

KEIO UNIVERSITY
MARKET QUALITY RESEARCH PROJECT
(A 21st Century Center of Excellence Project)

KUMQRP DISCUSSION PAPER SERIES

DP2005-004

The Core and Nucleolus Allocations in a
Public Goods Economy with Taxation

Toshiyuki Hirai*

Abstract

In this paper, we study a public goods economy with a certain tax system to finance public goods, and show that the core is equal to the set of nucleolus allocations defined in this economy. Coalitions are allowed to levy a tax upon the complementary coalitions and to improve current allocations under a rule restricting the resulting allocations, which corresponds to the effectiveness form due to Rosenthal [13]. We also show some sufficient conditions for the core to be nonempty.

* Graduate School of Economics, Keio University

Graduate School of Economics and Graduate School of Business and Commerce,
Keio University
2-15-45 Mita, Minato-ku, Tokyo 108-8345, Japan

THE CORE AND NUCLEOLUS ALLOCATIONS IN A PUBLIC GOODS ECONOMY WITH TAXATION

TOSHIYUKI HIRAI*

ABSTRACT. In this paper, we study a public goods economy with a certain tax system to finance public goods, and show that the core is equal to the set of nucleolus allocations defined in this economy. Coalitions are allowed to levy a tax upon the complementary coalitions and to improve current allocations under a rule restricting the resulting allocations, which corresponds to the effectiveness form due to Rosenthal [13]. We also show some sufficient conditions for the core to be nonempty.

1. INTRODUCCION

Since the core of an economy with public goods is quite large as indicated by Foley [4], there have been several proposals of new core concepts in the literature. Champsaur, Roberts and Rosenthal [2] assumed that each coalition can tax upon the complementary coalition. The core of Champsaur *et. al.* [2] coincides with the set of symmetric allocations of Foley's [4] core when the agent population is large. The core of the voting games based on the market systems of Kaneko [7], [8] and Hirokawa [5] coincides with the set of the market equilibrium allocations under the market systems. Nakayama [9] considered the public goods economy, where coalitions are allowed to tax upon the agents outside the coalition under a certain restriction on the level of public goods to be provided.

Date: June 21, 2005.

The author is grateful to Mikio Nakayama, Shinsuke Nakamura and Tomoki Inoue for the helpful comments.

*Graduate School of Economics, Keio University, 2-15-45 Mita, Tokyo 108-8345

E-mail Address: hirai@gs.econ.keio.ac.jp.

In this paper, we also approach the problem via a certain system of taxation and restriction on the public goods to be provided. What a coalition can achieve for itself will be determined by the system of taxation and the restriction, which is an application of the effectiveness form due to Rosenthal [13], and is an extension of Nakayama [9]. The purpose of this paper is to show that the core is equal to the set of *nucleolus allocations* defined in this economy, and this analysis sheds light on the prescriptive aspect of the core allocations of such an economy.

The nucleolus allocation is one that can be obtained analogously to the nucleolus of a TU-coalitional game, the definition of which is originally due to Schmeidler [14]. It was extended to the NTU-coalitional games by Kalai [6], and later by Nakayama [10]. There are, however, only a few applications to the market economy, one of which is that of McLean and Postlewaite [12] studying excess functions and nucleolus allocations for pure exchange economies.

The excess in our public goods economy is defined in an analogous way to that of TU-coalitional games, in terms of a coalitional net wealth given by the allocation. This coalitional wealth may vary according to the tax system and the restriction on the level of the public goods to be provided, and the nucleolus allocation is defined based on this excess. We will call the criterion leading to the nucleolus allocation the *minimization of the maximum excess*.

We will show that the nonempty core of the economy coincides with the set of the nucleolus allocations, if the tax system satisfies two properties called the *additivity* and the *tax feasibility*. We also show the nonemptiness of the core under some convexity assumptions and the properties on the tax system called the *additivity* and the *exact tax feasibility*.

In the next section, we introduce the basic model and the definitions. In sections 3 and 4, the core and the nucleolus allocation under a tax system are introduced. The main result will be stated in Section 5, and the nonemptiness of the core is shown in Section 6. Section 7 includes some examples of tax systems. In the final section, we conclude with some remarks.

2. THE BASIC MODEL

We consider a public goods economy consisting of n agents, m public goods and one private good. The set of the agents is denoted by $N = \{1, \dots, n\}$. The set of the projects for the public goods provision is denoted by $Y \subset \mathbb{R}_+^m$, where Y is assumed to be closed and $0 \in Y$. The cost of the public goods provision is represented by the function $c : Y \rightarrow \mathbb{R}_+$. We assume that c is continuous, monotone increasing (*i.e.* $y \geq y'$ and $y \neq y'$ imply $c(y) > c(y')$) and satisfies $c(0) = 0$. We also assume that c satisfies $c^{-1}([0, M])$ is compact for any $M \geq 0$, where c^{-1} denotes the inverse image. Each agent is characterized by the tuple $(X_i, \succsim_i, \omega_i)$, where $X_i = \mathbb{R}_+ \times Y$ is the consumption set, \succsim_i is the complete, transitive and reflexive preference relation of agent i on X_i , and $\omega_i \in \mathbb{R}_{++}$ is the amount of the private good initially endowed with agent i . We assume that for all $i \in N$, \succsim_i satisfies the continuity, the strict monotonicity in the private good (*i.e.* $x_i > x'_i$ implies $(x_i, y) \succ_i (x'_i, y)$ for any $y \in Y$) and the monotonicity in the public goods (*i.e.* $y \geq y'$ implies $(x_i, y) \succsim_i (x_i, y')$ for any $x_i \geq 0$). We also assume that the private good is indispensable to any agent (*i.e.* $(x_i, y) \succsim_i (0, y')$ for any $x_i \geq 0$ and any $y, y' \in Y$) for all $i \in N$. Let $E = \{N, Y, c, (X_i, \succsim_i, \omega_i)_{i \in N}\}$ be the public goods economy.

A nonempty subset S of N is called a coalition. We denote by \mathcal{N} the set of the coalitions. An allocation (x, y) is said to be feasible

if $\sum_{i \in N} x_i + c(y) \leq \sum_{i \in N} \omega_i$. Let A denote the set of the feasible allocations. A feasible allocation (x, y) is said to be (weakly) Pareto optimal if there exists no feasible allocation (x', y') such that (x'_i, y') \succ_i (x_i, y) for all $i \in N$. Let A^* denote the set of the Pareto optimal allocations.

3. THE CORE UNDER TAXATION AND RESTRICTION

We now introduce into the basic model the tax system and a certain restriction on the allocation, which are assumed to be governed by a public authority. Coalitions are allowed to propose an allocation under this system of taxation and restriction.

The tax system is given by the function $T : X \times (\mathcal{N} \cup \{\emptyset\}) \rightarrow \mathbb{R}_+$, which is assumed to be continuous and $T(x, y; \emptyset) = 0$ for any $(x, y) \in X$. It is also assumed that $T(x, y; S) \leq \sum_{i \in S} \omega_i$ for all $S \in \mathcal{N}$ and $(x, y) \in A$. The amount $T(x, y; S)$ represents the tax revenue *levied upon* S at (x, y) .

We introduce following properties of T .

ADD: (*Additivity*): For any $S, S' \in \mathcal{N}$ with $S \cap S' = \emptyset$, $T(x, y; S) + T(x, y; S') = T(x, y; S \cup S')$ for any $(x, y) \in X$.

TF: (*Tax Feasibility*): $T(x, y; N) \geq c(y)$ for any $(x, y) \in A$.

The additivity simply means that the amount of the tax levied upon the coalition is just the sum of the amount of the agents in the coalition. The tax feasibility requires that the total tax revenue be sufficient for the public goods provision.

The restriction is given by the correspondence $G : X \times \mathcal{N} \rightarrow Y$, which is assumed to be continuous, closed-valued, $y \in G(x, y; S)$ for any $(x, y; S) \in X \times \mathcal{N}$, and $G(x, y; N) = Y$ for any $(x, y) \in X$. We call G the *acceptable provision correspondence*. The role of the acceptable provision correspondence is to restrict the possible allocations proposed

by a coalition. Any such allocation is to be financed by the coalition with the increased wealth through the taxation upon the complementary coalition. Thus, it will be legitimate for the resulting public goods to be under partial control by the authority. For example, if the public goods should be no less preferable to the current ones, G will be given by $G(x, y; S) = \{y' \in Y \mid (x_i, y') \succsim_i (x_i, y) \text{ for all } i \in N\}$ for all $S \neq N$.

The public goods economy with the tax system and the acceptable provision correspondence is denoted by $\bar{E} = (E, T, G)$. Hereafter, we call \bar{E} simply the public goods economy.

We define the core of the public goods economy. First, we introduce the domination in \bar{E} .

Definition 1. *An allocation (x', y') is said to (T, G) -dominate (x, y) if*

- (i): $(x'_i, y') \succ_i (x_i, y)$ for all $i \in S$,
- (ii): $\sum_{i \in S} x'_i + c(y') \leq \sum_{i \in S} \omega_i + T(x, y; N \setminus S)$ and
- (iii): $y' \in G(x, y; S)$.

Definition 1(ii) describes that each coalition is allowed to tax upon the complementary coalition according to the current allocation. But, by Definition 1(iii), the new proposal of the public goods is restricted by the acceptable provision correspondence.

The core is the set of the undominated feasible allocations. Hence we define the (T, G) -core as follows.

Definition 2. *An allocation (x, y) is said to be in the (T, G) -core if*

- (iv): $(x, y) \in A$ and
- (v): *there exists no (x', y') which (T, G) -dominates (x, y) .*

The (T, G) -core of \bar{E} is denoted by $C(\bar{E})$. Note that $C(\bar{E}) \subset A^*$ follows from Definition 1 and Definition 2.

4. THE EFFECTIVE EXCESS AND THE NUCLEOLUS ALLOCATIONS

In this section, we define the function which determines the excess of each coalition. Since our game is in the effectiveness form due to Rosenthal [13], we call this function the effective excess function.

To define the effective excess function, we modify the function appeared in Champsaur [1]. First, we define the correspondence $U : X \times \mathcal{N} \rightarrow X$. U denotes the set of the allocations which are at least as preferred as the current allocation.

$$U(x, y; S) = \left\{ (x', y') \in X \mid \begin{array}{l} (x'_i, y') \succsim_i (x_i, y) \text{ for all } i \in S \text{ and} \\ y' \in G(x, y; S). \end{array} \right\}$$

The effective excess of each coalition is determined by the following function f . The effective excess is measured in terms of the private good.

$$f(x, y; S) = \sum_{i \in S} \omega_i + T(x, y; N \setminus S) - \min_{(x', y') \in U(x, y; S)} \left(\sum_{i \in S} x'_i + c(y') \right).$$

Note that the minimum can be attained by the continuity of c and \succsim_i . Moreover, the continuity of f on X can be shown by applying Berge's maximum theorem.

If S has a sufficient wealth to improve upon (x, y) , then $f(x, y; S) > 0$ and this means that S is dissatisfied with (x, y) . On the other hand, if S does not have a sufficient wealth to improve itself, then $f(x, y; S) \leq 0$ and S is satisfied with (x, y) .

Let $f^*(x, y) = (f(x, y; S_1), \dots, f(x, y; S_{2^n-2}))$ be a $(2^n - 2)$ -vector obtained by arranging $f(x, y; S)$ of all $S \in \mathcal{N} \setminus N$ in the non-increasing order. That is, $f_k^*(x, y) \geq f_{k'}^*(x, y)$ if $k < k'$. Note that f^* is also continuous by the continuity of f .

To define the nucleolus allocation, we need to introduce the lexicographic order denoted by \geq_{lex} . Let $x, y \in \mathbb{R}^l$, $x \geq_{lex} y$ if $x = y$ or there exists $k = 1, \dots, l$ such that $x_k > y_k$ and $x_{k'} = y_{k'}$ for all $k' < k$.

The nucleolus allocation is defined as follows.

Definition 3. A feasible allocation (x, y) is said to be a nucleolus allocation in \bar{E} if

- (vi): $(x, y) \in A^*$ and
- (vii): $f^*(x, y) \leq_{lex} f^*(x', y')$ for all $(x', y') \in A^*$.

The set of the nucleolus allocations in \bar{E} is denoted by $\nu(\bar{E})$. The nucleolus allocation perhaps should be called the prenucleolus allocation since our definition does not require any individual rationality.

Note that (vi) corresponds to the total rationality of TU-coalitional games since $f(x, y; N) = 0$ for any $(x, y) \in A^*$. We can therefore ignore the excess of N when we construct f^* by (vi).

Note also that it is possible to prove the nonemptiness of $\nu(\bar{E})$ in the same way to the nucleolus existence theorem in Schmeidler [14], since the effective excess function f^* is continuous on the compact set A^* .

Theorem 1. Assume that T satisfies **ADD** and **TF**. If $C(\bar{E}) \neq \emptyset$, then $\nu(\bar{E}) = C(\bar{E})$.

Proof. We first show that $(\bar{x}, \bar{y}) \in C(\bar{E})$ if and only if $(\bar{x}, \bar{y}) \in A$ and for all $S \in \mathcal{N}$,

$$\sum_{i \in S} \omega_i + T(\bar{x}, \bar{y}; N \setminus S) - \min_{(x', y') \in U(\bar{x}, \bar{y}; S)} \left(\sum_{i \in S} x'_i + c(y') \right) \leq 0. \quad (1)$$

The necessity of (1) immediately follows from the strict monotonicity of the preference in the private good. Thus, we show that (1) is sufficient.

Let (\bar{x}, \bar{y}) be a feasible allocation satisfying (1). Suppose that there exists some (x^*, y^*) which (T, G) -dominates (\bar{x}, \bar{y}) , then, there exists

some $S \in \mathcal{N}$ satisfying $(x_i^*, y^*) \succ_i (\bar{x}_i, \bar{y})$ for all $i \in S$, $\sum_{i \in S} \omega_i + T(\bar{x}, \bar{y}; N \setminus S) - (\sum_{i \in S} x_i^* + c(y^*)) \geq 0$ and $y^* \in G(\bar{x}, \bar{y}; S)$ by Definition 1. Since the private good is indispensable to all agents, we have $x_i^* > 0$ for all $i \in S$. Then by the continuity and the monotonicity of \succsim_i , there exists a sufficiently small $\varepsilon > 0$ such that $(x_i^* - \varepsilon, y^*) \succsim_i (\bar{x}_i, \bar{y})$ for all $i \in S$. It follows that $\sum_{i \in S} \omega_i + T(\bar{x}, \bar{y}; N \setminus S) - (\sum_{i \in S} (x_i^* - \varepsilon) + c(y^*)) > 0$, and thus, together with (1), we obtain

$$\sum_{i \in S} (x_i^* - \varepsilon) + c(y^*) < \min_{(x', y') \in U(\bar{x}, \bar{y}; S)} \left(\sum_{i \in S} x'_i + c(y') \right).$$

This is a contradiction since $((x_i^* - \varepsilon)_{i \in N}, y^*) \in U(\bar{x}, \bar{y}; S)$. Hence $(\bar{x}, \bar{y}) \in C(\bar{E})$, and (1) is sufficient.

Now, for any $(x, y) \in C(\bar{E})$, we have $f(x, y; S) \leq 0$ for all $S \in \mathcal{N}$. Fix any $(x', y') \in A^* \setminus C(\bar{E})$. Then, we have $f(x', y'; S) > 0$ for some $S \in \mathcal{N} \setminus N$. Thus, we obtain $(x, y) \in C(\bar{E})$ implies $f^*(x, y) <_{lex} f^*(x', y')$. That is, $(x', y') \notin \nu(\bar{E})$. Consequently, we obtain $\nu(\bar{E}) \subset C(\bar{E})$ from the assumption that $C(\bar{E}) \neq \emptyset$.

Next, we show that $C(\bar{E}) \subset \nu(\bar{E})$. We will verify that $f^*(x, y) = 0$ for any $(x, y) \in C(\bar{E})$ to show that f^* is lexicographically minimized at the allocations in $C(\bar{E})$.

Suppose that there exists some $(x, y) \in C(\bar{E})$ such that $f^*(x, y) \neq 0$. Then, since $(x, y) \in C(\bar{E})$ is equivalent to that (x, y) is feasible and satisfies (1), there exists some $S \in \mathcal{N} \setminus N$ such that $f(x, y; S) < 0$.

It follows that

$$\begin{aligned} \sum_{i \in S} \omega_i + T(x, y; N \setminus S) &< \min_{(x', y') \in U(x, y; S)} \left(\sum_{i \in S} x'_i + c(y') \right) \\ &\leq \sum_{i \in S} x_i + c(y). \end{aligned} \tag{2}$$

Moreover, by (1), we have $f(x, y; N \setminus S) \leq 0$. Hence in a similar way to inequality (2), we have

$$\sum_{i \in N \setminus S} \omega_i + T(x, y; S) \leq \sum_{i \in N \setminus S} x_i + c(y). \quad (3)$$

By adding (2) and (3), and **ADD**,

$$\sum_{i \in N} \omega_i + T(x, y; N) < \sum_{i \in N} x_i + 2c(y).$$

By **TF**, that is $T(x, y; N) \geq c(y)$, we obtain

$$\sum_{i \in N} \omega_i < \sum_{i \in N} x_i + c(y).$$

This contradicts that $(x, y) \in C(\bar{E})$ since (x, y) is not feasible. Hence $f^*(x, y) = 0$ for any $(x, y) \in C(\bar{E})$.

Thus, for any $(x, y) \in C(\bar{E})$,

$$f^*(x, y) =_{lex} f^*(x', y') \text{ for any } (x', y') \in C(\bar{E}) \text{ and,}$$

$$f^*(x, y) <_{lex} f^*(x'', y'') \text{ for any } (x'', y'') \in A^* \setminus C(\bar{E}).$$

Hence $C(\bar{E}) \subset \nu(\bar{E})$. □

5. THE NONEMPTINESS OF THE (T, G) -CORE

In the previous section, we assumed the nonemptiness of the (T, G) -core. We now turn to the question of existence. Before stating the existence theorem, we give a characterization of the (T, G) -core, which is a generalization of the theorem of Nakayama [9].

Proposition 1. *Let T satisfy **ADD** and **TF**, and $G(x, y; S) \subseteq \{y' \in Y \mid y' \geq y\}$ for any $(x, y) \in X$ and all $S \neq N$. An allocation (x^*, y^*) is in the (T, G) -core if and only if (x^*, y^*) is a Pareto optimal allocation satisfying $T(x^*, y^*; \{i\}) = \omega_i - x_i^*$ for all $i \in N$.*

Note that G can be a proper subset of $\{y' \in Y \mid y' \geq y\}$.

Proof. We first show the sufficiency. Let (x^*, y^*) be a Pareto optimal allocation satisfying $T(x^*, y^*; \{i\}) = \omega_i - x_i^*$ for all $i \in N$. Then, $f(x^*, y^*; N) = 0$ by the Pareto optimality of (x^*, y^*) .

Suppose there exists a coalition $S \neq N$ such that $f(x^*, y^*; S) > 0$. That is, there exists some $(x, y) \in U(x^*, y^*; S)$ such that

$$\sum_{i \in S} \omega_i + T(x^*, y^*; N \setminus S) - \left(\sum_{i \in S} x_i + c(y) \right) = \varepsilon > 0.$$

Define

$$\bar{x}_i = \begin{cases} x_i + \frac{\varepsilon}{n} & \text{if } i \in S, \\ x_i^* + \frac{\varepsilon}{n} & \text{otherwise.} \end{cases}$$

Then, $\sum_{i \in N} \omega_i = \sum_{i \in N} \bar{x}_i + c(y)$. Since $(x, y) \in U(x^*, y^*; S)$, we have $y \geq y^*$. Thus, by the monotonicity of the preference, we have $(\bar{x}_i, y) \succ_i (x_i^*, y^*)$ for all $i \in N$. This contradicts $(x^*, y^*) \in A^*$. Hence $f(x^*, y^*; S) \leq 0$ for all $S \in \mathcal{N}$ and this implies (x^*, y^*) is in the (T, G) -core by the proof of Theorem 1. (In fact, $f(x^*, y^*; S) = 0$ for all $S \in \mathcal{N}$.)

Now, we show the necessity. Let (x^*, y^*) be an allocation in the (T, G) -core. It follows that $(x^*, y^*) \in A^*$. Suppose that there exists some $i \in N$ such that $T(x^*, y^*; \{i\}) \neq \omega_i - x_i^*$. Without loss of generality, we may assume $T(x^*, y^*; \{i\}) > \omega_i - x_i^*$, by **ADD**, **TF** and the Pareto optimality of (x^*, y^*) .

Then, since $(x^*, y^*) \in A^*$, we have

$$\begin{aligned} & f(x^*, y^*; N \setminus \{i\}) \\ &= \sum_{k \in N \setminus \{i\}} \omega_k + T(x^*, y^*; \{i\}) - \min_{(x', y') \in U(x^*, y^*; N \setminus \{i\})} \left(\sum_{k \in N \setminus \{i\}} x'_k + c(y') \right) \\ &\geq \sum_{k \in N \setminus \{i\}} \omega_k + T(x^*, y^*; \{i\}) - \left(\sum_{k \in N \setminus \{i\}} x_k^* + c(y^*) \right) \\ &> \sum_{k \in N} \omega_k - \left(\sum_{k \in N} x_k^* + c(y^*) \right) \\ &= 0. \end{aligned}$$

This contradicts that (x^*, y^*) is in the (T, G) -core by the proof of Theorem 1. Hence $(x^*, y^*) \in C(E, T, G)$. □

To show the nonemptiness of the (T, G) -core, we assume the following additional conditions.

(A): The preference relation \succsim_i is convex for all $i \in N$.

(B): $Y = \mathbb{R}_+$.

(C): The cost function c is convex.

Moreover, we introduce a stronger form of **TF** requiring the taxation be exact.

ETF: (*Exact Tax Feasibility*): T satisfies **TF**, and $T(x, y; N) = c(y)$ for any (x, y) with $\sum_{i \in N} x_i + c(y) = \sum_{i \in N} \omega_i$.

Theorem 2. *Assume (A), (B) and (C). Let T satisfy **ADD** and **ETF**, and $G(x, y; S) \subseteq \{y' \in Y \mid y' \geq y\}$ for any $(x, y) \in X$ and all $S \neq N$. Then, the (T, G) -core is nonempty.*

Proof. It suffices to show that there exists some $(x^*, y^*) \in A^*$ satisfying $x_i^* = T(x^*, y^*; \{i\})$ for all $i \in N$ by Proposition 1.

For each $i \in N$, there exists a continuous and quasi-concave utility function u_i which represents \succsim_i . Without loss of generality, we may assume that u_i is normalized such as $u_i(0, 0) = 0$ for all $i \in N$.

Consider the following maximization problem (P) .

$$\begin{aligned} & \max h \\ (P) \quad & \text{s.t. there exists some } (x, y) \in A \text{ such that} \\ & u_i(x_i, y) \geq hr_i \text{ for all } i \in N, \end{aligned}$$

where $r = (r_1, \dots, r_n)$ is in $\Delta = \{r \in \mathbb{R}_+^n \mid \sum_{i \in N} r_i = 1\}$.

By the compactness of A and the continuity of u_i of each $i \in N$, (P) has a maximum solution for any $r \in \Delta$, and the maximum solution $h(r)$

is given by $h(r) = \max_{(x,y) \in A} \min_{i \in N} \left(\frac{u_i(x_i, y)}{r_i} \right)$, which is a continuous function on Δ . Then, we have

$$A^*(r) = \{(x, y) \in A \mid u_i(x_i, y) \geq h(r)r_i \text{ for all } i \in N\}$$

is a compact set for any $r \in \Delta$ since u_i is continuous for all $i \in N$. We prove the following claims on $A^*(r)$.

Claim 1. *Let $r \in \Delta$.*

- (I): *If $(x, y) \in A^*(r)$, then $(x, y) \in A^*$.*
- (II): *If $r_i = 0$, then $x_i = 0$ for any $(x, y) \in A^*(r)$.*
- (III): *$A^*(r)$ is a nonempty, convex-valued correspondence from Δ into A which has a closed graph.*

Proof. (I). Fix any $r \in \Delta$. Let $(x, y) \in A^*(r)$. Suppose there exists some $(x', y') \in A$ such that $u_i(x'_i, y') > u_i(x_i, y)$ for all $i \in N$. This immediately contradicts that $h(r)$ is the maximum solution in (P). Hence (x, y) is Pareto optimal.

(II). Let $r \in \Delta$ be $r_i = 0$ for some $i \in N$. Suppose $x_i = \varepsilon > 0$ for some $(x, y) \in A^*(r)$. Define $\bar{x}_i = 0$ and $\bar{x}_k = x_k + \frac{\varepsilon}{n-1}$ for all $k \neq i$. Then, since u_k is strictly monotone in the private good for all $k \in N$, $(\bar{x}, y) \in A$ and $u_k(\bar{x}_k, y) \geq (h(r) + \xi)r_k$ for all $k \in N$, where ξ is a sufficiently small positive real. This contradicts that $h(r)$ is the maximum solution in (P). Hence $x_i = 0$.

(III). Fix any $r \in \Delta$. The nonemptiness follows from the definition, and the convex-valuedness follows from the quasi-concavity of the utility functions. Fix any $(x, y) \in A$. Let $\{r^\mu\}$ be a sequence in Δ which converges to $r \in \Delta$, and $\{(x^\mu, y^\mu)\}$ be a sequence which converges to (x, y) and satisfies $(x^\mu, y^\mu) \in A^*(r^\mu)$ for all $\mu = 1, 2, \dots$. Suppose $(x, y) \notin A^*(r)$. It follows that $u_i(x_i, y) < h(r)r_i$ for some $i \in N$. By the continuity of u_i and h , there exists a sufficiently large $\bar{\mu}$ such that

$u_i(x_i^{\bar{\mu}}, y^{\bar{\mu}}) < h(r^{\bar{\mu}})r_i^{\bar{\mu}}$. This contradicts the choice of $\{(x^\mu, y^\mu)\}$. Hence $A^*(r)$ has a closed graph. \square

Define

$$z_i(x, y) = T(x, y; \{i\}) - (\omega_i - x_i)$$

and

$$q_i(r, x, y) = \frac{r_i + \max_{k \in N} z_k(x, y) - z_i(x, y)}{1 + \sum_{j \in N} (\max_{k \in N} z_k(x, y) - z_j(x, y))}, \quad (4)$$

for each $i \in N$. It can be easily checked that $q = (q_1, \dots, q_n)$ is a continuous, single-valued function on $\Delta \times A$ into Δ by the continuity of T . Thus, by **(III)**, $q(r, x, y) \times A(r)$ is a nonempty, compact and convex-valued correspondence which has a closed graph. Then, there exists a point such that

$$(r^*, x^*, y^*) \in q(r^*, x^*, y^*) \times A(r^*) \quad (5)$$

by Kakutani's fixed point theorem.

Now, we show that (x^*, y^*) has the desired properties. It suffices to show that $z_i(x^*, y^*) = 0$ for all $i \in N$. Note that $\sum_{i \in N} x_i^* + c(y^*) = \sum_{i \in N} \omega_i$ by the monotonicity of the preference and the Pareto optimality of (x^*, y^*) . Therefore, $T(x^*, y^*; N) = c(y^*)$ by **ETF**.

Let $r_i^* = 0$ for some $i \in N$. Then, we have $x_i^* = 0$ by **(II)**, and $\max_{k \in N} z_k(x^*, y^*) = z_i(x^*, y^*)$ by (4) and (5). It follows that

$$\begin{aligned} 0 &\geq T(x^*, y^*; \{i\}) - \omega_i \\ &\geq T(x^*, y^*; \{k\}) - (\omega_k - x_k^*) \end{aligned} \quad (6)$$

for all $k \in N$. Suppose at least one of the inequality in (6) for some $k \in N$ is strict, then by **ADD**,

$$0 > T(x^*, y^*; N) - \left(\sum_{i \in N} \omega_i - \sum_{i \in N} x_i^* \right),$$

and by **ETF**,

$$\sum_{i \in N} \omega_i > \sum_{i \in N} x_i^* + c(y^*).$$

This contradicts that (x^*, y^*) is Pareto optimal since the preference is strictly monotone in the private good. Hence $z_k(x^*, y^*) = 0$ for all $k \in N$.

Let $r_i^* > 0$ for all $i \in N$. We show that $\max_{k \in N} z_k(x^*, y^*) = z_i(x^*, y^*)$ for all $i \in N$. Suppose there exists some $i \in N$ such that $\max_{k \in N} z_k(x^*, y^*) > z_i(x^*, y^*)$. It follows that

$$\sum_{j \in N} (\max_{k \in N} z_k(x^*, y^*) - z_j(x^*, y^*)) > 0.$$

Since there exists some $i' \in N$ such that $\max_{k \in N} z_k(x^*, y^*) = z_{i'}(x^*, y^*)$, (4) and (5) imply that $r_{i'}^* = 0$. This is a contradiction. Hence we have $\max_{k \in N} z_k(x^*, y^*) = z_i(x^*, y^*)$ for all $i \in N$.

Next, we show that $z_i(x^*, y^*) = 0$ for all $i \in N$. Suppose not, then, (*case 1*) $z_i(x^*, y^*) > 0$ for all $i \in N$, or (*case 2*) $z_i(x^*, y^*) < 0$ for all $i \in N$. When (*case 1*) holds, $\sum_{i \in N} \omega_i > \sum_{i \in N} x_i^* + c(y^*)$ by **ADD** and **ETF**. This contradicts that $(x^*, y^*) \in A^*$ since u_i is strictly increasing in the private good. When (*case 2*) holds, $\sum_{i \in N} \omega_i < \sum_{i \in N} x_i^* + c(y^*)$ by **ADD** and **ETF**. This also contradicts that $(x^*, y^*) \in A^*$ since (x^*, y^*) is not feasible. Hence $z(x^*, y^*) = 0$ for all $i \in N$. \square

6. SOME SPECIFIC TAX SYSTEMS

In this section, we show that some specific tax systems appeared in the literature satisfy **ADD** and **ETF**.

6.1. The cost share system. A collection of functions $h = (h_1, \dots, h_n)$ where $h_i : Y \rightarrow \mathbb{R}_+$, $\forall i \in N$, which satisfies $\sum_{i \in N} h_i(y) = c(y)$ for any $y \in Y$ and $h_i(y) \leq \omega_i$ for any $y \in Y^*$ and for all $i \in N$, is a cost share system defined in Mas-Colell and Silvestre [11].

Let us define a tax system according to a cost share system h as follows.

$$T^h(x, y; S) = \sum_{i \in S} h_i(y),$$

for any $(x, y) \in X$ and for all $S \in \mathcal{N}$.

It follows from the definition that T^h satisfies **ADD** and **ETF**.

6.2. The proportional income tax system. We show that a certain proportional income tax system defined in Nakayama [9] also satisfies **ADD** and **ETF**.

The proportional income tax system is given as follows.

$$T^e(x, y; S) = \left(1 - \frac{\sum_{i \in N} x_i}{\sum_{i \in N} \omega_i}\right) \sum_{i \in S} \omega_i.$$

It is obvious that T^e satisfies **ADD**. Moreover, since $T^e(x, y; N) - c(y) = \sum_{i \in N} \omega_i - (\sum_{i \in N} x_i + c(y))$, it can be easily checked that T^e satisfies **ETF**.

7. CONCLUDING REMARKS

In this paper, we have defined the nucleolus allocation in a public goods economy with taxation. It is proved that the nonempty core coincides with the set of the nucleolus allocations in the public goods economy with taxation. We also show the nonemptiness of the core, and the core consists of the Pareto optimal allocations where each agent pays own tax payment exactly. Note that such an allocation is generally called the public competitive equilibrium, defined by Foley [3].

There is an objection to the nucleolus in TU coalitional games such that the excess of the coalitions are treated equally independently of the size of the coalition. The weighted nucleolus, defined with the weighted excess, is an answer to this objection. In our model, it has been shown in the proof of Theorem 1 that the excess of each coalition is 0 at any allocation in the (T, G) -core, and there is at least one coalition whose effective excess is greater than 0 at any allocation not in the (T, G) -core. Thus, our results apply to the weighted nucleolus allocations.

REFERENCES

- [1] Champsaur, P., "How to Share the Cost of a Public Good," *International Journal of Game Theory* **4** (1975), 113-129
- [2] Champsaur, P., D. J. Roberts and R. W. Rosenthal, "On Cores in Economies with Public Goods," *International Economic Review* **16** (1975), 751-764
- [3] Foley, D. K., "Resouce Allocation and the Public Sector," *Yale Economic Essays* **7** (1967), 45-98
- [4] Foley, D. K., "Lindahl's Solution and the Core of an Economy with Public Goods," *Econometrica* **38** (1970), 66-72
- [5] Hirokawa, M., "The Equivalence of the Cost Share Equilibria and the Core of a Voting Game in a Public Goods Economy," *Social Choice and Welfare* **9** (1992), 63-72
- [6] Kalai, E., "Excess Functions for Games without Side Payments," *SIAM Journal of Applied Mathematics* **29** (1975), 60-71
- [7] Kaneko, M., "The Ratio Equilibrium and a Voting Game in a Public Goods Economy," *Journal of Economic Theory* **16** (1977), 123-136
- [8] Kaneko, M., "The Ratio Equilibria and the Core of the Voting Game $V(N, W)$ in a Public Goods Economy," *Econometrica* **45** (1977), 1589-1594
- [9] Nakayama, M., "Proportional Income Taxes and Cores in a Public Goods Economy," *Journal of Economic Theory* **15** (1977), 295-300
- [10] Nakayama, M., "A Note on a Generalization of the Nucleolus to Games without Sidepayments," *International Journal of Game Theory* **12** (1983) 115-122
- [11] Mas-Colell, A. and J. Silvestre, "Cost Share Equilibria: A Lindahlian Approach," *Journal of Economic Theory* **47** (1989), 239-256
- [12] McLean, R.P. and A. Postlewaite, "Excess Functions and Nucleolus Allocations of Pure Exchange Economies," *Games and Economic Behavior* **1** (1989), 131-143
- [13] Rosenthal, R.W., "Cooperative Games in Effectiveness Form," *Journal of Economic Theory* **5** (1975), 88-101
- [14] Schmeidler, D., "The Nucleolus of a Characteristic Function Game," *SIAM Journal of Applied Mathematics* **17** (1969), 1163-1170