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On the Measurement of Welfare Changes at second-Best Optima

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ON THE MEASUREMENT OF WELFARE CHANGES AT SECOND-BEST OPTIMA

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We evaluate the welfare change from one second best position to another when a small change takes place in the distortionary constraint of the model. Simple formulae for welfare changes depending on inverse substitution coefficients, which turn out to be direct generalizations of those obtained in the partial equilibrium framework, are derived by evaluating losses at second best optimal points. The Antonelli-Hicks-Allen transformation will bring out analytical similarities among the theorems that arise from various second-best problems which are discussed in the literature.

1 Introduction

When not all of the agents in an economy act on a common efficiency price vector (which is proportional to the vector of marginal rates of substitution whenever the latter exists), the allocation is not Pareto optimal under the standard assumptions about the economic environment, including the convexity of preferences and technologies and the non-existence of external economies. The price distortion as described above is an unavoidable consequence of sales taxes and monopolistic pricing.

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Allais(1973) and Debreu(1951), among others, defined a measure of welfare loss in a non-optimal state by the proportion of the initial resources that would be unnecessary if the distortions were completely removed (see, also Kawamata(1974)). There are also related measures of loss based on the second order derivatives of the welfare function evaluated at the initial optimal point (see, Hotelling(1951), Boiteaux(1951) and Harberger(1964).The relationships among these and some other measures of loss are discussed by Diewert(1981).

The chief purpose of this paper is to evaluate the welfare change from one second best position to another when there takes place a small change in the distortionary constraint of the model. We measure the loss by the amount of a single good, say the numeraire, that would just compensate for a one unit change in the distortionary constraint. Clearly, the idea underlying this measure is quite similar to those of Debreu and Allais.

Diewert(1981) conjectured that the welfare loss will increase if substitutability in consumption increases. A similar result was established by Samuelson (2001) in a economic model where the representative consumer has a CES utility function and a sales tax is imposed to finance a good for public service. We will show that simple formulae for welfare changes depending on inverse substitution coefficients can be derived when losses are evaluated at second best optimal points. The formulae for welfare changes are usually very complicated, and to simplify the results and facilitate comparisons they need to be evaluated at the first best points in the previous work (cf. e.g., Corlett and Hague(1953-54), Atkinson and Stiglitz(1980) and Diewert(1981) and Diewert, Turunen-Red and Woodland(1989).

The formulae for welfare changes at second-best optima take on somewhat different forms depending on the nature of the constraints. In the case where there is a single exogenously given ad valorem distortion of the Lipsey-Lancaster(1956-57) type, the measure may be expressed in terms of tax revenue and the demand and supply elasticities. If there is a constraint on the government budget as in the classical model of Ramsey(1927), extensively studied by Diamond and Mirrelees(1971) among others, it can be expressed by a dimensionless number depending only on the elasticities. Similar problems arise in international economics and public economics. We refer to Lloyd(1974) and Dixit(1975) for special features of the models. The Antonelli-Hicks-Allen transformation will help to bring out analytical similarities among the theorems that arise from various second-best problems discussed in the literature. This approach is especially useful to evaluate the welfare loss at non-Pareto optimal points where standard duality formula for the consumption theory is not directly applicable. Early literature using the Antonelli matrix in the second-best analysis includes Lloyd(1977), Kawamata(1977) and Deaton(1981).

2 Partial Equilibrium Analysis

Before presenting the formal model and stating general results on the welfare effects of changes in distortionary constraints, we will demonstrate the solutions of our problems in a partial equilibrium framework. Although the analysis in this section uses the notion of consumer's surplus, our formal analysis and results in the later sections do not rely on this concept.

In the standard partial equilibrium framework, let x be the amount of output, and $p = p(x)$ and $q = q(x)$ be the inverse demand and supply functions with $p'(x) < 0$ and $q'(x) > 0$. Suppose that two curves intersect at a point E which defines the competitive output $x(0)$ and the competitive price $p(x(0))$(See Fig.1).

(Fig 1 about here)

We are interested in evaluating the changes of the deadweight loss in three different situations. In each case, the deadweight loss is expressed as

$$\mathcal{L}(x^*(\alpha)) = \int_{x^*(\alpha)}^{x(0)} (p(x) - q(x))dx, \tag{1}$$

where $x(0)$ is the competitive output level and $x^* = x^*(\alpha)$ with $x^*(\alpha) < x(0)$ is the actual output level dependent on a parameter α.

Differentiating $\mathcal{L}(x^*(\alpha))$ with respect to α we have

$$d\mathcal{L}(x^*(\alpha))/d\alpha = -\left(p(x^*) - q(x^*)\right) \cdot dx^*/d\alpha \tag{2}$$

In the following we will consider three particular types of distortion arising from (i) imposition of a specific tax, (ii) imposition of an ad valorem tax and (iii) Ramsey type tax revenue requirement.

(i) Let the specific tax rate s be given parametrically. Corresponding to each s in the range $0 < s < p(0) - q(0)$, is a unique $x = x(s)$ such that

$p(x) - q(x) = s$. This relation implies that $x'(s) = 1/(p' - q')$. Inserting this into (2), the derivative of the deadweight loss $\mathcal{L}(x(s))$ with respect to s is computed as

$$d\mathcal{L}/ds = s/(p' - q') \tag{3}$$

where all functions are evaluated at $x(s)$. This case has been studied by Atkinson and Stiglitz(1980, p.367-369). Note that the dimension of $d\mathcal{L}/ds$ is the same as that of the output level x.

(ii) For each ad valorem tax rate t, $0 < t < (p(0) - q(0))/q(0)$, there is a unique $x = x(t)$ such that $t = (p(x) - q(x))/q(x)$. It follows from this that $x'(t) = q/(p' - (1+t)q')$. Hence the derivative of the deadweight loss $\mathcal{L}(x(t))$ with respect to t is computed as

$$\begin{aligned} \frac{d\mathcal{L}}{dt} &= \frac{(p-q)q}{(1+t)q' - p'} \\ &= \frac{(p-q)x}{(1+t)(1/\varepsilon + 1/\eta)} \end{aligned} \tag{4}$$

where all functions are evaluated at $x(t)$ and $1/\varepsilon$ and $1/\eta$ are the elasticities of inverse demand and supply functions (i.e., ε and η are the elasticities of demand and supply functions). Thus the change in the deadweight loss is related positively to the tax revenue and negatively to the sum of inverse elasticities. We also note that the dimension of $d\mathcal{L}/dt$ is the same as that of the tax revenue.

(iii) Suppose that the government has to raise a fixed amount $R > 0$ of tax revenue by distortionary taxation. This implies that the output level $x = x(R)$ should be chosen so that $(p(x) - q(x))x = R$.

This solution exists if R is given to be not greater than the maximum of the tax revenue function. But there remains a complication in this case due to the fact that there may be more than one solution to the above equation. Disregarding abnormal situations, we shall confine ourselves to the range of R (and x) where tax revenue is increasing in x (See, Samuelson(2002) for a comment on the decling branch of the tax burden curve).

Thus let $x > 0$ be the output level where the local maximum of the tax revenue function is attained closest to the competitive output $x(0)$, and

$\underline{R} > 0$ be the corresponding tax revenue. In the interval $[\underline{x}, x(0)]$ tax revenue is a decreasing function of x and we have $x'(R) = 1/\left((p-q)+(p'-q')x\right)$. The derivative of the deadweight loss $\mathcal{L}(x(R))$ with respect to R $(0 < R < \underline{R})$ is now computed as

$$\begin{aligned}\frac{d\mathcal{L}}{dR} &= \frac{-(p-q)}{(p-q)+(p'-q')x} \\ &= \frac{t}{1/\varepsilon(1+t)+1/\eta - t} \quad , \end{aligned} \tag{5}$$

where functions are evaluated at $x(R)$. Notice that the sum of the first two terms in the denominator represents the elasticities of "price-wedge function" $p-q$ and that the whole expression is independent of the units of the output level and the two prices.

It turns out that we can define a measure of welfare (which does not rely on the concept of consumer's surplus) so that similar, simple formulae can be derived for the general second-best models that have appeared in the literature. Theorem 1 in section 4 generalizes the above formula (4) for the problem of Lipsey and Lancaster (1956-57) with an a priori specified ad valorem distortion. Theorem 2 in section 5 does the same to formula (3) for the similar problem with a specific distortion. The formula corresponding to (5) will be obtained for the classical problem of Ramsey (1927) with a tax revenue constraint. We shall also examine the conditions under which the measure (as expressed by one of the above formulae) would faithfully reflect the true changes in the welfare (utilities of the consumer).

3 Basic Assumptions

We will consider a closed economy. Let $x = (x_1, ..., x_n)$ be the production vector and $\omega = (\omega_1, ..., \omega_n)$ the vector of initial endowment of commodities. The amounts of consumption may then be expressed as $x + \omega$.

For expository convenience we suppose that there is a representative consumer and the production technology of the whole economy is given. We assume that the utility of the consumer can be expressed by a real valued function

$$z = f(x + \omega), \tag{6}$$

and the production technology by

$$g(x) = 0, \tag{7}$$

another real valued function. It is implicit in this formulation that there is no distortion within the production sector (and consumption sector). Only the inter-sectoral distortion will be considered in this paper.

We suppose that f and g have continuous partial derivatives up to the second order, to be denoted as $f_i = \partial f/\partial x_i$ and $f_{ij} = \partial^2 f/\partial x_j \partial x_i$ etc. It is assumed that

$$f_i \geq 0,\ g_i \leq 0 \qquad (i = 1, 2..., n) \tag{8}$$

with strict inequality holding for $i = 1$ and n. It is also assumed that f is a strictly quasi-concave function and g a strictly quasi-convex function.

Let us denote the consumer's (resp. producer's) marginal rate of substitution of commodity i for commodity n by $P^i(x)$ (resp. $Q^i(x)$):

$$\begin{aligned} P^i(x) &= f_i(x)/f_n(x) \qquad (i = 1, 2, ..., n) \\ Q^i(x) &= g_i(x)/g_n(x) \qquad (i = 1, 2, ..., n). \end{aligned} \tag{9}$$

Let i, j and n be the indices of three different commodities, with n being used as numeraire. According to Hicks-Allen (1934), commodity i is a substitute for (resp. a complement to) j if $P^i(x)$ is decreased (resp. increased) when j is substituted for the numeraire in such a way as to leave the consumer at the same utility level as before.

Formally, commodity i is a substitute for or complement to commodity j depending on whether

$$a_{ij}(x) = P^i_j(x) - P^j(x)P^i_n(x) \tag{10}$$

is negative or positive (where $P^i_j(x)$ denotes the partial derivative of $P^i(x)$ with respect to the j-th argument). The first term of (10) measures how $P^i(x)$ (the marginal rate of substitution of commodity i for commodity n) changes due to the change in the consumption of commodity j and the second term measures the compensating change in the amount of the numeraire good to keep the consumer in the same utility level. It is clear that the second term vanishes if the utility function is linear in x_n, (that is if $u(x_1, \cdots, x_n)$ may be written as $v(x_1, x_2, \cdots, x_{n-1}) + x_n$). It is known that Hicks-Allen definition of substitutability coincides with the Slutzky-hicks definition for the three commodity case.

It will be assumed that the Antonelli matrix $A = (a_{ij}) \quad (i, j = 1, 2..., n-1)$ is symmetric and negative definite. It is known (see, Samuelson(1950) and Katzner (1970)) that A is the inverse of Slutzky-Hicks substitution matrix of the first $n-1$ commodities and so the above conditions on A hold, in particular, if the Slutzky matrix is symmetric and negative definite.

For the producer we shall similarly define

$$b_{ij}(x) = Q_j^i(x) - Q^j(x)Q_n^i(x) \tag{11}$$

and assume that i is symmetric and positive definite. In the case where production technology is linear, $B = (b_{ij})$ is a 0 matrix

4 Ad valorem Distortion and Welfare Loss

Although we have analyzed the specific distortion first and then ad valorem distortion in Section2, we will reverse the order in the following analysis. The ad valorem case is studied in this section and the spucific case is studied in the next section. This is due to expository convenience; once the ad valorem case is studied, key equations for the specific case may be written with minor changes but not the other way round.

Let us assume that there is a given ad valorem distortion $t_1 \neq 0$ between commodities 1 and n. The situation typically arrises when the n-th commodity is untaxable and used as numeraire and an ad valorem tax is imposed on commodity 1. By definition, t_1 must satisfy

$$P^1(x + \omega) = (1 + t_1)\, Q^1(x) \tag{12}$$

at the final allocation. For definiteness we shall assume that t_1 is positive.

Following Lipsey and Lancaster (1956-57), we consider the problem of maximizing the objective function f with respect to $(x_1, \cdots, x_n)$ subject to the production and institutional constraints (7) and (12). It is convenient to denote the optimal ad valorem distortion by t_i. By definition t_i satisfies

$$P^i\left(x+\omega\right)=\left(1+t_i\right)Q^i\left(x\right) \quad , \quad \left(i\in I\right) \tag{13}$$

where the functions are evaluated at the second best optimal and I is the set of indices such that $Q^i(x)\neq 0$ (i.e., $g_i\neq 0$). We will assume that $I=\{1,2,\cdots,n-1\}$ in the following discussion (see, Remark 1 (iv)).

Expressing the Lagrangean of the problem as

$$z=f\left(x+\omega\right)-\alpha g\left(x\right)+\beta\left(P^1\left(x+\omega\right)-\left(1+t_1\right)Q^1\left(x\right)\right) \tag{14}$$

we obtain the following first order conditions:

$$z_r=f_r\left(x+\omega\right)-\alpha g_r\left(x\right)+\beta\left(P_r^1\left(x+\omega\right)-\left(1+t_1\right)Q_r^1\left(x\right)\right)=0 \qquad \left(r=1,2,...,n\right) \quad . \tag{15}$$

In view of definitions (9), we may express them in the following matrix form:

$$\begin{pmatrix} Q^r & P_r^1-\left(1+t_1\right)Q_r^1 \\ 1 & P_n^1-\left(1+t_1\right)Q_n^1 \end{pmatrix}\begin{pmatrix} \alpha g_n \\ \beta \end{pmatrix} = \begin{pmatrix} P^r \\ 1 \end{pmatrix} f_n \qquad \left(r=1,2,...,n-1\right) \tag{16}$$

Let $\Delta\left(r\right)$, where r range from 2 to $n\left(r=1,2,...,n-1\right)$, denote the 2×2 matrix in (16). This is expressed as

$$\begin{aligned}\Delta\left(r\right)&=\begin{vmatrix} Q^r & P_r^1 \\ 1 & P_n^1 \end{vmatrix}-\left(1+t_1\right)\begin{vmatrix} Q^r & Q_r^1 \\ 1 & Q_n^1 \end{vmatrix} \\ &=\left(1+t_1\right)b_{1r}-a_{1r}-\left(P^r-Q^r\right)P_n^1 \quad .\end{aligned} \tag{17}$$

In particular, we have

$$\Delta\left(1\right)=\left(1+t_1\right)b_{11}-a_{11}-t_1Q^1P_n^1 \quad , \tag{18}$$

or alternatively,

$$\Delta\left(1\right)=\left(1+t_1\right)b_{11}-(a_{11}+t_1P_1^1)/\left(1+t_1\right) \quad . \tag{19}$$

It is convenient to summarize some of the implications of our analysis before we proceed.

Remarks 1

(i) We have $P_1^1 = (f_1 f_{n1} - f_n f_{11}) \, / \, (f_n)^2$ and $P_n^1 = (f_n f_{1n} - f_1 f_{nn}) \, / \, (f_n)^2$, by definition. If there are only two commodities 1 and n, the condition $P_1^1 < 0$ (resp. $P_n^1 > 0$) implies that commodity n (resp. commodity 1) is normal in the sense that the income term of the corresponding Slutzky equation is positive, or that the income consumption curve (Hicks (1946), pp27-28) between the two commodities slopes upward and to the right. Similar implications are true even when there are more than two commodities so long as the consumption of commodities other than 1 and n (resp. 1) is normal against commodity 1 (resp. n) if $P_1^1 < 0$ (resp. $P_n^1 > 0$).

(ii) It can readily be shown that

$$a_{11} = - \begin{vmatrix} 0 & f_1 & f_n \\ f_1 & f_{11} & f_{1n} \\ f_n & f_{n1} & f_{nn} \end{vmatrix} \, / \, (f_n)^3 \, ,$$

which is negative under our assumptions. Hence we must have either $P_1^1 < 0$ or $P_n^1 > 0$. Similarly, our assumption that $b_{11} > 0$ implies that either $Q_1^1 > 0$ or $Q_n^1 < 0$.

(iii) From (19) and Remark (i), it follows that $\Delta(1) > 0$ whenever $P_1^1 < 0$, i.e., whenever commodity n is normal against commodity 1.

(iv) It is possible to show (although the result will not be needed for the main discussions) that the optimal ad valorem distortion t_i $(i \in I)$ of the problem may be expressed as

$$\frac{t_i}{t_1} = \frac{a_{1i} - (1 + t_1) \, b_{1i}}{a_{11} - (1 + t_1) \, b_{11}} \cdot \frac{Q'}{Q^i} \tag{20}$$

(cf. Kawamata [12] equation (15)). Hence $t_1 > 0$ implies $t_1 > t_i > 0$ if commodity i $(i \in I)$ is a substitute for 1 when n is used as numeraire (see, Kawamata [12]). Using the above formula we can immediately infer that

$$\Delta(r) = \frac{a_{1r} - (1 + t_1) \, b_{1r}}{a_{11} - (1 + t_1) \, b_{11}} \cdot \Delta(1) \quad , \quad (r \in I) \, . \tag{21}$$

¤

Assume that commodity n is normal against commodity 1. Then $\Delta(1)$ is positive and so from (16) we find that

$$\beta = \frac{-t_1 Q^1 f_n}{\Delta(1)} \tag{22}$$

is negative.

We now proceed to study the welfare effect of changes in the initial endowment ω_i and the given distortion t_1. Our solution vector x determined by (7) ,(9),(12)and(15) are now considered to be a vector of functions of these two parameters which can be found by applying the implicit function theorem.

We first differentiate (14) with respect to ω_i, remembering that α and β as well as x are continuously differentiable functions of ω_i in the neighborhood of the original second best state. We can thus evaluate the marginal utility of ω_i at the second best allocation as

$$\begin{aligned}\frac{\partial z}{\partial \omega_i} &= \sum_{r=1}^{n} z_r \frac{\partial x_r}{\partial \omega_i} + f_i - \beta P_i^1 \\ &= f_i - \beta P_i^1 \ , \ (i = 1, 2, \cdots, n),\end{aligned} \tag{23}$$

using (15). Since $\beta < 0$ we infer that the "social" marginal utility of ω_i is greater than "private" marginal utility if and only if ω_i raises P^1:

<u>Proposition 1</u>

If commodity n is normal against commodity 1 then $\partial z / \partial \omega_i$ is greater than or smaller than $f_i = \partial f / \partial \omega_i$ depending on whether P_i^1 is negative or positive. In particular, $\partial z / \partial \omega_1 < f_1$ and $\partial z / \partial \omega_n > f_n$.

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From (18), (22) and (23) it is easy to derive

$$\frac{\partial z}{\partial \omega_n} = \frac{(a_{11} - (1 + t_1)\, b_{11}) f_n}{-\Delta(1)} \quad , \tag{24}$$

a result which will be used in a moment. A similar result for $\partial z / \partial \omega_i$ follows from (20).

We next consider the effect of a change in t_1, remembering that x, α and β are functions of t_1 and obtain

$$
\begin{aligned}
\frac{\partial z}{\partial t_i} &= \sum_{r=1}^{n} z_r \frac{\partial x_r}{\partial t_1} + \beta Q^1 \\
&= \beta Q^1 \\
&= -\left(P^1 - Q^1\right) Q^1 f_n / \Delta(1),
\end{aligned} \tag{25}
$$

using (15) and (22).

We define the inverse elasticity of demand ε_i^- and that of supply η_i^- by

$$
\varepsilon_i^- = -a_{ii} x_i / P^i \tag{26}
$$

and

$$
\eta_i^- = b_{ii} x_i / Q^i \tag{27}
$$

for each $i = 1, 2, \cdots, n-1$ such that $P^i \neq 0$ and $Q^i \neq 0$.

Given an ad valorem distortion $t_1 > 0$ we measure the economic loss due to a marginal change in t_1 by the amount of ω_n that would just compensate for it. Our first Theorem states that the loss that accompanies the change can be evaluated in terms of the revenue from the distortion (tax revenue) and the sum of inverse demand and inverse supply elasticities: Roughly, this means that welfare loss increases with effective tax revenue and decreases with the sum of the inverse demand and supply elacsticities.

Theorem 1

Suppose that commodity n is normal against 1 (i.e., $P_1^1 < 0$) and that the ad valorem distortion is at a level $t_1 > 0$. Then the rate of change of ω_n that would just compensate for a marginal change in t_1 is given by $\left(P^1 - Q^1\right) x_1 / \left(1 + t_1\right) \left(\varepsilon_i^- + \eta_i^-\right)$.

Proof) By (24) and (25) we have

$$
\left.\frac{\partial \omega_n}{\partial t_1}\right|_{z\,:\,\text{const}} = -\frac{\partial z}{\partial t_1} / \frac{\partial z}{\partial \omega_n}
$$

$$
= \frac{-\left(P^1 - Q^1\right) Q^1}{a_{11} - \left(1 + t_1\right) b_{11}}
$$

$$
= \frac{\left(P^1 - Q^1\right) x_1}{\left(1 + t_1\right)\left(\varepsilon_i^- + \eta_i^-\right)} \tag{28}
$$

as was to be shown.

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Remarks 2

Using the fact that the Antonelli matrix is the inverse of the $(n-1)\times(n-1)$ Slutzky matrix of the first $(n-1)$ goods, it is possible to express the inverse elasticities in terms of the ordinary elasticities. For the three commodity case, this is especially simple if the utility function is of form $U(X, Y) + Z$. Denoting ordinary elasticities of demand as $\varepsilon_{XX}(p, q) = pX_p/X$, $\varepsilon_{XY}(p, q) = -pX_q/Y$ etc. and inverse elasticities of demand as $\varepsilon^-_{XX}(X, Y) = -XP_X/P$, $\varepsilon^-_{XY}(X, Y) = -XP_Y/Q$ etc., we have $\varepsilon^-_{XX} = \varepsilon_{YY}/\Delta, \varepsilon^-_{XY} = \varepsilon_{YX}/\Delta, \varepsilon^-_{YX} = \varepsilon_{XY}/\Delta$ and $\varepsilon^-_{YY} = \varepsilon_{XX}/\Delta$, where $\Delta = \varepsilon_{XX}\varepsilon_{YY} - \varepsilon_{YX}\varepsilon_{XY}.$

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5 Specific Distortion and Welfare Loss

In this Section we consider the case where a specific distortion of a given rate $s_1 > 0$ is imposed on commodity 1 in term of commodity n. The requirement is as in the case of ad valorem distortions. The previous condition (12) must now be replaced by

$$
P^1(x + \omega) - Q^1(x) = s_1. \tag{29}
$$

Our problem now is to maximize the objective function (1) with respect to $(x_1, \cdots, x_n)$ subject to the constraints (7) and (29). Let us write the Lagrangean of the problem as

$$
z = f(x + \omega) - \alpha g(x) - \beta\left(P^1(x + \omega) - Q^1(x) - s_1\right). \tag{14'}
$$

We then obtain the first order conditions

$$z_r = f_r\left(x+\omega\right) - \alpha g_r\left(x\right) - \beta\left(P_r^i\left(x+\omega\right) - Q_r^i\left(x\right)\right) = 0, \qquad (r = 1, 2, ..., n) \;.$$

This is of the same form as (15) for $t_1 = 0$. Hence (16) must hold for $t_1 = 0$. Therefore the last expression of (17) now reduces to

$$\Delta(r) = b_{1r} - a_{ir} - s_r P_n^1 \quad (r = 1, 2, ..., n-1)\,, \tag{31}$$

where s_r is the specific distortion defined by

$$s_r = P^r\left(x+\omega\right) - Q^r(x) \quad , \quad (r = 1, 2, ..., n-1)\,. \tag{32}$$

In particular, we have

$$\Delta\left(1\right) = b_{11} - a_{11} - s_1 P_1^1, \tag{33}$$

or alternatively,

$$\Delta\left(1\right) = b_{11} - \left(Q^1 a_{11} + s_1 P_1^1\right)/P^1. \tag{34}$$

<u>Remarks 3</u>

(i) From (33), it is clear that $\Delta(1)$ is positive if commodity n is normal against commodity 1. (i.e., if $P_1^1 < 0$).

(ii) It is possible to show using the first order conditions that the optimal distortion s_i $(i = 1, 2, ..., n)$ of the present problem can be expressed as

$$s_i = \left(a_{1i} - b_{1i}\right) s_1 / \left(a_{11} - b_{11}\right) \tag{35}$$

(see Kawamata (1977), equation (18)). It therefore follows that

$$\Delta(r) = \left(a_{1r} - b_{1r}\right) / \left(a_{11} - b_{11}\right) \Delta(1) \;,\; (r = 2, 3, \cdots, n-1). \tag{36}$$

¤

Assume that commodity n is normal against commodity 1. Then, $\Delta(1)$ is positive and so we may express β as

$$\beta = -s_1 f_n / \Delta(1) \tag{37}$$

using (16) for $t_1 = 0$.

The welfare effect of a change in ω_i may be evaluated as

$$\frac{\partial z}{\partial \omega_i} = f_i - \beta P_i^1 \ , \ (i = 1, 2, \cdots, n) \tag{38}$$

in the same way as we derived (23). Since $\beta < 0$ under our assumption, Proposition 1 is true for the specific case also.

We note also that for $i = n$ (38) may be written as

$$\frac{\partial z}{\partial \omega_n} = \frac{(b_{11} - a_{11}) f_n}{\Delta(1)} \tag{39}$$

using (37). On the other hand, the welfare effect of a change in s_1 may be evaluated as

$$\begin{aligned} \frac{\partial z}{\partial s_1} &= \beta \\ &= \frac{-s_1 f_n}{\Delta(1)} \end{aligned} \tag{40}$$

in the same way as we derived (25).

We now state:

Theorem 2

Suppose that commodity n is normal against 1 (i.e. $P_1^1 < 0$), and that the specific distortion (tax rate) is at a level $s_1 > 0$. The rate of change of ω_n that would justly compensate for a marginal change in s_1 is given by $s_1 / (b_{11} - a_{11})$.

Proof) By (39) and (40) we have

$$\begin{aligned} \left. \frac{\partial \omega_n}{\partial s_i} \right|_{z:\ \text{const}} &= -\frac{\partial z}{\partial s_i} / \frac{\partial z}{\partial \omega_n} \\ &= \frac{s_1}{b_{11} - a_{11}} \end{aligned} \tag{41}$$

as was to be shown.

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6 Ramsey's Problem

In this section we consider the situation in which the government raises a fixed amount of revenue by means of distortionary taxation. The government's problem is to maximize the object function

$$z = f(x + \omega) \tag{6}$$

with respect to $(x_1, \cdots, x_n)$ subject to the production technology constraint

$$g(x) = 0, \tag{7}$$

and the revenue restriction

$$\sum_{i=1}^{n-1} \left(P^i (x + \omega) - Q^i (x) \right) x_i = R \ , \tag{42}$$

where $R \geq 0$ is the exogenously specified amount of taxation revenue to be raised. Commodity n is assumed to be untaxable. We suppose that $R \geq 0$ is such that the set of feasible allocations (namely, x which satisfies (7) and (42) is not empty).

This problem was first studied in the classical paper by Ramsey (1927) and has been modified and extended by other economists. Major contributions in related fields are listed in Atkinson and Stiglitz (1980) and Sheshinski (1986). See, also Baumol and Willing (1982). Our primary concern here is to examine the sensitivity of welfare to the changes in R and ω_n (the initial endowment of commodity n).

We write the Lagrangean of the problem as

$$z = f\,(x + \omega) - \alpha g(x) + \beta (\sum_{i=1}^{n-1} \left(P^i (x + \omega) - Q^i (x) \right) x_i - R \ ) \tag{43}$$

and derive the following first order conditions:

$$z_r = f_r\,(x + \omega) - \alpha g_r(x) + \beta (\sum_{i=1}^{n-1} \left(P^i_r (x + \omega) - Q^i_r (x) \right) x_i - P^r (x + \omega) - Q^r (x))$$

$$= 0$$

$$(r = 1, 2, ..., n) \tag{44}$$

This may be expressed in matrix form as

$$\begin{pmatrix} P^1 & Q^1 & \sum_i \left(P_1^i - Q_1^i\right) x_i + \left(P^1 - Q^1\right) \\ P^r & Q^r & \sum_i \left(P_r^i - Q_r^i\right) x_i + \left(P^r - Q^r\right) \\ 1 & 1 & \sum_i \left(P_n^i - Q_n^i\right) x_i \end{pmatrix} \begin{pmatrix} -f_n \\ \alpha g_n \\ \beta \end{pmatrix} = 0$$

$$(r = 2, 3, ..., n-1)\,, \tag{45}$$

where each summation i runs from 1 to $n-1$.

This implies that, for each r, the determinant of the 3×3 matrix on the left-hand side is zero; hence we have

$$\begin{vmatrix} P^1 & Q^1 & \sum_i \left(a_{i1} - b_{i1}\right) x_i \\ P^r & Q^r & \sum_i \left(a_{ir} - b_{ir}\right) x_i \\ 1 & 1 & 0 \end{vmatrix} = 0 \tag{46}$$

Denoting the optimal ad valorem tax rate by $t_r \quad (r = 2, ..., n-1)$, we have

$$t_r Q^r \sum_{i=1}^{n-1} \left(a_{i1} - b_{i1}\right) x_i = t_1 Q^1 \sum_{i=1}^{n-1} \left(a_{ir} - b_{ir}\right) x_i$$

$$(r = 2, ..., n-1)\,. \tag{47}$$

Hence

$$\theta = \sum_{i=1}^{n-1} \left(a_{ir} - b_{ir}\right) x_i / t_r Q^r \tag{48}$$

is independent of r.

<u>Lemma 1</u>
If $R > 0$ then $\theta < 0$

Proof)

$$\begin{aligned} R\theta &= \sum_{r=1}^{n-1} t_r Q^r x^r \theta \\ &= x' \left(A - B\right) x, \end{aligned} \tag{49}$$

where A (resp. B) is the Antonelli-matrix for consumption (resp. production). Since A is negative definite and B is positive definite, the result follows.

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A direct consequence of the above analysis is the following n-commodity version of Ramsey's formula originally stated in this form for one commodity case:

Theorem 3

An optimal tax rule for Ramsey's problem is given by

$$t_r = -\left(\sum_{i=1}^{n-1} \varepsilon_{ir}^{-} + \sum_{i=1}^{n-1} \eta_{ir}^{-}\right) / (\theta + \sum_{i=1}^{n-1} \varepsilon_{ir}^{-})$$
$$(r = 1, 2, ..., n-1) \tag{50}$$

where ε_{ir}^{-} and η_{ir}^{-} are inverse demand and supply elasticities defined by $\varepsilon_{ir}^{-} = -a_{ir}x_i/P^r$ and $\eta_{ir}^{-} = b_{ir}x_i/Q^r \quad (i, r = 1, 2, ...n-1)$.

Proof) In view of (48) and the definition of inverse elasticities we can write

$$\theta = -\left(\sum\nolimits_i (1 + t_r)\, \varepsilon_{ir}^{-} + \sum\nolimits_i \eta_{ir}^{-}\right) / t_r \quad .$$

We only need to solve for t_r to obtain the desired formula.

We next proceed to express β in terms of x. For this we rewrite (45) in the following way:

$$\begin{pmatrix} Q^r & \sum_i \left(P_r^i - Q_r^i\right) x_i + \left(P^r - Q^r\right) \\ 1 & \sum_i \left(P_n^i - Q_n^i\right) x_i \end{pmatrix} \begin{pmatrix} \alpha g_n \\ \beta \end{pmatrix} = \begin{pmatrix} P^r \\ 1 \end{pmatrix} f_n$$
$$(r = 2, 3 \cdots, n-1) \tag{51}$$

For each $r = 1, 2, ..., n-1$, let $\Delta(r)$ be the determinant of the 2×2 matrix on the left-hand side of (51). Then we have

$$\begin{aligned} \Delta(r) &= -\sum\nolimits_i (P_r^i - P^r P_n^i) x_i + \sum\nolimits_i (Q_r^i - Q^r Q_n^i) x^i - (P^r - Q^r)(1 + \sum\nolimits_i P_n^i x_i) \\ &= \sum\nolimits_i (b_{ir} - a_{ir}) x_i - (P^r - Q^r)(1 + \sum\nolimits_i P_n^i x_i) \\ &= -t_r Q^r (\theta + 1 + \sum\nolimits_i P_n^i x_i) \end{aligned} \tag{52}$$

using (48).

Now if $\theta + 1 + \sum_i P_n^i x_i$ is non-zero, $\Delta(r)$ is non-zero(See the Remark below). Hence from (51) we can solve for β as

$$\begin{aligned}\beta &= \frac{-(P^r - Q^r) f_n}{\Delta(r)} \\ &= \frac{f_n}{\theta + 1 + \sum_i P_n^i x_i}.\end{aligned} \tag{53}$$

This Lagrange multiplier shows how welfare changes in response to a change in R. Indeed, by differentiating (43) with respect to R and using (44) we have

$$\begin{aligned}\frac{\partial z}{\partial R} &= \sum\nolimits_r z_r \frac{\partial x_r}{\partial R} + \beta \\ &= \beta,\end{aligned} \tag{54}$$

where the summation r runs from 1 to n.

This shows that β and $\partial z/\partial R$ have the same sign. As in the one commodity case (see Section 2 example (iii)), however, the tax revenue may exhibit rather complicated behavior as a function of the amount of consumption x or as a function of initial endowment, ω. Disregarding abnormal situations, we shall confine ourselves to the range of R (and x) where the constraint is effective in the sense that $\partial z/\partial R < 0$. This means that we consider the part where the Laffer curve is decreasing.

Remark 4

If $R = 0$, the allocation which is realized is Pareto optimal (where $P^i = Q^i$ for $i = 1, 2, ..., n-1$). We can show that if R is close to zero (and hence the resulting allocation is close to a Pareto optimal one in the regular case), the constraint is actually effective. Indeed using Lemma 1 we have

$$R(\theta + 1 + \sum\nolimits_i P_n^i x_i) = x'(A - B)x + (P - Q)'x(1 + \sum\nolimits_i P_n^i x_i)$$

where $P = (P^1, ..., P^{n-1})$ and $Q = (Q^1, ..., Q^{n-1})$.

In the above equation the first term on the right-hand side is negative and the second term is zero at the Pareto optimal point. Hence by continuity if $R > 0$ is sufficiently small then $\theta + 1 + \sum_i P_n^i x_i < 0$, and $\partial z/\partial R = \beta < 0$. ¤

We next analyze the effect of a change in ω_i. Differentiating (43) with respect to ω_i and using (44) and (53) we obtain

$$\frac{\partial z}{\partial \omega_i} = \sum_r z_r \frac{\partial x_r}{\partial \omega_i} + f_i - \beta \sum_r P_i^r x^r$$
$$= f_i - \beta \sum_r P_i^r x^r \ ,$$
$$(i = 1, 2, ..., n), \tag{55}$$

where the summation i runs from 1 to n.

In view of (53) it then follows that

$$\frac{\partial z}{\partial \omega_n} = \frac{(\theta + 1) f_n}{\theta + 1 + \sum_i P_n^i x_i}$$
$$= \beta(\theta + 1). \tag{56}$$

The above analysis implies a result that corresponds to Proposition 1 in Section 4.

Proposition 2

If all of the non-numeraire goods are normal against the numeraire good n then $\partial z / \partial \omega_n > f_n$, i.e., the social utility of the commodity n is greater than its private marginal utility.

Proof) Notice that β is negative by the previous argument and that P_n^i is positive if commodity i is normal against commodity n, since Remark 1(i) applies for commodities i and n as well as for commodities 1 and n. The proof of the proposition is then obvious from (55).

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We now proceed to measure the welfare loss that would result from a small increase in R by the amount of ω_n that would just compensate for the initial change. The next theorem asserts that this loss can be expressed in terms of the inverse demand and supply elasticities.

Theorem 4

If $R > 0$ the rate of change in ω_n that would just compensate for a marginal change in R is given by

$$-1/(\theta+1) = t_r/(\sum_{i=1}^{n}(1+t_r)\varepsilon_{ir}^{-} + \sum_{i=1}^{n}\eta_{ir}^{-} - t_r)\ ,\ (r = 2, 3\cdots n-1).$$

If R is sufficiently small (so that the constraint is effective), and all of the non-numeraire consumption goods are normal against the numeraire good then the above measure is positive.

Proof) By (54) and (56), we have

$$\left.\frac{\partial \omega_n}{\partial R}\right|_{z\,:\,\mathrm{const}} = -\frac{\partial z}{\partial R}/\frac{\partial z}{\partial \omega_n} = -1/(\theta+1).$$

Hence the first part of the theorem follows from the relation in the proof of Ramsey's formula. The second part of the theorem is clear from Lemma 2 and the assumption that $P_n^i > 0$ and $x_i \geq 0$ for any non-numeraire consumption good i.

We note that the above measure reduces to formula (5) in Section 2 when $n = 1$. It clearly indicates how inverse elasticities affect the welfare loss.

7 Conclusion

We have provided a formula for the welfare change arising from a change in a distortion when all other distortions are adjusted second-best optimally. We measured the welfare loss in a non-optimal state by the amount of the numeraire that would just compensate for a one unit change in the given distortion.

We have evaluated the welfare loss in three different situations; first, when the original distortion is specific distortion, second, when it is an ad valorem distortion and third where there is Ramsey type revenue constraint. Our analysis confirms the conjecture by Diewert(1981) that welfare loss will increase if substitutability in consumption increases. This also support a similar view expressed by Samuelson(2001) for a public good economy. Similar formulae could be obtained for cases when several distortions are exogenously given. But the corresponding formula would become very complicated unless exogenously given distortions are chosen (second-best) optimally in some senses.

The formula for the welfare loss for each case is expressed in terms of inverse elasticities of Antonelli matrices. The result may be expressed using ordinally demand and supply elasticities. The expression takes on a simple form in the three commodity case with a quasi-linear utility function (see Remark 2). To our knowledge, there has not been a general study of sensitivity of welfare changes, apart from that near the first-best point. We hope that the present study gives information about the strength of various price distinctions from the view-point of welfare loss.

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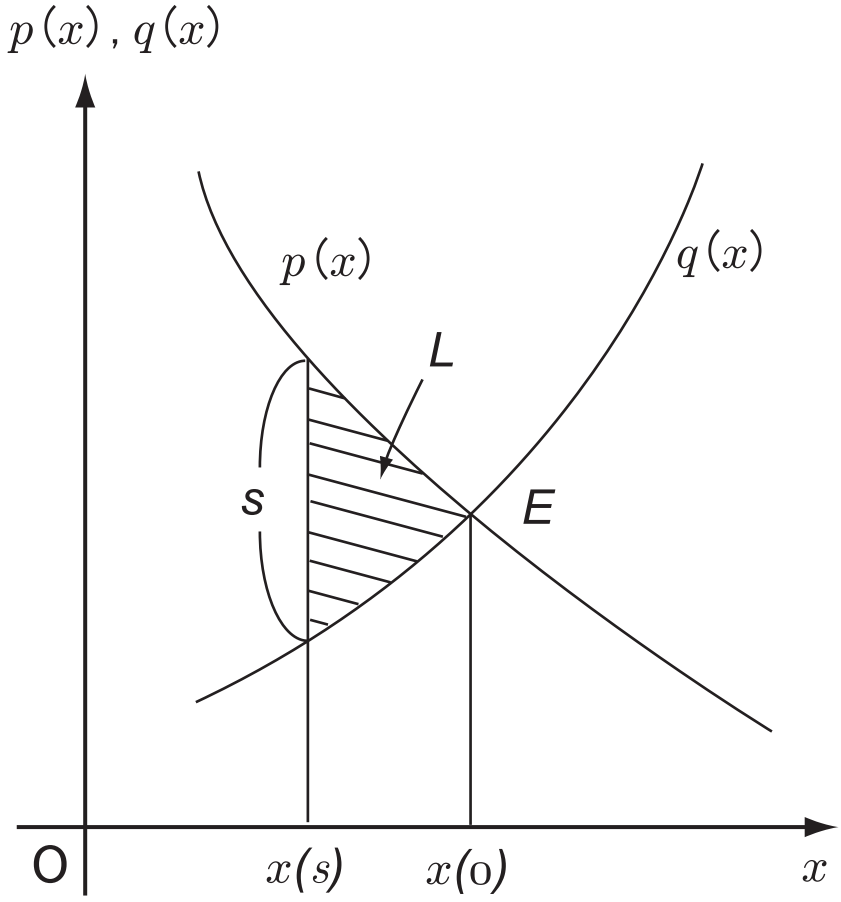

Fig. 1