N$ ($\emptyset \neq S \neq N$) and $a^{N\setminus S} \in A^{N\setminus S}$. The *reduced game of* G *with respect to* list $(S, a^{N\setminus S})$, denoted by $G_*(S, a^{N\setminus S})$, is a strategic form game such that

- the set of player is S,
- the set of strategies of $i \in S$ is X^i, and
- the payoff function of $i \in S$, u_*^i, is defined by for all $x^S \in X^S$

  $$u_*^i(x^S) := u^i(x^S, y_*^{N\setminus S}),$$

  where $y_*^{N\setminus S}$, called the associated strategy for x^S, is the maximum of feasible solutions, $y^{N\setminus S} \in X^{N\setminus}$, of the following system:

$$y^i = 0 \text{ or } u^i(x^S, y^{N\setminus S}) \geq a^i \text{ for all } i \in N \setminus S. \tag{2}$$

From the discussion of the previous section, for all $x^S \in X^S$ and all $i \in S$, $u_*^i(x^S)$ is well defined, and, for all feasible $y^{N\setminus S}$ of system (2), $u_*^i(x^S) \geq u^i(x^S, y^{N\setminus S})$.

First, following proposition holds:

Proposition 2
If a strategic game G satisfies **SOR**, then for all $S \subset N$ and $a^{N\setminus S} \in A^{N\setminus S}$, $G_*(S, a^{N\setminus S})$ also satisfies **SOR**.

Proof: Let $i, j \in S$ ($i \neq j$), $x^i, y^i \in X^i$ ($x^i \geq y^i$), $z^{S\setminus\{i\}} \in X^{S\setminus\{i\}}$. Suppose that $x^{N\setminus S}, y^{N\setminus S} \in X^{N\setminus S}$ are the associated strategies for $(z^{S\setminus\{i\}}, x^i)$ and $(z^{S\setminus\{i\}}, y^i)$ respectively. They determine the payoff vectors $u_*^S(z^{S\setminus\{i\}}, x^i)$ and $u_*^S(z^{S\setminus\{i\}}, y^i)$ respectively, where u_*^S is the payoff function of $G_*(S, a^{N\setminus S})$.
Then, from **SOR**, for all $k \in N \setminus S$ with $y^k \neq 0$,

$$u_k(z^{S\setminus\{i\}}, x^i, y^{N\setminus S}) \geq u_k(z^{S\setminus\{i\}}, y^i, y^{N\setminus S}) \geq a_k.$$

Then, from the definition of the maximum, $x^{N\setminus S} \geq y^{N\setminus S}$. Thus,

$$u_*^j(z^{S\setminus\{i\}}, x^i) = u^j(z^{S\setminus\{i\}}, x^i, x^{N\setminus S}) \geq u^j(z^{S\setminus\{i\}}, y^i, y^{N\setminus S}) = u_*^j(z^{S\setminus\{i\}}, y^i).$$

Thus, $G_*(S, a^{N\setminus S})$ satisfies **SOR**. $\square$

From this proposition, it can be easily shown that the α-coalitional game of $G_*(S, a^{N\setminus S})$, denoted by $V_{**\alpha}$, satisfies that: for all $T \subset S$,

$$V_{**\alpha}(T) = \bigcup_{R\subset N\setminus S.} \bigcup_{c^{N\setminus S}\in A^{N\setminus S}.} \{c^S : (c^S, c^{N\setminus S}) \in V_\alpha(T \cup R), c^R \geq a^R.\}. \tag{3}$$

Furthermore, consider the case when

$$a^{N\setminus S} \in \hat{V}_\alpha(N \setminus S).$$

Let $T \subset S$ be nonempty coalition. Suppose that $c^S \in V_{**\alpha}(T)$. Then, from equation (3),

$$(c^S, a^{N\setminus S}) \in V_\alpha(T \cup R) \text{ for some } R \subset N \setminus S.$$

Moreover, from lemma 1,

$$V_\alpha(T \cup R) \cap V_\alpha(N \setminus S) \subset V_\alpha((N \setminus S) \cup T) \text{ for all } R \subset N \setminus S.$$

Thus, we obtain the result that, for all nonempty $T \subset S$,

$$V_{**\alpha}(T) = \bigcup_{c^{N\setminus S}\in A^{N\setminus S}} \{c^S : (c^S, c^{N\setminus S}) \in V_\alpha((N \setminus S) \cup T), c^{N\setminus S} \geq a^{N\setminus S}.\}. \tag{4}$$

Especially,

$$V_{**\alpha}(S) = \bigcup_{c^{N\setminus S}\in A^{N\setminus S}} \{c^S : (c^S, c^{N\setminus S}) \in V_\alpha(N), c^{N\setminus S} \geq a^{N\setminus S}.\}. \tag{5}$$

Thus, the notion of the reduced games of our version is consistent with that of Greenberg and Peleg.

In every reduced game, the maximum improving coalition for any given payoff vector can be found by algorithm 1 because a MGSP of a reduced game can be converted into a MGSP of the original game; that is, we obtain the following proposition.

Proposition 3

Let $S \subset N$, $a^{N\setminus S} \in A^{N\setminus S}$, and, for all $j \in S$, u_*^j be the payoff function of $G_*(S, a^{N\setminus S})$. Then, x_*^S is the maximum of feasible solutions, $x^S \in X^S$, of system

$$\text{for all } i \in S, \quad x^i = 0 \text{ or } u_*^j(x^S) > a^i \tag{6}$$

if and only if, for some $y_*^{N\setminus S}$, $(x_*^S, y_*^{N\setminus S})$ is the maximum of feasible solutions, $(x^S, y^{N\setminus S}) \in X^N$, of system

$$\begin{cases} \text{for all } i \in S, & x^i = 0 \text{ or } u^i(x^S, y^{N\setminus S}) > a^i, \\ \text{for all } i \in N \setminus S, & y^i = 0 \text{ or } u^i(x^S, y^{N\setminus S}) \geq a^i. \end{cases} \tag{7}$$

Proof: Let x_*^S be the maximum solution of (6), $x_*^{N\setminus S}$ the associated strategy of x_*^S, and z_*^N be the maximum solution of (7). We shall prove that $x_*^N = z_*^N$. Because z_*^N is the maximum of (7) and x_*^N also satisfies (7), we have that

$$z_*^N \geq x_*^N.$$

Because z_*^N is the maximum of (7), $z_*^{N\setminus S}$ is the maximum of feasible solutions, $y^{N\setminus S} \in X^{N\setminus S}$, of the following system:

$$\text{for all } i \in N \setminus S, \quad y^i = 0 \text{ or } u^i(z_*^S, y^{N\setminus S}) \geq a^i.$$

Thus, z_*^S is also a feasible solution of system (6). Thus, $x_*^S \geq z_*^S$. It follows, moreover, that

$$\text{for all } i \in N \setminus S, \quad x_*^i = 0 \text{ or } u^i(x_*^S, z_*^{N\setminus S}) \geq a^i.$$

Thus, $x_*^{N\setminus S} \geq z_*^{N\setminus S}$ because $x_*^{N\setminus S}$ is the maximum of feasible solutions, $x^{N\setminus S} \in X^{N\setminus S}$, of the system

$$\text{for all } i \in N \setminus S, \quad x^i = 0 \text{ or } u^i(x_*^S, x^{N\setminus S}) \geq a^i.$$

□

4.2 Algorithm to find a member of the α-core

In Greenberg (1985) and Peleg (1986), it has been shown that, if a game is ordinally convex, the coordinate of a payoff vector in the core of a subgame and one in the core of the associated reduced game is in the core of the original game under additional boundary conditions. We show that **SOR** strengthens the result:

Proposition 4
Let S be a nonempty subset of N, and let $T \subset S$. Consider a subgame $G(N \setminus S)$, a reduced game $F = G_*(S, a^{N\setminus S})$, and a subgame of F, $F(T)$. If $a^{N\setminus S}$ is in the α-core of $G(N \setminus S)$ and a^T is in the α-core of $F(T)$, then $(a^T, a^{N\setminus S})$ is in the α-core of $G(T \cup (N \setminus S))$.

Proof: Let V_α be the α-coalitional game of G, $V_{**\alpha}$ that of $G_*(S, a^{N\setminus S})$.
From the assumption,

$$a^{N\setminus S} \in \hat{V}_\alpha(N \setminus S), \quad a^T \in \hat{V}_{**\alpha}(T).$$

Thus, from equation (4), we obtain that,

$$(a^T, a^{N\setminus S}) \in \hat{V}_\alpha(T \cup (N \setminus S)).$$

Suppose to the contrary that there exist $S_* = T_* \cup R \neq \emptyset$ $(T_* \subset T,\ R \subset N \setminus S)$, and $b^N \in V_\alpha(T_* \cup R)$ such that $b^{S_*} \gg a^{S_*}$.
Then, from equation (3),

$$b^{T_*} \in \hat{V}_{\alpha**}(T_*) \text{ and } b^{T_*} \gg a^{T_*}.$$

Thus, $T_* = \emptyset$ because a^T is in the α-core of $F(T)$. However, then,

$$b^N \in V_\alpha(R) \text{ and } b^R \gg a^R$$

, which contradicts that $a^{N\setminus S}$ is in the α-core of $G(N \setminus S)$. $\square$

Therefore, if we can efficiently find a subgame $F(T)$ of any given reduced game F and its α-core strategy y^T, then by repeating it we can obtain the α-core of the original game. To obtain $F(T)$, we will repeat to find the maximum improving coalitions and the associated strategies. First, for a given feasible payoff vector (the reference point), if it is not in the α-core of F, we find the maximum improving coalition and the associated strategy, (T_1, y^{1,T_1}). If y^{1,T_1} is not an α-core strategy of $F(T_1)$, next we find the maximum improving coalition and the associated strategy

(T_2, y^{2,T_2}) for y^{1,T_1} in $F(T_1)$, and so on. Because the game is finite, this process eventually terminates and we obtain $F(T)$ and y^T.

Let $N_0 \subset N$ ($N_0 \neq N$) be an empty or nonempty coalition, and $a^N \in A^N$ be a payoff vector. Then following algorithm finds a subgame of a reduced game $G_*(N \setminus N_0, a^{N_0})$ and its α-core.

Algorithm 2

Input: game G with **SOR**, $N_0 \neq N$, $a^N \in V_\alpha(N)$

Output: $\boldsymbol{S}$, $\boldsymbol{T} \subset N \setminus N_0$, $\boldsymbol{y}^N$, $\boldsymbol{z}^N \in X^N$, $\boldsymbol{u}^{N\setminus N_0} \in A^{N\setminus N_0}$.

Step1 Let $\boldsymbol{y}^N$ be the maximum of feasible solutions, $y^N \in X^N$, of system (1a):

$$\text{for all } j \in N, \qquad u^j(y^N) \geq a^j \text{ or } y^j = 0^j; \tag{1a}$$

Set $\boldsymbol{S} := N \setminus N_0$; Set $\boldsymbol{u}^j := a^j$ for all $j \in N \setminus N_0$; **Go to** Step 2;

Step 2 Let $\boldsymbol{z}^N$ be the maximum element of feasible solutions, $z^N \in X^N$, of system (2a):

$$\begin{cases} \text{for all } j \in N_0, & u^j(z^N) \geq a^j \text{ or } z^j = 0^j, \\ \text{for all } j \in \boldsymbol{S}, & u^j(z^N) > \boldsymbol{u}^j \text{ or } z^j = 0^j, \\ \text{for all } j \in N \setminus (\boldsymbol{S} \cup N_0), & z^j = 0^j; \end{cases} \tag{2a}$$

Set $\boldsymbol{T} := \{j \in \boldsymbol{S} : u^j(\boldsymbol{z}^N) > \boldsymbol{u}^j\}$;
If $\boldsymbol{T} = \emptyset$ then stop, else go to Step 3;

Step 3 Set $\boldsymbol{S} := \boldsymbol{T}$; Set $\boldsymbol{y}^j := \boldsymbol{z}^j$ for all $j \in N$;
Set $\boldsymbol{u}^j := u^j(\boldsymbol{z}^N)$ for all $j \in \boldsymbol{T}$; **go to** Step 2.

First, note that the existence and efficient computation of a maximum of feasible solutions of (1a) and that of (2a) are guaranteed from proposition 1. Thus, any step of algorithm 2 is well defined. Second, in step 2, the algorithm finds the maximum improving coalition, $\boldsymbol{T}$, and the associated strategy, $\boldsymbol{z}^{\boldsymbol{T}}$, for $\boldsymbol{u}^{\boldsymbol{S}}$ in subgame $F(\boldsymbol{S})$ of $F := G_*(S, a^{N\setminus S})$.

First of all, we prove that algorithm 2 terminates within a polynomial time.

Lemma 2

(i) Both of $\boldsymbol{y}^N$ and $\boldsymbol{z}^N$ do not increase any of their coordinates.

(ii) Algorithm 2 terminates within $O(n \cdot \sum m^i)$.

Proof: (i). The indicator vector of $\boldsymbol{S}$ does not increase any of its coordinates, and $\boldsymbol{u}^{N \setminus N_0}$ does not decrease any of its coordinates.
Thus, constraint (2a) is strengthened every time it is updated.
Since $\boldsymbol{z}^N$ is defined as a maximum, thus $\boldsymbol{z}^N$ and $\boldsymbol{y}^N$ do not increase any of their coordinates.
(ii). Consider an execution of Step 2. Let z_*^N be the newly updated value of $\boldsymbol{z}^N$ in this execution , y^N, $u^{N\setminus N_0}$ and S the current values of $\boldsymbol{y}^N$, $\boldsymbol{u}^{N\setminus N_0}$ and $\boldsymbol{S}$ respectively. Because constraint (2a) is strengthened every time it is updated and $\boldsymbol{y}^N$ follows $\boldsymbol{z}^N$ in any execution of Step 3, we have that

$$y^N \geq z_*^N.$$

Thus, to find z_*^N, we can use y^N, instead of $(m^i)_{i \in N}$, as the starting point h^N of algorithm 1. It follows that this execution of Step 2 of algorithm 2 can be done within $O(n \cdot \max\{1, \sum(y^i - z_*^i)\})$ steps.
If $\sum_{i \in N}(y^i - z_*^i) = 0$ (that is, $y^N = z_*^N$) and this execution is not the first time to execute Step 2, then

$$T_* := \{j \in S : u^j(z_*^N) > u^j\} = \{j \in S : u^j(z_*^N) > u^j(y^N)\} = \emptyset.$$

That is, then this execution of Step 2 is the last one.
Because $y^N \geq z_*^N$ and $\boldsymbol{y}^N$ follows $\boldsymbol{z}^N$ in any execution of Step 3, algorithm 2 terminates after $O(n \cdot \sum_{i \in N} m^i)$ steps.$\Box$

The following theorem holds:

Theorem 1
Let S_*, y_*^N be the output of algorithm 2 for $\boldsymbol{S}$ and $\boldsymbol{y}^N$. Then, $S_* \neq \emptyset$, and $u^{S_*}(y_*^N)$ is in the α-core of subgame $F(S_*)$ of reduced game $F := G(N \setminus N_0, a^{N_0})$. Especially, if a^{N_0} is in the α-core of $G(N_0)$, then $(u^{S*}(y_*^N), a^{N_0})$ is in the α-core of $G(S_* \cup N_0)$ and $(y_*^{S_*}, y_*^{N_0})$ is the associated α-core strategy.

Proof: Let V_{**} be the α-coalitional game of $F(S_*)$.

It is obvious that $y_*^{N\setminus(S_* \cup N_0)} = 0^{N\setminus(S_* \cup N_0)}$ and that, for all $i \in N_0$, $y_*^i = 0^i$ or $u^i(y_*^N) \geq a^i$. Thus, $u^{S_*}(y_*^N) \in \hat{V}_{**\alpha}(S_*)$. Furthermore $\boldsymbol{T} = \emptyset$ and $\boldsymbol{S} \neq \emptyset$ when the computation terminates. As maximum improving coalition is empty, from corollary 1, $u^{S_*}(y_*^N)$ is in the α-core of $F(S_*)$.

Next, suppose that a^{N_0} is in the α-core of $G(N_0)$. From proposition 4, we can see that $(u^{S_*}(y_*^N), a^{N_0})$ is in the α-core of $G(S_* \cup N_0)$. Note that $y_*^{N_0 \cup S_*}$ is the maximum solution of a MGSP and that, $a^{N_0} \in V_\alpha(N_0)$. Thus, from proposition 1 (iii) we obtain that $u^{N_0}(y_*^N) \geq a^{N_0}$. Thus, $(y_*^S, y_*^{N_0})$ is an α-core strategy of $G(S_* \cup N_0)$ which guarantees $(u^{S_*}(y_*^N), a^{N_0})$. $\Box$.

Let $b^N \in A^N$ be a payoff vector in $V_\alpha(N)$, called *the reference point.* The reference point can be obtained by any stratgy $x^N \in X^N$ by "$b^N := u^N(x^N)$". Now, we obtain an efficient method to find α-core strategies.

Algorithm 3

Input: game G with **SOR**, reference point $b^N \in V_\alpha(N)$,

Output: $\boldsymbol{Q} \subset N$, $\boldsymbol{k}$: positive integer, $S_k \subset N$ for $k = 1, 2, \ldots, \boldsymbol{k}$
$\boldsymbol{w}^N \in X^N$, $\boldsymbol{c}^N \in A^N$

Step 1 Set $\boldsymbol{Q} := \emptyset$; Set $\boldsymbol{c}^N ;= b^N$; Set $\boldsymbol{k} := 0$; **Go to** 2;

Step 2 while $\boldsymbol{Q} \neq N$**do begin**

Set $\boldsymbol{k} := \boldsymbol{k} + 1$;

Let $(\boldsymbol{y^N}, \boldsymbol{S})$ be the output of algorithm 2 when $N_0 = \boldsymbol{Q}$ and $a^N = \boldsymbol{c}^N$, and set

- $S_{\boldsymbol{k}} := \boldsymbol{S} \subset N \setminus \boldsymbol{Q}$ and
- $\boldsymbol{w}^N := \boldsymbol{y}^N$;

Set, for all $j \in (N \setminus \boldsymbol{Q})$, $\boldsymbol{c}^j := u^j(\boldsymbol{w}^{\boldsymbol{Q} \cup S^{\boldsymbol{k}}}, 0^{N \setminus (\boldsymbol{Q} \cup S^{\boldsymbol{k}})})$;

Set $\boldsymbol{Q} := \boldsymbol{Q} \cup S_{\boldsymbol{k}}$ **end**.

Theorem 2

Algorithm 3 terminates within $O(n^2 \cdot \sum m^i)$ steps. The output $\boldsymbol{c}^N$ is in the α-core of G and $\boldsymbol{w}^N$ is an α-core strategy of G.

Proof: From theorem 1, it can be easily shown recursively that, every time immediately after $\boldsymbol{Q}$ is updated, (i) $u^{\boldsymbol{Q}}(w^{\boldsymbol{Q}}, 0^{N \setminus \boldsymbol{Q}}) \geq \boldsymbol{c}^{\boldsymbol{Q}}$, (ii) $\boldsymbol{c}^N \in V_\alpha(\boldsymbol{Q}) \cap V_\alpha(N)$, where $V_\alpha(\emptyset) := A^N$ and (iii) $\boldsymbol{c}^{\boldsymbol{T}}$ is in the α-core of $G(\boldsymbol{Q})$. Thus, we obtain that, every time immediately after $\boldsymbol{Q}$ is updated, $\boldsymbol{w}^{\boldsymbol{Q}}$ is an α-core strategy of $G(\boldsymbol{Q})$. Because the time complexity of algorithm 2 is $O(n \cdot \sum_N m^i)$, algorithm 3 terminates after $O(n^2 \cdot \sum_N m^i)$.$\Box$

5 Discussion

A game with punishment-dominance relation has many α-core strategies, while algorithm 3 can find only one of them for a given reference point. In this section, we discuss the properties of the α-core strategy obtained by our algorithm. In the previous section, the reference point, b^N, which is a part of the input of algorithm 3 is supposed to be arbitrarily chosen. However, in this section, it turns out that he reference point has important relations with the α-core strategy obtained by the algorithm.

5.1 Dominating strategy

The vNM stable set is the most traditional solution concept of n-person cooperative game, which is introduced by Von Neuman and Moregenstren (1944). Aumann and Peleg (1960) have discussed the stable set in the situations without side payments. Peleg (1986) has shown that the core of an ordinally convex game is the unique vNM stable set: for any $b \in V(N)$ not in the core ,there exist coalition S and payoff vector c^N in the core such that $c^N \in V(S)$ and $c^S \gg b^S$ if V is ordinally convex. We call such S a *dominating coalition* for b^N, and c^N a *dominating payoff vector* for b^N here. We show that, for the reference point b^N, the algorithm 3 finds a dominating coalition and a dominating payoff vector if b^N is not in the α-core.

Theorem 3
Let $b^N \in V_\alpha(N)$ be a vector not in the α-core. Then the output $\boldsymbol{c}^N = c^N$ of algorithm 3 is a dominating payoff vector for the reference point, b^N, and S_1 is the associated dominating coalition.

Proof: From theorem 2, c^N is in the α-core. Furthermore $\boldsymbol{c}^j$ becomes c^j when $j \in S_{\boldsymbol{k}}$ for current $\boldsymbol{k}$. Thus, from the proof of theorem 2, we have the result that, for all k, c^{S_k} is in the α-core of $G(S_k)$. Thus, $c^N \in V_\alpha(S_1)$. Then, it suffices to show that $c^{S_1} \gg b^{S_1}$.

Consider the first execution of step 2 of algorithm 3. We consider algorithm 2, which is a subroutine of algorithm 3 to execute the step 2 of algorithm 3.

Let $(S_*, y_*^N, z_*^N, u_*^N)$ be the output value of $(\boldsymbol{S}, \boldsymbol{y}^N, \boldsymbol{z}^N, \boldsymbol{u}^N)$ by algorithm 2 when $N_0 = \emptyset$ and $a^N = b^N$; that is, $S = S_1$ and $u_*^S = c^S$. Just before the first execution of step 2 of algorithm 2, $\boldsymbol{u}^{\boldsymbol{S}} = b^N$, which is not in the α-core of $F(\boldsymbol{S}) = G$. Then, from corollary 1, $\boldsymbol{T}$ is set to be the maximum improving coalition for b^N in step 2 of algorithm 2, which is nonempty. Thus, S is determined not in step 1 of algorithm 2 but in step 3 of algorithm 2. Here, note that, in algorithm3 for all $i \in S$, $\boldsymbol{u}^i$ is strictly increasing. Thus, we obtain that $c^i = u^i_* = u^i(y^N_*) > a^i = b^i$ for all $i \in S_* = S_1$. $\square$

5.2 Symmetric strategy

From the normative point of view, we will focus on the symmetric element in the α-core. To my best knowledge, the symmetry of a strategy in ordinal sense has not been defined, while the notion seems to be obvious. Thus, we define the symmetry of a strategy profile.

A *permutation* of N is one to one mapping from N onto N. Let $\mathcal{S}(G)$ be a set of all permutation σ such that

(i) for all $i \in N$, $X^i = X^{\sigma(i)}$ and ,

(ii) for all $y^N, z^N \in X^N$ and all $i \in N$, $u^i(y^N \cdot \sigma) \geq u^i(z^N \cdot \sigma)$
if and only if $u^{\sigma(i)}(y^N) \geq u^{\sigma(i)}(z^N)$,
where, for all $x^N \in X^N$ $(x^N \cdot \sigma)$ is a strategy profile defined by, for all $i \in N$, $(x \cdot \sigma)^i := x^{\sigma(i)}$.

That is, for any $\sigma \in \mathcal{S}(G)$, an ordered set of players $(1, 2, \ldots, n)$ for any player $j \in N$ is equivalent to $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ for player $\sigma(j)$. It can be easily shown that $\mathcal{S}(G)$ is a subgroup. Thus, $\mathcal{S}(G)$ induces a partition $\mathcal{P}(G)$ of N such that, for all $i, j \in N$, there exists $S \in \mathcal{P}(G)$ such that $i, j \in S$ if and only if there exists $\sigma \in \mathcal{S}(G)$ such that $i = \sigma(j)$.

We say that two players $i, j \in N$ are in an *equivalent position* of G if $i, j \in S$ for some $S \in \mathcal{P}(G)$. A strategy profile $x^N \in X^N$ of G is *symmetric* if for all $S \in \mathcal{P}(G)$ and all $i, j \in S$, $x^i = x^j$. It follows that, in any symmetric strategy, one of a pair of players in an equivalent position enjoys the same utility level as the other.

Note that we do not know $\mathcal{S}(G)$ in advance, and we cannot specify the associated partition, $\mathcal{P}(G)$, in any polynomial time. Thus, we have to make a method which treats, without completely checking that they are equivalent, a pair of players equally if they are in an equivalent position. That is, we obtain:

Proposition 5
Algorithm 3 yields a symmetric α-core strategy of G if the reference point is given by $b^N := u^N(x^N)$ for some symmetric strategy x^N.

Proof:
Then, algorithm 3 treats equally any pair of players in equivalent position, and thus the output is symmetric. Thus, algorithm 3 gives a symmetric α-core strategy. $\square$

6 Marginal worth vectors and the nonemptiness of the α-core

Theorem 2 implies that the α-core of a game with **PD+ F** is nonempty. Remember that throughout this paper, the completeness of the preference is not assumed. Thus, this result does not follow directly from Scarf (1967), Greenberg (1985), Peleg (1986), and Masuzawa (2003). The aim of the section is to extend this result for the games with (**PD+C**).

The nonemptiness theorem will be obtained through *marginal worth vector*, which is a payoff vector that gives every player his marginal contribution according to any given ordering. Let us formally define *marginal worth vectors*. Let $V : 2^N \setminus \{\emptyset\} \to A^N$ be an NTU coalitional game. Given any liner ordering $\succ$ of N, a payoff vector $a_*^N \in A^N$ is a *marginal worth vector with respect to* $\succ$, if for all i, a_*^i is a maximal element with respect to $\geq^i$ of the following set

$$\Big\{a^N : a^N \in V(\{j : i \succ j\} \cup \{i\}) \text{ and } a^j \geq a_*^j \text{ for all } j \text{ such that } i \succ j\Big\}.$$

Because $\geq^i$ is not necessarily complete, the marginal worth vector with respect to $\succ$ is not uniquely determined.

For all ordering $\succ$, a marginal worth vector with respect to $\succ$ can be obtained efficiently:

Theorem 4
Let $G = (N, (X^i)_{i\in N}, (u^i)_{i\in N})$ be a finite strategic form game which satisfies **SOR**. For all ordering $\succ$ of N, a marginal worth vector of the α-coalitional game can be obtained in $O(n^3 \cdot \max m^i)$ steps.

Proof: Let $j \in N$ be any player, and $a^T \in A^T$ be a vector in $V_\alpha(T)$, where $T := N \setminus \{j\}$. Let $x_*^N \in X^N$ be a solution of the following problem and a_*^j an optimum value.

> **P1**: Find a strategy profile $x^N \in X^N$ such that $u^j(x^N)$ is a maximal element of $\{u^j(x^N) : u^i(x^N) \geq a^i$ for all $i \in T.\}$

From equation (4) of subsection 4.1, a vector $a^j \in A^i$ is an optimum value if and only if it is in the α-core of $G_*(\{j\}, a^T)$. From theorem 1, **P1** can be solved by algorithm 2 in $O(k^2 \cdot \max m^i)$ for a k-person game with **SOR**. Thus, for any game with **SOR**, the marginal worth vector can be found within $O(n^3 \cdot \max m^i)$ because $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$. □

Next, we prove that all marginal worth vector is in the α-core of games with **PD+C**. Note that the core of an ordinally convex game does not always include all marginal worth vectors even if the preferences are all complete. (See, Hendricx, Borm and Timmer (2002)). On the contrary, in the context of TU coalitional game, the core of a convex game includes all marginal worth vectors (Shapley (1971) and Ichiishi (1981)). However, we obtain:

Theorem 5
If a strategic form game G satisfies conditions **C** and **PD**, then, for any liner ordering $\succ$ of N, any of the corresponding marginal worth vectors is in the α-core.

Proof : Consider **P1** in the proof of theorem 4. It is sufficient to prove that $(a^j, a^{N\setminus\{j\}})$ is in the α-core of G if $a^{N\setminus\{j\}}$ is in the α-core of $G(T)$.

First, we consider the case when **PD+F** is satisfied. The statement is not related to computational issues at all. Thus, without any loss of generalities of the statement, we can assume **SOR**. Clearly a^j is in the α-core of $G_*(\{j\}, a^T)$. Thus, from proposition 4, (a^T, a^j) is in the α-core of G. Especially, x_*^N is an α-core strategy of G.

Now, let us consider game with (**PD+C**). For all $i \in N$, let $0^i \in X^i$ be a strategy such that for all $x^i \in X^i$, $P(0^i, x^i)$. Because a^T is in the α-core of $G(T)$, there exists $w^N \in X^N$ such that $u^T(w^N) \geq a^T$ and $w^j = 0^j$. Suppose to the contrary that there exist $y^N \in X^N$ and $S \neq \emptyset$ such that $u^S(y^N) \gg a^S$ and $y^{N\setminus S} = 0^{N\setminus S}$. Then, consider a game $G^* = (N, (X^{*i})_{i\in N}, (u^i)_{i\in N})$ such that $X^{*i} = \{0^i, w^i, x_*^i, y^i\}$ for all $i \in N$. That is, in game G^*, the strategy space is restricted to be finite. Then, x_*^N is also an optimum solution of **P1** for G^* , and a^T is in the α-core of $G^*(T)$. Thus, from the discussion above, (a^T, a^j) is in the α-core of G^*, which contradicts the definition of y^N. □

i	d^i	e^i		i	d^i	e^i		i	d^i	e^i		i	d^i	e^i
1	3.00	21		6	3.20	21		11	4.15	20		16	5.50	19
2	3.00	21		7	3.30	21		12	4.40	20		17	5.80	19
3	3.00	21		8	3.50	21		13	4.65	20		18	6.10	19
4	3.00	21		9	3.70	21		14	4.90	20		19	6.40	19
5	3.10	21		10	3.90	20		15	5.20	19		20	6.40	19

Table 1:

7 A numerical example

We discuss a voluntary contribution game with one private good and one public good to illustrate our algorithms. Let N be a set of consumer. Every player i has m^i units of the private good as an initial allocation and chooses, x^i, the contribution level of the private good for the public good provision. As a result, player i consumes $(m^i - x^i)$ units of the private good and $f(\sum_{j\in N} x^j)$ units of the public good, where f is a production function from an input of the private good into an output of the public good. To sum up, a voluntary contribution game is a list $(N, (X^i)_{i\in N}, (u^i)_{i\in N})$ such that $X^i = \{0, 1, \ldots, m^i\}$, and, for all $x^N \in X^N$, $u^i(x^N) := v^i(m^i - x^i, f(\sum_{j\in N} x^j))$, where v^i is an increasing utility function of i. Clearly, this game satisfies **SOR**.

Let us consider a case when

$$N := \{1, 2, \ldots, 20\}, \quad X^i := \{0, 1, \ldots, 15\}, \text{for all } i \in N,$$
$$\text{and} \qquad v^i(x, y) = d^i \cdot (30 \cdot y)^{1/2} + e^i \cdot x + 0.2 \cdot (85 + x)^{1/2} \text{ for all } i \in N,$$

where $(d^i)_{i\in N}$ and $(e^i)_{i\in N}$ are defined by Table 1. As $d^1 = d^2 = d^3 = d^4 = 3$, and $e^1 =^2= e^3 = e^4 = 21$, then any pair of $\{1, 2, 3, 4\}$ is in the equivalent position.

To illustrate the algorithms introduced, consider, for example, strategy profile $w_1^N = (w_1^1, w_1^2, \ldots, w_1^n) := (13, 13, \ldots, 13)$. First, following the discussion of section 3, we can see that w_1 is not an α-core strategy of the game. Indeed, for the reference point $u^N(w_1)$, algorithm 1 yields strategy profile w_2^N such that:

$$w_2^i := \begin{cases} 12 & \text{for} \quad i = 1, 2, \ldots, 9 \\ 11 & \text{for} \quad i = 10, 11, \ldots, 20 \end{cases}$$

Thus, the largest improving coalition for $u^N(w_1^N)$ is

$$S := \{i : u^i(w_2^N) > u^i(w_1^N)\} = N,$$

and it improves upon w_1^N via w_2^N. Similarly, N improves upon $u^N(w_2^N)$ via w_3^N defined by

$$w_3^i := \begin{cases} 10 & \text{for} \quad i = 1, 2, \ldots, 9 \\ 8 & \text{for} \quad i = 10, 11, \ldots, 15 \\ 7 & \text{for} \quad i = 16, 17, \ldots, 20 \end{cases}$$

Furthermore, if the reference point is given by $a^N := u^N(w_3^N)$, algorithm 1 yields strategy profile

$$w_4^N = (3, 3, 3, 3, 3, 3, 2, 2, 2, 0, 0, \ldots, 0).$$

The largest improving coalition for $u^N(w_3^N)$ is $\{i \in N : u^i(w_4^N) > u^i(w_3^N)\} = \{1, 2, \ldots, 9\}$, which improves upon w_3^N via w_4^S. Thus, we can see that w_3^N is weekly Pareto-efficient but not an α-core strategy.

In turn, let us find a dominating coalition and the dominating payoff vector in the α-core for $u^N(w_1^N)$, and a strategy profile which induces the payoff vector. The result of the computation by algorithm 3 for the reference point $u^N(w_1^N)$ is following:

$$S_1 = \{1, 2, \ldots, 9\}, \quad S_2 = \{10, 11, \ldots, 20\}$$

$$c^{*i} = \begin{cases} u^i(z^{*N}) & \text{if } i = 1, 2, \ldots, 9 \\ u^i(y^{*,N}) & \text{if } i = 10, 11, \ldots, 20 \end{cases}$$

where $y^{*,N}$, which is an α-core strategy, and $z^{*,N}$ are given by

$$y^{*i} = \begin{cases} 7 & \text{if } i = 10 \\ 8 & \text{if } i = 7, 11, 12, 15, 16, \ldots, 20 \\ 9 & \text{otherwise.} \end{cases} \qquad z^{*i} = \begin{cases} 2 & \text{if } i = 1, 2, \ldots, 6 \\ 1 & \text{if } i = 7, 8, 9 \\ 0 & \text{otherwise} \end{cases}$$

Thus, $S_1 = \{1, 2, \ldots, 9\}$ is a dominating coalition , c^{*N} is a dominating payoff vector in the α-core, and coalition S_1 dominates $u^N(w_1^N)$ by strategy z^{*S}. Note that $y^{*,N}$ is a symmetric strategy because the reference point, w_1^N, is symmetric.

On the other hand, for $\succ$ such that $n \succ n-1 \succ \cdots \succ 1$, the associated marginal worth vector is unique and given by:

$$(317, 319.23, 340.24, 340.24, 341.01, 345.56, 368.12,$$
$$371.22, 379.55, 390.07, 402.27, 429.52, 436.77, 444.02,$$
$$475.05, 526.11, 539.15, 585.13, 659.63, 681.27),$$

which is given by strategy profile

$$(9, 9, 8, 8, 8, 8, 7, 8, 8, 8, 8, 7, 8, 8, 7, 6, 6, 5, 2, 1).$$

Note that it is not symmetric. Indeed, player 2 and 3 is in an equivalent position of G while they choose different strategies.

8 Concluding remarks

In this paper, we have discussed the computation of a payoff vector in the α-core and the associated strategy profile of games with punishment-dominance relations. The time complexity of the algorithm is $O(n^3 \max\{m_i\})$, where n is the number of the set of players and m_i is that of strategies of player i. We can obtain, by our method, a dominating coalition and an dominating imputation in the α-core for any given payoff vector. it has been also shown that all marginal worth vectors of any game in the class is in the α-core.

A methodological problem remains to be discussed. Our algorithm takes strategic game G as an input which is given by an oracle. However, the strategic game is too complex in general, because the domain of a payoff function is X^N, the size of which increases exponentially as the number of players increases. Thus, it is difficult to realize the oracle as a long list. That is, we can not input such a long list in the memory of the calculator before the computation.

If u_i can be characterized by a few of parameters and $u_i(x)$ can be computed efficiently from these parameters, the size of input can be small enough to be inputted in the memory. However, this is not often the case, because, as is in the example of the previous section, sometimes u_i represents a utility level which the consumption of a commodity vector brings to him/her and there is no reason why u_i is characterized by a few of parameters.

However, our final goal is not to compute numerical examples by a calculator but to give a method for players in real life to obtain an outcome in the α-core. If we use the algorithm as such a method, we do not necessarily confront such a problem. In such a case, the data of utility function is initially in minds of players, and in communication process, to check $u_i(x^N) > u_i(y^N)$, we have only to ask player i, "do you prefer x^N to y^N?". In other words, the set of players plays the role of the oracle. As such a method, some problem may arise. The time complexity of our algorithm may be not small enough, or some player may have incentives to tell lies. However, they are beyond the scope of this paper and the author believe that they do not decrease the significance of our results.

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